Susan Larsen

Content

2 The Metric tensor and Vector Transformations. 3
2.1 Properties 3
2.2 The Time-Signature of a Metric 3
2.2.1 Negative Time-Signature 3
2.2.2 Positive Time-Signature 3
2.2.3 Change of sign 4
2.3 The physical units of various quantities 4
2.4 The conversion factors between length and mass: 5
2.5 The conversion factor between Gab and Tab 5
2.6 A few remarks on geometry 5
2.6.1 Length in spacetime compared to Euclidian space. 5
2.6.2 The circle. 6
2.7 Flat Minkowski space-time: The space-time of special relativity 6
2.7.1 Flat Spacetime in Spherical polar coordinates 6
2.7.2 Lorentz Boosts – important features in special relativity 7
2.8 The Rindler Space-time. 8
2.8.1 How to find the coordinate transformation – the Jacobian matrix 8
2.9 Other realizations of the flat space-time. 9
2.9.1 Coordinate transformations 9
2.9.2 The Penrose Diagram for Flat Space-time 10
2.10 Local inertial frames 11
2.10.1 The metric of a Sphere at the North Pole. 11
2.11 Static Weak Field Metric. 13
2.11.1 Rates of Emission and Reception 13
2.11.2 Newton's second equation: 13
2.12 A Fifth Dimension? 14
2.13 Flat Space 15
2.13.1 How to find the coordinate transformation – the Jacobian matrix 16
2.13.2 Flat space with a singularity 16
2.13.3 Three-dimensional flat space in spherical coordinates and vector transformation 17
2.14 The line-element and metric of an ellipsoid 17
2.15 Length, Area, Volume and Four-Volume for Diagonal Metrics 18
2.15.1 Area and Volume Elements of a Sphere 18

2.15.2 The dimensions of a peanut 19
2.15.3 The dimensions of an egg 19
2.15.4 Distance, Area, Volume and four-volume of a metric 20
2.15.5 Distance, Area and Volume in the Curved Space of a Constant Density Spherical Star or a Homogenous Closed Universe 20
2.16 Inverting a metric tensor. 21
2.16.1 Inverting a Diagonal tensor. 21
2.16.2 Inverting a non-diagonal two-dimensional tensor. 22
2.16.3 Inverting a non-diagonal four-dimensional tensor. 22
2.16.4 Example: The Gödel Metric tensor. 22
2.16.5 The inverse metric of the Kerr Spinning Black Hole 23
2.17 Conformal space-time 24
Citerede værker 25

Space-time		Line-element	Chap-ter
Egg geometry	ds^2	$= a^2[(\cos^2\theta + 4\sin^2\theta)d\theta^2 + \sin^2\theta\, d\phi^2]$	2
Ellipsoid	ds^2	$= adx^2 + bdy^2 + cdz^2$	2
Example: Four-dimensional space-time	ds^2	$= -(1-Ar^2)^2dt^2 + (1-Ar^2)^2dr^2 + r^2d\theta^2 + r^2\sin^2\theta\, d\phi^2$	2
A Fifth dimension	ds^2	$= -dx^2 + dx^2 + dy^2 + dz^2 + R^2d\Omega^2$	2
Flat Minkowsky space-time	ds^2	$= -dt^2 + dx^2 + dy^2 + dz^2$	2
Flat Minkowsky space-time in polar coordinates	ds^2	$= -dt^2 + dr^2 + r^2d\theta^2 + r^2\sin^2\theta\, d\phi$	2, 10
Flat space-time in Eddington Finkelstein coordinates	ds^2	$= -dv^2 + 2dvdr + r^2d\theta^2 + r^2\sin^2\theta\, d\phi$	2
General four-dimensional diagonal metric	ds^2	$= g_{00}dx^0dx^0 + g_{11}dx^1dx^1 + g_{22}dx^2dx^2 + g_{33}dx^3dx^3$	2
Gödel metric	ds^2	$= \frac{1}{2\omega^2}\left((dt + e^xdz)^2 - dx^2 - dy^2 - \frac{1}{2}e^{2x}dz^2\right)$	2,9
Homogenous closed universe	dS^2	$= \frac{1}{1-\left(\frac{r}{a}\right)^2}dr^2 + r^2d\theta^2 + r^2\sin^2\theta\, d\phi^2$	2
Kerr Spinning black hole	ds^2	$= \left(1-\frac{2mr}{\Sigma}\right)dt^2 + \frac{4amr\sin^2\theta}{\Sigma}dtd\phi - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2 - \left(r^2 + a^2 + \frac{2a^2mr\sin^2\theta}{\Sigma}\right)\sin^2\theta\, d\phi^2$	2, 11
North pole	ds^2	$= \left(1-\frac{y^2}{3a^2}\right)dx^2 + \frac{xy}{3a^2}dxdy + \left(1-\frac{x^2}{3a^2}\right)dy^2$	2
Peanut geometry	ds^2	$= a^2d\theta^2 + a^2f^2(\theta)d\phi^2f(\theta)$	2
Rindler metric	ds^2	$= -X^2dT^2 + dX^2$	2,3,4,7

Static Weak field	ds^2	$= -\left(1+\frac{2\Phi(x^i)}{c^2}\right)(cdt)^2 + \left(1-\frac{2\Phi(x^i)}{c^2}\right)(dx^2+dy^2+dz^2)$	2
Three-dimensional flat space in polar coordinates	ds^2	$= dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$	2,7, 13
Two-dimensional flat space in polar coordinates	dS^2	$= dr^2 + r^2 d\phi^2$	2,5

2 The Metric tensor and Vector Transformations.

2.1 [a]Properties

$$g_{ab}g^{bc} = \delta^c_a$$

2.2 [b]The Time-Signature of a Metric

A metric can have either positive or negative time-signature and we always have to be careful in which signature we are working because it affects various quantities.

2.2.1 [1]Negative Time-Signature

Minkowsky space:

$$ds^2 = \eta_{ab} dx^a dx^b$$

$$\eta_{ab} = \begin{Bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{Bmatrix}$$

Geodesics:

$ds^2 < 0$ geodesics are time-like
$ds^2 > 0$ geodesics are space-like
$ds^2 = 0$ null-vector[2]

Proper time:

$d\tau^2 = -ds^2$
$\Rightarrow$ $d\tau^2 > 0$ geodesics are time-like
$d\tau^2 < 0$ geodesics are space-like

The four velocity is time-like

$$u \cdot u = -1$$

2.2.2 [3] [4]Positive Time-Signature

Minkowsky space:

$$ds^2 = \eta_{ab} dx^a dx^b$$

[1] Often you will see that this is called *positive signature* because this refers to the sum of the diagonal
[2] About the null-vector I would like to quote Roger Penrose (Penrose, 2004, s. 414): "… unlike the case for a massive particle $\int ds$ is zero for a world line of a photon (so non-coincident point on the world-line can be 'zero distance' apart). This would also be true for any other particle that travels with the speed of light. The time 'experienced' by such a particle would always be zero, no matter how far it travels!"
[3] Often you will see that this is called *negative signature* because this refers to the sum of the diagonal
[4] About the positive time-signature I would like to quote Roger Penrose (Penrose, 2004, s. 413): "… $ds^2 = dt^2 - dx^2 - dy^2 - dz^2$ is more directly physical, because it is positive along the timelike curves that are the allowable worldliness of massive particles, the integral $\int ds$ being directly interpretable as the actual physical time measured by an ideal clock …"

$$\eta_{ab} = \begin{Bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{Bmatrix}$$

Geodesics:

$ds^2 > 0$ geodesics are time-like

$ds^2 < 0$ geodesics are space-like

$ds^2 = 0$ null-vector

Proper time:

$$d\tau^2 = ds^2$$

$\Rightarrow \quad d\tau^2 > 0$ geodesics are time-like

$d\tau^2 < 0$ geodesics are space-like

The four velocity is time-like

$$u \cdot u = 1$$

[5]It is important to notice that in Special Relativity the sign convention and notation for time-like trajectories is

$$d\tau^2 = dt^2 - dx^2 - dy^2 - dz^2$$

where τ can be interpreted as the local time-coordinate for the moving particle.

2.2.3 Change of sign

What happens to various quantities when a metric g_{ab} changes time-signature.

The metric tensor	g_{ab}		Changes sign
The line element	ds^2	$= g_{ab}dx^a dx^b$	Changes sign
Christoffel symbols	$\Gamma^a{}_{bc}$	$= \frac{1}{2} g^{ad}\left(\frac{\partial g_{db}}{\partial x^c} + \frac{\partial g_{dc}}{\partial x^b} - \frac{\partial g_{bc}}{\partial x^d}\right)$	No change
	Γ_{abc}	$= \frac{1}{2}(\partial_a g_{bc} + \partial_b g_{ca} - \partial_c g_{ab})$	Changes sign
Riemann tensor	$R^a{}_{bcd}$	$= \partial_c \Gamma^a{}_{bd} - \partial_d \Gamma^a{}_{bc} + \Gamma^e{}_{bd}\Gamma^a{}_{ec} - \Gamma^e{}_{bc}\Gamma^a{}_{ed}$	No change
	R_{abcd}	$\equiv g_{ae}R^e{}_{bcd}$	Changes sign
Ricci tensor	R_{ab}	$= R^c{}_{acb}$	No change
[6]Ricci scalar	R	$= g^{ab}R_{ab}$	Changes sign
Einstein tensor	G_{ab}	$= R_{ab} - \frac{1}{2} g_{ab}R$	No change
Energy tensor	$8\pi G T_{ab}$	$= G_{ab}$	No change
Cosmological constant, Λ	G_{ab}	$= R_{ab} - \frac{1}{2} g_{ab}R + g_{ab}\Lambda$	Changes sign

2.3 The physical units of various quantities

Doing the theoretical calculations it is most convenient to work in geometrical units, i.e. $c = G = 1$. But what are the dimensions of the various quantities.

$$[g_{ab}] = 1$$

$$[\Gamma^a{}_{bc}] = \left[\frac{\partial g_{ab}}{\partial x_i}\right] = length^{-1}$$

[5] I recommend the video: https://www.youtube.com/watch?v=Cg2tOUTE2F4 and it's Q&A

[6] The Ricci scalar is also referred to as the curvature of the space, where $R > 0$ and $R < 0$ represents positive and negative curvature. But also here you have to be careful because the sign of R depends on the signature. Look at the cover of (Charles Misner, 2017) to see a list of sign conventions.

$$\begin{aligned}
[R^a{}_{bcd}] &= [\Gamma^a{}_{bc}]^2 = length^{-2} \\
[R_{ab}] &= [R^a{}_{bcd}] = length^{-2} \\
[R] &= [R^a{}_{bcd}] = length^{-2} \\
[G_{ab}] &= [R^a{}_{bcd}] = length^{-2} \\
[\Lambda] &= [G_{ab}] = length^{-2} \\
[T_{ab}] &= energy * voloume^{-1} \\
&= mass * (length * time^{-1})^2 * length^{-3} = mass * length^{-1} * time^{-2}
\end{aligned}$$

2.4 The conversion factors between length and mass:

Recall

$$\begin{aligned}
[G] &= m^3 kg^{-1} s^{-2} \\
[c] &= ms^{-1}
\end{aligned}$$

Now

$$\begin{aligned}
[lenght] &= [mass]^a [G]^b [c]^c \\
\Rightarrow \quad m &= (kg)^a (m^3 kg^{-1} s^{-2})^b (ms^{-1})^c = (kg)^{a-b} (m)^{3b+c} (s)^{-2b-c} \\
\Rightarrow \quad a - b &= 0 \\
3b + c &= 1 \\
-2b - c &= 0 \\
\Rightarrow \quad a &= 1 \\
b &= 1 \\
c &= -2 \\
\Rightarrow \quad length(m) &= \frac{mass(kg) G}{c^2}
\end{aligned}$$

2.5 The conversion factor between G_{ab} and T_{ab}

We look at

$$\begin{aligned}
[G_{ab}] &= [c^i G^j T_{ab}] \\
\Rightarrow \quad length^{-2} &= (lengt * time^{-1})^i (length^3 * mass^{-1} * time^{-2})^j mass * length^{-1} * time^{-2} \\
&= length^{i+3j-1} * time^{-i-2j-2} * mass^{-j+1} \\
\Rightarrow \quad j &= 1 \\
i &= -4 \\
\Rightarrow \quad [G_{ab}] &= [c^{-4} G T_{ab}]
\end{aligned}$$

2.6 A few remarks on geometry

2.6.1 [c]Length in spacetime compared to Euclidian space.

In two-dimensional Euclidian space the distance between two events $(x_1, \quad x_2)$ and $(x'_1, \quad x'_2)$ is defined as

$$|x|^2 = (x'_1 - x_1)^2 + (x'_2 - x_2)^2$$

In a two-dimensional spacetime[7] the distance between two events $(x^t, \quad x^1)$ and $(x^{t'}, \quad x^{1'})$ is defined as

$$(\Delta s)^2 = -\left(x^{t'} - x^t\right)^2 + \left(x^{1'} - x^1\right)^2$$

We choose the three points (events)

$$\begin{aligned}
A &= (1, \quad 7) \\
B &= (4, \quad 10) \\
C &= (7, \quad 7)
\end{aligned}$$

And compare the distances between the events in the Euclidian and the two-dimensional spacetime respectively.

[7] $\eta_{ab} = \begin{Bmatrix} -1 & \\ & 1 \end{Bmatrix}$

Euclidian		Two-dimensional spacetime		
$\lvert AB\rvert^2$	$= (4-1)^2 + (10-7)^2 = 18$	$(\Delta s_{AB})^2$	$= -(4-1)^2 + (10-7)^2 = 0$	Null-vector
$\lvert AC\rvert^2$	$= (7-1)^2 + (7-7)^2 = 36$	$(\Delta s_{AC})^2$	$= -(7-1)^2 + (7-7)^2 = -36$	Time-like
$\lvert BC\rvert^2$	$= (7-4)^2 + (7-10)^2 = 18$	$(\Delta s_{BC})^2$	$= -(7-4)^2 + (7-10)^2 = 0$	Null-vector

2.6.1.1 *Change of signature*

It is instructive to change the signature[8] an copy the calculation i.e.

$$(\Delta s)^2 = \left(x^{t'} - x^t\right)^2 - \left(x^{1'} - x^1\right)^2$$

Euclidian		Two-dimensional spacetime		
$\lvert AB\rvert^2$	$= (4-1)^2 + (10-7)^2 = 18$	$(\Delta s_{AB})^2$	$= (4-1)^2 - (10-7)^2 = 0$	Null-vector
$\lvert AC\rvert^2$	$= (7-1)^2 + (7-7)^2 = 36$	$(\Delta s_{AC})^2$	$= (7-1)^2 - (7-7)^2 = 36$	Time-like
$\lvert BC\rvert^2$	$= (7-4)^2 + (7-10)^2 = 18$	$(\Delta s_{BC})^2$	$= (7-4)^2 - (7-10)^2 = 0$	Null-vector

And notice that

$$|AB| + |BC| > |AC|$$
$$\Delta s_{AB} + \Delta s_{BC} < \Delta s_{AC}$$

So, if we imagine the point ABC is a triangle, the distance along AC is the shortest in the Euclidian space, as we would expect, but in the space-time it is not.

2.6.2 [d]The circle

The circle in the Euclidian plane is described by

$$r^2 = x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta$$
$$x = r\cos\theta$$
$$y = r\sin\theta$$

The analogy to the circle in the two-dimensional spacetime is

$$r^2 = -t^2 + x^2$$
$$t = r\sinh\theta$$
$$x = r\cosh\theta$$

Introducing hyperbolical geometry

2.7 [e]Flat Minkowski space-time: The space-time of special relativity

Flat Minkowski spacetime is the mathematical setting in which Einstein's special theory of relativity is most conveniently formulated. In Cartesian coordinates with $c = 1$ the line element is

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

and the metric

$$g_{\mu\nu} = \begin{Bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Bmatrix}$$

$\mathrm{d}s^2 < 0$: time-like, inside the light cone
$\mathrm{d}s = 0$: null-vector, on the light cone
$\mathrm{d}s^2 > 0$: space-like, outside the light cone

2.7.1 [f]Flat Spacetime in Spherical polar coordinates

The spherical part of the metric can be transformed[9] into spherical polar coordinates by

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$
$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$

---

[8] $\eta_{ab} = \begin{Bmatrix} 1 & \\ & -1 \end{Bmatrix}$

[9] Notice: No mixing of space and time in this transformation

$$
\begin{aligned}
z &= r\cos\theta \\
\Rightarrow \quad dx &= \sin\theta\cos\phi\,dr + r\cos\theta\cos\phi\,d\theta - r\sin\theta\sin\phi\,d\phi \\
dy &= \sin\theta\sin\phi\,dr + r\cos\theta\sin\phi\,d\theta + r\sin\theta\cos\phi\,d\phi \\
dz &= \cos\theta\,dr - r\sin\theta\,d\theta \\
\Rightarrow \quad dx^2 &= (\sin\theta\cos\phi\,dr + r\cos\theta\cos\phi\,d\theta - r\sin\theta\sin\phi\,d\phi)^2 \\
&= \sin^2\theta\cos^2\phi\,dr^2 + 2r\sin\theta\cos\theta\cos^2\phi\,drd\theta - 2r\sin^2\theta\sin\phi\cos\phi\,drd\phi \\
&\qquad + r^2\cos^2\theta\cos^2\phi\,d\theta^2 - 2r^2\sin\theta\cos\theta\sin\phi\cos\phi\,d\theta d\phi \\
&\qquad + r^2\sin^2\theta\sin^2\phi\,d\phi^2 \\
dy^2 &= (\sin\theta\sin\phi\,dr + r\cos\theta\sin\phi\,d\theta + r\sin\theta\cos\phi\,d\phi)^2 \\
&= \sin^2\theta\sin^2\phi\,dr^2 + 2r\sin\theta\cos\theta\sin^2\phi\,drd\theta + 2r\sin^2\theta\sin\phi\cos\phi\,drd\phi \\
&\qquad + r^2\cos^2\theta\sin^2\phi\,d\theta^2 + 2r^2\cos\theta\sin\theta\cos\phi\sin\phi\,d\theta d\phi \\
&\qquad + r^2\sin^2\theta\cos^2\phi\,d\phi^2 \\
dz^2 &= (\cos\theta\,dr - r\sin\theta\,d\theta)^2 = \cos^2\theta\,dr^2 + r^2\sin^2\theta\,d\theta^2 - 2r\cos\theta\sin\theta\,d\theta dr \\
\Rightarrow \quad ds^2 &= {}^{10\ 11\ 12\ 13\ 14} - dt^2 + dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\,d\phi^2 \\
g_{\mu\nu} &= \begin{Bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{Bmatrix}
\end{aligned}
$$

2.7.2 [g]Lorentz Boosts – important features in special relativity

In this chapter we will look at an important transformation that mixes both space and time – Lorentz boosts. Lorentz boosts is a hyperbolic Lorentz transformation[15], which mixes space and time.
We can connect the two systems $(t, \quad x, \quad y, \quad z)$ and $(t', \quad x', \quad y', \quad z')$ with the transformations

$$t' = t\cosh\theta - x\sinh\theta \tag{2.1.}$$
$$x' = -t\sinh\theta + x\cosh\theta \tag{2.2.}$$
$$y' = y \tag{2.3.}$$
$$z' = z \tag{2.4.}$$

Where the parameter θ is the hyperbolic angle which can vary from $-\infty$ to $+\infty$ and is related to the constant relative velocity (v) of the two systems. It can be found if we look at a particle at rest in $x' = 0$

$$\Rightarrow \quad x' = -t\sinh\theta + x\cosh\theta = 0$$
$$\Rightarrow \quad v = \frac{dx}{dt} = \frac{x}{t} = \frac{\sinh\theta}{\cosh\theta} = \tanh\theta$$

And θ can in this context be treated as a constant, some like to think of it as a rotation between the two systems.
Next we check if ds^2 is conserved
We need

$$dt' = \cosh\theta\,dt - \sinh\theta\,dx$$

---

[10] $= \sin^2\theta\cos^2\phi\,dr^2 + 2r\sin\theta\cos\theta\cos^2\phi\,drd\theta - 2r\sin^2\theta\sin\phi\cos\phi\,drd\phi + r^2\cos^2\theta\cos^2\phi\,d\theta^2 - 2r^2\sin\theta\cos\theta\sin\phi\cos\phi\,d\theta d\phi + r^2\sin^2\theta\sin^2\phi\,d\phi^2 + \sin^2\theta\sin^2\phi\,dr^2 + 2r\sin\theta\cos\theta\sin^2\phi\,drd\theta + 2r\sin^2\theta\sin\phi\cos\phi\,drd\phi + r^2\cos^2\theta\sin^2\phi\,d\theta^2 + 2r^2\cos\theta\sin\theta\cos\phi\sin\phi\,d\theta d\phi + r^2\sin^2\theta\cos^2\phi\,d\phi^2 + \cos^2\theta\,dr^2 - 2r\cos\theta\sin\theta\,drd\theta + r^2\sin^2\theta\,d\theta^2 =$

[11] $= \sin^2\theta\cos^2\phi\,dr^2 + 2r\sin\theta\cos\theta\cos^2\phi\,drd\theta + r^2\cos^2\theta\cos^2\phi\,d\theta^2 + r^2\sin^2\theta\sin^2\phi\,d\phi^2 + \sin^2\theta\sin^2\phi\,dr^2 + 2r\sin\theta\cos\theta\sin^2\phi\,drd\theta + r^2\cos^2\theta\sin^2\phi\,d\theta^2 + r^2\sin^2\theta\cos^2\phi\,d\phi^2 + \cos^2\theta\,dr^2 - 2r\cos\theta\sin\theta\,drd\theta + r^2\sin^2\theta\,d\theta^2$

[12] $= (\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta)dr^2 + (2r\sin\theta\cos\theta\cos^2\phi - 2r\cos\theta\sin\theta + 2r\sin\theta\cos\theta\sin^2\phi)drd\theta + (r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta)d\theta^2 + (r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi)d\phi^2 =$

[13] $= (\sin^2\theta(\cos^2\phi + \sin^2\phi) + \cos^2\theta)dr^2 + (2r\sin\theta\cos\theta(\cos^2\phi + \sin^2\phi) - 2r\cos\theta\sin\theta)drd\theta + r^2(\cos^2\theta(\cos^2\phi + \sin^2\phi) + \sin^2\theta)d\theta^2 + r^2\sin^2\theta(\sin^2\phi + \cos^2\phi)d\phi^2 =$

[14] $= (\sin^2\theta + \cos^2\theta)dr^2 + (2r\sin\theta\cos\theta - 2r\cos\theta\sin\theta)drd\theta + r^2(\cos^2\theta + \sin^2\theta)d\theta^2 + r^2\sin^2\theta\,d\phi^2 =$

[15] Remember that a Lorentz transformation conserves the line element ds^2

$$
\begin{aligned}
dx' &= -\sinh\theta\, dt + \cosh\theta\, dx \\
dy' &= dy \\
dz' &= dz \\
dt'^2 &= {}^{16}\cosh^2\theta\, dt^2 + \sinh^2\theta\, dx^2 - 2\cosh\theta\sinh\theta\, dtdx \\
dx'^2 &= {}^{17}\sinh^2\theta\, dt^2 + \cosh^2\theta\, dx^2 - 2\cosh\theta\sinh\theta\, dtdx \\
dy'^2 &= dy^2 \\
dz'^2 &= dz^2 \\
ds'^2 &= -dt'^2 + dx'^2 + dy'^2 + dz'^2 \\
&= {}^{18} - (\cosh^2\theta - \sinh^2\theta)dt^2 + (\cosh^2\theta - \sinh^2\theta)dx^2 + dy^2 + dz^2 \\
&= -dt^2 + dx^2 + dy^2 + dz^2
\end{aligned}
$$

Finally we rewrite eq. (2.1.), eq. (2.2.), eq. (2.3.) and eq. (2.4.) in the more familiar form, choosing

$$\gamma = {}^{19}\frac{1}{\sqrt{1-v^2}} = {}^{20}\cosh\theta$$

$$t' = \gamma(t - vx) = \cosh\theta\left(t - \frac{\sinh\theta}{\cosh\theta}x\right) = t\cosh\theta - x\sinh\theta \qquad (2.1.)$$

$$x' = \gamma(x - vt) = \cosh\theta\left(x - \frac{\sinh\theta}{\cosh\theta}t\right) = -t\sinh\theta + x\cosh\theta \qquad (2.2.)$$

$$y' = y \qquad (2.3.)$$

$$z' = z \qquad (2.4.)$$

So by requiring two things: 1) ds^2 has to be conserved, 2) Mixing of space and time, we get the well-known transformation equations in special relativity.

2.8 [h]The Rindler Space-time

Yet another realization of flat space-time in two dimensions is the line element

$$
\begin{aligned}
ds^2 &= -X^2 dT^2 + dX^2 \\
g_{\mu\nu} &= \begin{Bmatrix} -X^2 & 0 \\ 0 & 1 \end{Bmatrix}
\end{aligned}
$$

This can be found from the coordinate transformation

$$
\begin{aligned}
t &= X\sinh(T) \\
x &= X\cosh(T) \\
\Rightarrow \quad dt &= X\cosh(T)\, dT + \sinh(T)\, dX \\
dx &= X\sinh(T)\, dT + \cosh(T)\, dX \\
\Rightarrow \quad ds^2 &= -dt^2 + dx^2 = -(\sinh(T)\, dX + X\cosh(T)\, dT)^2 + (\cosh(T)\, dX + X\sinh(T)\, dT)^2 \\
&= {}^{21}(\cosh^2(T) - \sinh^2(T))dX^2 - X^2(\cosh^2(T) - \sinh^2(T))dT^2 = dX^2 - X^2 dT^2
\end{aligned}
$$

2.8.1 How to find the coordinate transformation – the Jacobian matrix

The coordinate transformation can be written in terms of the Jacobian matrix

---

[16] $= (\cosh\theta\, dt - \sinh\theta\, dx)^2 =$

[17] $= (-\sinh\theta\, dt + \cosh\theta\, dx)^2 =$

[18] $= -(\cosh^2\theta\, dt^2 + \sinh^2\theta\, dx^2 - 2\cosh\theta\sinh\theta\, dtdx) + \sinh^2\theta\, dt^2 + \cosh^2\theta\, dx^2 - 2\cosh\theta\sinh\theta\, dtdx + dy^2 + dz^2 =$

[19] Actually it is $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ but we choose to work in a frame were $c = 1$

[20] $1 - v^2 = 1 - \tanh^2\theta = 1 - \left(\frac{\sinh\theta}{\cosh\theta}\right)^2 = 1 - \frac{\cosh^2\theta - 1}{\cosh^2\theta} = \frac{1}{\cosh^2\theta}$

[21] $= -\sinh^2(T)\, dX^2 - X^2\cosh^2(T)\, dT^2 - 2X\sinh(T)\cosh(T) dX\, dT + \cosh^2(T)\, dX^2 + X^2\sinh^2(T)\, dT^2 + 2X\cosh(T)\sinh(T)\, dXdT =$

$$\begin{Bmatrix} dt \\ dx \end{Bmatrix} = \begin{Bmatrix} \frac{\partial t}{\partial T} & \frac{\partial t}{\partial X} \\ \frac{\partial x}{\partial T} & \frac{\partial x}{\partial X} \end{Bmatrix} \begin{Bmatrix} dT \\ dX \end{Bmatrix}$$

$$\Rightarrow \quad dt = \frac{\partial t}{\partial T} dT + \frac{\partial t}{\partial X} dX$$

$$dx = \frac{\partial x}{\partial T} dT + \frac{\partial x}{\partial X} dX$$

$$\Rightarrow \quad dt^2 = \left(\frac{\partial t}{\partial T} dT + \frac{\partial t}{\partial X} dX\right)^2 = \left(\frac{\partial t}{\partial T}\right)^2 dT^2 + \left(\frac{\partial t}{\partial X}\right)^2 dX^2 + 2\frac{\partial t}{\partial T}\frac{\partial t}{\partial X} dTdX$$

$$dx^2 = \left(\frac{\partial x}{\partial T} dT + \frac{\partial x}{\partial X} dX\right)^2 = \left(\frac{\partial x}{\partial T}\right)^2 dT^2 + \left(\frac{\partial x}{\partial X}\right)^2 dX^2 + 2\frac{\partial x}{\partial T}\frac{\partial x}{\partial X} dTdX$$

$$\Rightarrow \quad ds^2 = -dt^2 + dx^2$$

$$= \left(\left(\frac{\partial x}{\partial T}\right)^2 - \left(\frac{\partial t}{\partial T}\right)^2\right) dT^2 + \left(\left(\frac{\partial x}{\partial X}\right)^2 - \left(\frac{\partial t}{\partial X}\right)^2\right) dX^2$$

$$= -X^2 dT^2 + dX^2$$

$$\Rightarrow \quad -X^2 = \left(\frac{\partial x}{\partial T}\right)^2 - \left(\frac{\partial t}{\partial T}\right)^2 \tag{2.5.}$$

$$1 = \left(\frac{\partial x}{\partial X}\right)^2 - \left(\frac{\partial t}{\partial X}\right)^2 \tag{2.6.}$$

If you look at eq. (2.6.) the solution is obviously hyperbolic i.e.

$$\Rightarrow \quad t = X \sinh(T)$$

$$x = X \cosh(T)$$

$$\Rightarrow \quad -X^2 = \left(\frac{\partial x}{\partial T}\right)^2 - \left(\frac{\partial t}{\partial T}\right)^2 = (X\sinh(T))^2 - (X\cosh(T))^2 = -X^2(\cosh^2(T) - \sinh^2(T))$$

$$= -X^2$$

$$1 = \left(\frac{\partial x}{\partial X}\right)^2 - \left(\frac{\partial t}{\partial X}\right)^2 = \cosh^2(T) - \sinh^2(T) = 1$$

2.9 Other realizations of the flat space-time

2.9.1 [22] [i]Coordinate transformations

The following line element corresponds to flat space-time

$$ds^2 = -dt^2 + 2dxdt + dy^2 + dz^2$$

with the metric tensor

$$g_{ab} = \begin{Bmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Bmatrix}$$

Find a coordinate transformation that puts the line element in the usual flat space-time form

$$ds^2 = -dt'^2 + dx'^2 + dy'^2 + dz'^2$$

We want to find the matrix, $\Lambda_{ab}^{a'b'}$ that transforms $g_{a'b'}$ into g_{ab}

$$g_{ab} = \Lambda_{ab}^{a'b'} g_{a'b'}$$

We have

$$\Lambda_{ab}^{a'b'} = \Lambda_{ab}^{a'b'} g_{a'b'} g^{a'b'} = g_{ab} g^{a'b'} = \begin{Bmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Bmatrix} \begin{Bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Bmatrix}$$

[22] Notice: Mixing of space and time in this transformation

$$= {}^{23}\begin{Bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Bmatrix}$$

$$\Rightarrow \quad t = t' - x'$$
$$x = t'$$
$$y = y'$$
$$z = z'$$
$$\Rightarrow \quad dt = dt' - dx'$$
$$dx = dt'$$
$$dy = dy'$$
$$dz = dz'$$
$$\Rightarrow \quad ds^2 = -dt^2 + 2dxdt + dy^2 + dz^2 = -(dt' - dx')^2 + 2(dt')(dt' - dx') + dy'^2 + dz'^2$$
$$= -dt'^2 - dx'^2 + 2dt'dx' + 2dt'^2 - 2dt'dx' + dy'^2 + dz'^2$$
$$= {}^{24} - dx'^2 + dt'^2 + dy'^2 + dz'^2$$

2.9.2 [j]The Penrose Diagram for Flat Space-time

A Penrose diagram is a method to map the infinite coordinates such as t, with the range $-\infty < t < +\infty$, and r, with the range $0 < r < +\infty$, into to coordinates with finite ranges.
Begin with the flat space-time line element in spherical polar coordinates

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

and replace t and r by the coordinates

$$u = t - r \qquad v = t + r$$
$$t = \frac{1}{2}(u + v) \qquad r = \frac{1}{2}(u - v)$$
$$dt = \frac{1}{2}(du + dv) \quad dr = \frac{1}{2}(du - dv)$$
$$\Rightarrow \quad ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$
$$= -\left(\frac{1}{2}(du + dv)\right)^2 + \left(\frac{1}{2}(du - dv)\right)^2 + \left(\frac{1}{2}(u - v)\right)^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$$
$$= -dudv + \frac{1}{4}(u - v)^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$$

The (u, v) axes are rotated with respect to the (t, r) axes by 45°. Radial light rays travel on lines of either constant u or constant v. Remember radial light rays has constant θ and ϕ and $ds^2 = 0$, which leaves us with $dudv = 0$.
Make a further transformation of u and v to new coordinates u' and v'.

$$u' = \tan^{-1} u \qquad v' = \tan^{-1} v$$
$$u = \tan u' \qquad v = \tan v'$$
$$du = (1 + \tan^2 u')du' \quad dv = (1 + \tan^2 v')dv'$$
$$\Rightarrow \quad ds^2 = -dudv + \frac{1}{4}\big((u - v)\big)^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$$
$$= -(1 + \tan^2 u')(1 + \tan^2 v')du'dv' + \frac{1}{4}\big((\tan u' - \tan v')\big)^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$$

Map these coordinates into a (t', r') diagram, where

$$u' = \tan^{-1} u = t' - r' \quad v' = \tan^{-1} v = t' + r'$$

---

[23] Notice: The transformation matrix is not necessarily symmetric.

[24] We check: $g_{ab} = \Lambda^{a'b'}_{ab} g_{a'b'} = \begin{Bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Bmatrix} \begin{Bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Bmatrix} = \begin{Bmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Bmatrix}$

Because $\tan^{-1} x$ lies between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$ the ranges for (u', v') and (t', r') are finite. This is another example of how the infinite coordinates (t, r) is mapped into a finite region.

2.10 [k]Local inertial frames

A local inertial frame is defined by the conditions

$$\begin{aligned} g'_{\alpha\beta}(x'_P) &= \eta_{\alpha\beta} \\ \left.\frac{\partial g'_{\alpha\beta}}{\partial x'^\gamma}\right|_{x=x_P} &= 0 \end{aligned}$$

This means that if you have a system described by a metric $g_{\alpha\beta}$it can locally in a point P be transformed into the flat space-time metric $\eta_{\alpha\beta}$. Furthermore the first derivatives of the transformed metric vanish. This is best illustrated by an example.

2.10.1 [l]The metric of a Sphere at the North Pole.

The line element of the geometry of a sphere with radius a is

$$dS^2 = a^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

At the north pole $\theta = 0$, and the metric doesn't look like the metric of a flat plane at all. Can we find a coordinate transformation so that

$$\begin{aligned} dS^2 &= dx^2 + dy^2 \\ g_{ij} &= \begin{Bmatrix} 1 & 0 \\ 0 & 1 \end{Bmatrix} \end{aligned}$$

[m]<u>The Riemann normal coordinates</u>

$$\begin{aligned} x &= a\theta\cos\phi \\ y &= a\theta\sin\phi \end{aligned}$$

At the north pole both x and y are zero.

Next we calculate

$$\begin{aligned} x^2 + y^2 &= (a\theta\cos\phi)^2 + (a\theta\sin\phi)^2 = a^2\theta^2(\cos^2\phi + \sin^2\phi) = a^2\theta^2 \\ \Rightarrow \quad \theta &= \frac{1}{a}\sqrt{x^2+y^2} \\ \Rightarrow \quad \frac{y}{x} &= \frac{a\theta\sin\phi}{a\theta\cos\phi} = \tan\phi \\ \Rightarrow \quad \phi &= \tan^{-1}\left(\frac{y}{x}\right) \end{aligned}$$

The differentials

$$\begin{aligned} d\theta &= d\left(\frac{1}{a}\sqrt{x^2+y^2}\right) = \frac{1}{a}\frac{1}{2\sqrt{x^2+y^2}}(2xdx + 2ydy) \\ &= \frac{1}{a}\frac{1}{\sqrt{x^2+y^2}}(xdx + ydy) \\ d(\tan\phi) &= d\left(\frac{y}{x}\right) \\ \Rightarrow \quad (1+\tan^2\phi)d\phi &= \frac{1}{x}dy - \frac{y}{x^2}dx \\ \Rightarrow \quad \left(1 + \left(\frac{y}{x}\right)^2\right)d\phi &= \frac{1}{x}dy - \frac{y}{x^2}dx \\ \Rightarrow \quad (x^2+y^2)d\phi &= xdy - ydx \\ \Rightarrow \quad d\phi &= \frac{xdy - ydx}{x^2+y^2} \end{aligned}$$

The line element

$$dS^2 = a^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$= a^2\left(\left(\frac{1}{a}\frac{1}{\sqrt{x^2+y^2}}(xdx+ydy)\right)^2 + \sin^2\theta\left(\frac{xdy-ydx}{x^2+y^2}\right)^2\right)$$

$$= {}^{25\ 26}\left(\frac{x^2}{x^2+y^2}+\frac{y^2}{(x^2+y^2)^2}a^2\sin^2\left(\frac{1}{a}\sqrt{x^2+y^2}\right)\right)dx^2 + \left(\frac{y^2}{x^2+y^2}+\frac{x^2}{(x^2+y^2)^2}a^2\sin^2\left(\frac{1}{a}\sqrt{x^2+y^2}\right)\right)dy^2 + 2\left(\frac{xy}{x^2+y^2}-\frac{xy}{(x^2+y^2)^2}a^2\sin^2\left(\frac{1}{a}\sqrt{x^2+y^2}\right)\right)dxdy$$

The metric tensor

$$g_{xx} = \frac{x^2}{x^2+y^2}+\frac{y^2}{(x^2+y^2)^2}a^2\sin^2\left(\frac{1}{a}\sqrt{x^2+y^2}\right)$$

$$g_{yy} = \frac{y^2}{x^2+y^2}+\frac{x^2}{(x^2+y^2)^2}a^2\sin^2\left(\frac{1}{a}\sqrt{x^2+y^2}\right)$$

$$g_{xy} = g_{yx} = \frac{xy}{x^2+y^2}-\frac{xy}{(x^2+y^2)^2}a^2\sin^2\left(\frac{1}{a}\sqrt{x^2+y^2}\right)$$

If we evaluate these around the north pole where x and y are small[27]

$$g_{xx} = 1-\frac{y^2}{3a^2}$$

$$g_{yy} = 1-\frac{x^2}{3a^2}$$

$$g_{xy} = g_{yx} = \frac{xy}{3a^2}$$

$$\Rightarrow \quad ds^2 = \left(1-\frac{y^2}{3a^2}\right)dx^2 + 2\frac{xy}{3a^2}dxdy + \left(1-\frac{x^2}{3a^2}\right)dy^2$$

At the north pole i.e. $(x,y)=(0,0)$

$$g_{\alpha\beta}\big((x,y)=(0,0)\big) = \eta_{\alpha\beta}$$

$$\left.\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\right|_{(x,y)=(0,0)} = 0$$

We have proved that the sphere is locally flat at the North Pole. Of course, this is true for any set of coordinates on the sphere, because it is simply a matter of rotation.

---

[25] $= \frac{1}{x^2+y^2}(x^2dx^2+y^2dy^2+2xydxdy)+\left(\frac{a\sin\theta}{x^2+y^2}\right)^2(x^2dy^2+y^2dx^2-2xydxdy) =$

[26] $= \frac{1}{x^2+y^2}\left(\left(x^2+\frac{a^2\sin^2\theta}{x^2+y^2}y^2\right)dx^2+\left(y^2+\frac{a^2\sin^2\theta}{x^2+y^2}x^2\right)dy^2+2xy\left(1-\frac{a^2\sin^2\theta}{x^2+y^2}\right)dxdy\right) =$

[27] Use wxMaxima http://maxima.sourceforge.net/to evaluate the Taylor polynomials to second order:

```
g_xx(x,y) := x^2/(x^2+y^2)+a^2*sin(1/a*sqrt(x^2+y^2))*sin(1/a*sqrt(x^2+y^2))*y^2/(x^2+y^2)^2;
g_yy(x,y) := y^2/(x^2+y^2)+a^2*sin(1/a*sqrt(x^2+y^2))*sin(1/a*sqrt(x^2+y^2))*x^2/(x^2+y^2)^2;
g_xy(x,y) :=x*y/(x^2+y^2)-a^2*sin(1/a*sqrt(x^2+y^2))*sin(1/a*sqrt(x^2+y^2))*x*y/(x^2+y^2)^2;
taylor(g_xx(x,y),[x,y],[0,0],[2,2]);
taylor(g_yy(x,y),[x,y],[0,0],[2,2]);
taylor(g_xy(x,y),[x,y],[0,0],[2,2]);
g_xx(x,y)=1-y^2/(3*a^2)+...
g_yy(x,y)= 1-x^2/(3*a^2)+...
g_xy(x,y) =(y*x)/(3*a^2)+...
```

2.11 [n]Static Weak Field Metric

The line element is predicted by general relativity for small curvatures produced by time-independent weak sources, and it is a good approximation to the curved spacetime geometry produced by the Sun.

$$ds^2 = -\left(1+\frac{2\Phi(\vec{x})}{c^2}\right)(cdt)^2 + \left(1-\frac{2\Phi(\vec{x})}{c^2}\right)(dx^2+dy^2+dz^2)$$

$\Phi(x^i)$ is a function of position satisfying the Newtonian field equation° $\nabla^2\Phi(\vec{x}) = 4\pi G\mu(\vec{x})$and assumed to vanish at infinity. For example outside Earth $\Phi(r) = -\frac{GM_\oplus}{r}$.

2.11.1 [p]Rates of Emission and Reception

We look at a system where two light signals are emitted in a system A, described by a world line (ct, x_A), with a proper time separation $\Delta\tau_A$. We want to predict: what is the proper time separation $\Delta\tau_B$ in a system B, described by a world line (ct, x_B) in a static weak field limit where $\frac{\Phi}{c^2} \ll 1$. This implies $dx = 0$, $\Phi(x^i) = \Phi(x_i, 0,0) = \Phi_i$ and $d\tau^2 = -\frac{ds^2}{c^2}$. Also notice that because the metric is independent of the coordinate t, Δt is the same in both systems. This leads to

$$d\tau_i{}^2 = \left(1+\frac{2\Phi_i}{c^2}\right)(dt)^2$$

$$\Rightarrow \quad \Delta\tau_A = \sqrt{\left(1+\frac{2\Phi_A}{c^2}\right)}\Delta t$$

$$\Delta\tau_A \sim^{28} \left(1+\frac{\Phi_A}{c^2}\right)\Delta t$$

$$\Delta\tau_B \sim \left(1+\frac{\Phi_B}{c^2}\right)\Delta t$$

Eliminating Δt we get

$$\Delta\tau_B = \frac{\left(1+\frac{\Phi_B}{c^2}\right)}{\left(1+\frac{\Phi_A}{c^2}\right)}\Delta\tau_A \sim^{29} \left(1+\frac{\Phi_B}{c^2}\right)\left(1-\frac{\Phi_A}{c^2}\right)\Delta\tau_A$$

$$\Delta\tau_B \sim \left(1+\frac{\Phi_B-\Phi_A}{c^2}\right)\Delta\tau_A$$

which tells us the observed fact, that when the receiver B is at a higher gravitational potential than the emitter A, the signals will be received at a slower rate than they were emitted and vice versa.

2.11.2 [q]Newton's second equation:

Next we want to look at the motion of a particle in this weak field geometry and show how it leads to Newton's second equation.

[r]We use the Lagrange method of extremal path

First we find the proper time τ_{AB} between two points A and B.

$$\tau_{AB} = \int_A^B d\tau = \int_A^B \left(-\frac{ds^2}{c^2}\right)^{\frac{1}{2}}$$

$$= \int_A^B \left(\left(1+\frac{2\Phi(\vec{x})}{c^2}\right)dt^2 - \frac{1}{c^2}\left(1-\frac{2\Phi(\vec{x})}{c^2}\right)(dx^2+dy^2+dz^2)\right)^{\frac{1}{2}}$$

---

[28] $\sqrt{1+x} \sim 1+\frac{1}{2}x$ if $x \ll 1$

[29] $\frac{1}{1+x} \sim 1-x$ if $x \ll 1$

$$= \int_A^B dt \left(\left(1 + \frac{2\Phi(\vec{x})}{c^2} \right) - \frac{1}{c^2} \left(1 - \frac{2\Phi(\vec{x})}{c^2} \right) \left(\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 + \left(\frac{dz}{dt} \right)^2 \right) \right)^{\frac{1}{2}}$$

$$= \int_A^B dt \left(\left(1 + \frac{2\Phi(\vec{x})}{c^2} \right) - \frac{1}{c^2} \left(1 - \frac{2\Phi(\vec{x})}{c^2} \right) \vec{V}^2 \right)^{\frac{1}{2}}$$

$$= \int_A^B dt \left(1 + \frac{2\Phi(\vec{x})}{c^2} - \frac{\vec{V}^2}{c^2} + \frac{2\Phi(\vec{x})\vec{V}^2}{c^4} \right)^{\frac{1}{2}} \approx^{30} \int_A^B dt \left(1 + \frac{2\Phi(\vec{x})}{c^2} - \frac{\vec{V}^2}{c^2} \right)^{\frac{1}{2}}$$

$$\approx^{31} \int_A^B dt \left(1 + \frac{1}{2} \left(\frac{2\Phi(\vec{x})}{c^2} - \frac{\vec{V}^2}{c^2} \right) \right) = \int_A^B dt \left(1 + \left(\frac{\Phi(\vec{x})}{c^2} - \frac{1}{2} \frac{\vec{V}^2}{c^2} \right) \right)$$

The Lagrange is $(c = 1)$

$$L = 1 - \frac{1}{2}\vec{V}^2 + \Phi(\vec{x}) = 1 - \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 + \Phi(\vec{x})$$

The Lagrange equation

$$-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) + \frac{\partial L}{dx^i} = 0$$

$$\frac{\partial L}{\partial \dot{x}} = \frac{\partial(1 - \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 + \Phi(\vec{x}))}{\partial \dot{x}} = -\dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = -\ddot{x}$$

$$\frac{\partial L}{dx} = \frac{\partial\left(1 - \frac{1}{2}\vec{V}^2 + \Phi(\vec{x})\right)}{dx} = \frac{\partial \Phi(\vec{x})}{\partial x}$$

$$\Rightarrow \quad -(-\ddot{x}) + \frac{\partial \Phi(\vec{x})}{\partial x} = 0$$

$$\Rightarrow \quad \frac{d^2\vec{x}}{dt^2} = -\nabla\Phi(\vec{x})$$

Multiplying by m on both sides

$$m\frac{d^2\vec{x}}{dt^2} = -m\nabla\Phi(\vec{x})$$

We recognize Newton's second equation

$$m\vec{a} = \vec{F}$$

We can interpret this as a particle of mass m moving along a geodesic in a spacetime created by a gravitational field $\Phi(\vec{x})$.

2.12 ^{s}A Fifth Dimension?

This is maybe not that relevant in the GR context, but when I came across this curious "back on the envelope" calculation on how to use a some simple space-time calculation including a fifth dimension, I just had to present it here.

To describe the consequences of a fifth dimension we can use the familiar Minkowsky spacetime plus an extra spatial coordinate. We imagine the extra dimension as a small circle with angular coordinate Ω $(0 \le \Omega \le 2\pi)$ and a fixed radius R

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 + R^2 d\Omega^2$$

[30] Emitting elements of higher order

[31] $\sqrt{1+x} \sim 1 + \frac{1}{2}x$ if $x \ll 1$

$$= g_{AB} dx^A dx^B$$

$$g_{AB} = \begin{Bmatrix} -1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & R^2 \end{Bmatrix}$$

To detect the fifth dimension we can imagine an electromagnetic wave with the five-dimensional wave vector k^A. This wave can be described by

$$\Phi_k(x) \propto \cos(k \cdot x)$$

Where

$$k^A = \left(k^t, \vec{k}, k^\Omega\right) = {}^{32}\left(\omega, \vec{k}, k^\Omega\right)$$

The wave form implies that the wave has to be periodic in Ω with period 2π i.e.

$$\left(k_\Omega \cdot x^\Omega\right) = g_{\Omega\Omega} k^\Omega x^\Omega$$

Now assuming that $x^\Omega = 2\pi$ we find

$$R^2 k^\Omega 2\pi = n \cdot 2\pi$$

$$\Rightarrow \quad k^\Omega = \frac{n}{R^2}$$

As in the four-dimensional case we assume conservation of five-vector i.e.

$$k \cdot k = g_{AB} k^A k^B = -\omega^2 + \overrightarrow{k^2} + R^2 \left(k^\Omega\right)^2 = -\omega^2 + \overrightarrow{k^2} + R^2 \left(\frac{n}{R^2}\right)^2$$

$$= -\omega^2 + \overrightarrow{k^2} + \left(\frac{n}{R}\right)^2 = 0$$

$$\Rightarrow \quad \omega^2 = \overrightarrow{k^2} + \left(\frac{n}{R}\right)^2$$

If $n = 0$ we are back to the well-known four-dimensional case were

$$E = \hbar\omega = \hbar\vec{k}$$

In order to detect the fifth dimension we need an extra energy of minimum

$$E' \geq \frac{\hbar}{R}$$

If R is of the order of the Planck length[t]:

$$l_{Pl} = \sqrt{\frac{\hbar G}{c^3}} \approx 1.61 \cdot 10^{-35} m$$

With the corresponding Planck energy[u]:

$$E_{Pl} = \sqrt{\frac{\hbar c^5}{G}} \approx 1.22 \cdot 10^{19} GeV$$

We can see that the needed energy E' is much larger than any accelerator energy produced today[33], and we will not be able to detect these small extra dimensions with our current technology, should they exist.

2.13 Flat Space

Flat space in Cartesian coordinates

$$ds^2 = dx^2 + dy^2$$

Flat space in polar coordinates

$$ds^2 = dr^2 + r^2 d\phi^2$$

---

[32] Remember the four-impulse: $p^\alpha = (E, \vec{p}) = \left(\hbar\omega, \hbar\vec{k}\right) \Rightarrow k^t = \omega$

[33] The LHC in CERN has reached 13 TeV in summer 2017 - https://arstechnica.com/science/2017/05/the-lhc-is-starting-another-year-of-high-energy-physics/

2.13.1 [v]How to find the coordinate transformation – the Jacobian matrix

The coordinate transformation can be written in terms of the Jacobian matrix

$$\begin{Bmatrix} dx \\ dy \end{Bmatrix} = \begin{Bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} \end{Bmatrix} \begin{Bmatrix} dr \\ d\phi \end{Bmatrix}$$

$$\Rightarrow \quad dx = \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \phi} d\phi$$

$$dy = \frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \phi} d\phi$$

$$\Rightarrow \quad dx^2 = \left(\frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \phi} d\phi\right)^2 = \left(\frac{\partial x}{\partial r}\right)^2 dr^2 + \left(\frac{\partial x}{\partial \phi}\right)^2 d\phi^2 + 2\frac{\partial x}{\partial r}\frac{\partial x}{\partial \phi} dr d\phi$$

$$dy^2 = \left(\frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \phi} d\phi\right)^2 = \left(\frac{\partial y}{\partial r}\right)^2 dr^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 d\phi^2 + 2\frac{\partial y}{\partial r}\frac{\partial y}{\partial \phi} dr d\phi$$

$$\Rightarrow \quad ds^2 = dx^2 + dy^2$$

$$= \left(\left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2\right) dr^2 + \left(\left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2\right) d\phi^2 + 2\left(\frac{\partial x}{\partial r}\frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \phi}\right) dr d\phi$$

$$= dr^2 + r^2 d\phi^2$$

$$\Rightarrow \quad 1 = \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 \tag{2.7.}$$

$$r^2 = \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 \tag{2.8.}$$

$$0 = \frac{\partial x}{\partial r}\frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \phi} \tag{2.9.}$$

If you look at eq. (2.7.) the solution is obviously a trigonometric function i.e.

$$\Rightarrow \quad x = r\cos\phi$$

$$y = r\sin\phi$$

$$\Rightarrow \quad 1 = (\cos\phi)^2 + (\sin\phi)^2 = 1 \tag{2.7.}$$

$$r^2 = (-r\sin\phi)^2 + (r\cos\phi)^2 = r^2 \tag{2.8.}$$

$$0 = \cos\phi\,(-r\sin\phi) + \sin\phi\, r\cos\phi = 0 \tag{2.9.}$$

2.13.2 [w]Flat space with a singularity

Look at the line element of the two-dimensional plane in polar coordinates ($\theta = 0$)

$$ds^2 = dr^2 + r^2 d\phi^2$$

and make the transformation, for some constant a

$$r = \frac{a^2}{r'}$$

$$\Rightarrow \quad dr = d\left(\frac{a^2}{r'}\right) = -\frac{a^2}{r'^2} dr'$$

$$\Rightarrow \quad dr^2 = \frac{a^4}{r'^4} dr'^2$$

$$\Rightarrow \quad dS^2 = \frac{a^4}{r'^4} dr'^2 + \left(\frac{a^2}{r'}\right)^2 d\phi^2 = \frac{a^4}{r'^4}(dr'^2 + r'^2 d\phi^2)$$

This line element blows up at $r' = 0$. Not because something physically interesting happens here, but simply because the coordinate transformation $r = \frac{a^2}{r'}$ has mapped all the points at $r \to \infty$ into $r' = 0$.

[x]We can show that that the distance between $r' = 0$ and a point with any finite value of r' is infinite, which corresponds to the distance between some finite value of r and $r \to \infty$:

$$\int dS = \int_0^{r'} \sqrt{\frac{a^4}{r'^4}(dr'^2 + r'^2 d\phi^2)} = a^2 \int_0^{r'} dr' \sqrt{\frac{1}{r'^4}\left(1 + r'^2\left(\frac{d\phi}{dr'}\right)^2\right)} = {}^{34}a^2 \int_0^{r'} \frac{1}{r'^2} dr'$$
$$= \left[-\frac{a^2}{r'}\right]_0^{r'} \to \infty$$

2.13.3 [y]Three-dimensional flat space in spherical coordinates and vector transformation

The line element

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

The metric tensor

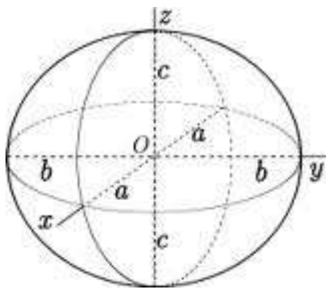
$$g_{ab} = \begin{Bmatrix} 1 & & \\ & r^2 & \\ & & r^2 \sin^2\theta \end{Bmatrix}$$

If

$$X^a = \left(r, \frac{1}{r\sin\theta}, \frac{1}{\cos^2\theta}\right)$$
$$X_a = g_{ab} X^a$$
$$\Rightarrow \quad X_r = g_{rr} X^r = (1)(r) = r$$
$$X_\theta = g_{\theta\theta} X^\theta = (r^2)\left(\frac{1}{\mathrm{r}\sin\theta}\right) = \frac{r}{\sin\theta}$$
$$X_\phi = g_{\phi\phi} X^\phi = (r^2 \sin^2\theta)\left(\frac{1}{\cos^2\theta}\right) = r^2 \tan^2\theta$$
$$\Rightarrow \quad X^a = \left(r, \frac{r}{\sin\theta}, r^2 \tan^2\theta\right)$$

2.14 The line-element and metric of an ellipsoid

The line-element of an ellipsoid in Cartesian coordinates



$$ds^2 = adx^2 + bdy^2 + cdz^2$$

We use the parameterization

$$x = \cos\phi \sin\theta$$
$$y = \sin\phi \sin\theta$$
$$z = \cos\theta$$

With $0 \le \phi \le 2\pi$ in the xy-plane and $0 \le \theta \le \pi$ where the z-axis corresponds to $\theta = 0$.

$$dx = a(\cos\phi \cos\theta \, d\theta - \sin\phi \sin\theta \, d\phi)$$
$$dy = b(\sin\phi \cos\theta \, d\theta + \cos\phi \sin\theta \, d\phi)$$
$$dz = -c \sin\theta \, d\theta$$
$$dx^2 = a^2(\cos\phi \cos\theta \, d\theta - \sin\phi \sin\theta \, d\phi)^2$$
$$= a^2[\cos^2\phi \cos^2\theta \, d\theta^2 + \sin^2\phi \sin^2\theta \, d\phi^2 - 2\cos\phi \cos\theta \sin\phi \sin\theta \, d\theta d\phi]$$
$$dy^2 = b[\sin^2\phi \cos^2\theta \, d\theta^2 + \cos^2\phi \sin^2\theta \, d\phi^2 + 2\sin\phi \cos\theta \cos\phi \sin\theta \, d\theta d\phi]$$
$$dz^2 = c^2 \sin^2\theta \, d\theta^2$$

[34] $\frac{d\phi}{dr'} = 0$

Collecting the results in terms of $d\theta^2$, $d\phi^2$ and $d\theta d\phi$ we get the line element

$$ds^2 = [\cos^2\theta\,(a^2\cos^2\phi + b^2\sin^2\phi) + c^2\sin^2\theta]d\theta^2 + \sin^2\theta\,(a^2\sin^2\phi + b^2\cos^2\phi)d\phi^2 + 2(b^2 - a^2)(\cos\phi\cos\theta\sin\phi\sin\theta)d\theta d\phi$$

and the metric tensor

$$g_{ab} = \begin{Bmatrix} \cos^2\theta\,(a^2\cos^2\phi + b^2\sin^2\phi) + c^2\sin^2\theta & (b^2 - a^2)(\cos\phi\cos\theta\sin\phi\sin\theta) \\ (b^2 - a^2)(\cos\phi\cos\theta\sin\phi\sin\theta) & \sin^2\theta\,(a^2\sin^2\phi + b^2\cos^2\phi) \end{Bmatrix}$$

As a curious observation, you can now calculate the line-element and metric tensor of an idealized egg. For an idealized egg we can choose $a = b = \frac{1}{2}c$

$$\Rightarrow \quad ds^2 = a^2[(\cos^2\theta + 4\sin^2\theta)d\theta^2 + \sin^2\theta\, d\phi^2]$$

$$g_{ab} = a^2\begin{Bmatrix} \cos^2\theta + 4\sin^2\theta & 0 \\ 0 & \sin^2\theta \end{Bmatrix}$$

2.15 ᶻLength, Area, Volume and Four-Volume for Diagonal Metrics

For a diagonal metric of the type

$$ds^2 = g_{00}dx^0dx^0 + g_{11}dx^1dx^1 + g_{22}dx^2dx^2 + g_{33}dx^3dx^3$$

you can define proper length elements in the various coordinates as

$$dl^1 = \sqrt{g_{11}}dx^1$$
$$dl^2 = \sqrt{g_{22}}dx^2$$
$$dl^3 = \sqrt{g_{33}}dx^3$$

The area element

$$dA = dl^1dl^2 = \sqrt{g_{11}g_{22}}dx^1dx^2$$

Notice, that it is not always the g_{11} and g_{22} that are involved in the calculation of the area. As we shall see below it can also be the g_{22} and g_{33}

The three-volume element

$$d\mathcal{V} = dl^1dl^2dl^3 = \sqrt{g_{11}g_{22}g_{33}}dx^1dx^2dx^3$$

The four-volume element

$$dv = \sqrt{-g_{00}g_{11}g_{22}g_{33}}dx^0dx^1dx^2dx^3$$

In the case of a non-diagonal metric the four-volume element is

$$dv = \sqrt{-g}d^4x$$

where g is the determinant of the matrix $g_{\alpha\beta}$

2.15.1 ᵆArea and Volume Elements of a Sphere

The line element of flat space-time in polar coordinates

$$ds^2 = -dt^2 + dr^2 + r^2d\theta^2 + r^2\sin^2\theta\, d\phi$$

The proper length elements

$$dl^1 = dr$$
$$dl^2 = rd\theta$$
$$dl^3 = r\sin\theta\, d\phi$$

The area element

$$dA = dl^2dl^3 = r^2\sin\theta\, d\theta d\phi$$

The area

$$A = \int dA = \int\int r^2\sin\theta\, d\theta d\phi = r^2\int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi = 2\pi r^2[-\cos\theta]_0^\pi$$
$$= 2\pi r^2\big(-(-1-1)\big) = 4\pi r^2$$

The three-volume element

$$d\mathcal{V} = dl^1dl^2dl^3 = drdA = r^2\sin\theta\, drd\theta d\phi$$

The three-volume

$$\mathcal{V} = \int d\mathcal{V} = \int\int\int r^2 \sin\theta \, dr d\theta d\phi = \int\int r^2 dr dA = 4\pi \int_0^r r^2 dr = 4\pi \left[\frac{1}{3} r^3\right]_0^r = \frac{4\pi}{3} r^3$$

Collection the results

$$A = 4\pi r^2$$

$$\mathcal{V} = \frac{4\pi}{3} r^3$$

We recognize the familiar values.

2.15.2 ᶿThe dimensions of a peanut

The line-element of a peanut geometry is

$$ds^2 = a^2 d\theta^2 + a^2 f^2(\theta) d\phi^2 f(\theta) = \sin\theta \left(1 - \frac{3}{4} \sin^2\theta\right)$$

$$dl^1 = a d\theta$$

$$dl^2 = a f(\theta) d\phi$$

The distance from pole to pole ($\phi = 0$)

$$d = \int dl^1 = \int_0^\pi a d\theta = a\pi$$

The circumference at a constant angle θ

$$C = \int dl^2 = \int_0^{2\pi} a f(\theta) d\phi = a f(\theta) \int_0^{2\pi} d\phi = 2\pi a f(\theta) = 2\pi a \sin\theta \left(1 - \frac{3}{4} \sin^2\theta\right)$$

At the center or equator $\theta = \frac{\pi}{2}$

$$C = 2\pi a \sin\left(\frac{\pi}{2}\right) \left(1 - \frac{3}{4} \sin^2\left(\frac{\pi}{2}\right)\right) = 2\pi a \left(1 - \frac{3}{4}\right) = \frac{\pi}{2} a$$

The area of a peanut

$$A = \int\int dl^1 dl^2 = a^2 \int_0^\pi f(\theta) d\theta \int_0^{2\pi} d\phi = 2\pi a^2 \int_0^\pi \sin\theta \left(1 - \frac{3}{4} \sin^2\theta\right) d\theta$$

$$= 2\pi a^2 \int_0^\pi \left(1 - \frac{3}{4}(1 - \cos^2\theta)\right) \sin\theta \, d\theta = -2\pi a^2 \int_1^{-1} \left(1 - \frac{3}{4}(1 - x^2)\right) dx$$

$$= -\frac{\pi}{2} a^2 \int_1^{-1} (1 + 3x^2) dx = -\frac{\pi}{2} a^2 [x + x^3]_1^{-1} = -\frac{\pi}{2} a^2 (-1 - 1 - 1 - 1) = 2\pi a^2$$

2.15.3 ᵃThe dimensions of an egg

The line-element

$$ds^2 = a^2[(\cos^2\theta + 4\sin^2\theta) d\theta^2 + \sin^2\theta \, d\phi^2]$$

The circumference of the egg at constant θ

$$C(\theta) = \int_0^{2\pi} a \sin\theta \, d\phi = 2\pi a \sin\theta$$

The distance from pole to pole

$$d_{pole-to-pole} = \int_0^\pi a \sqrt{\cos^2\theta + 4\sin^2\theta} \, d\theta = a \int_0^\pi \sqrt{1 + 3\sin^2\theta} \, d\theta = 4{,}84a = 0{,}77 * 2\pi a$$

The ratio of the biggest circle around the axis to the pole-to-pole distance is

$$\frac{C\left(\theta = \frac{\pi}{2}\right)}{d_{pole-to-pole}} = \frac{2\pi a}{0{,}77 * 2\pi a} = 1{,}3$$

The surface area of an egg

$$A = a^2 \int_0^\pi \sin\theta \sqrt{1 + 3\sin^2\theta} \, d\theta \int_0^{2\pi} d\phi = 3{,}41 * 2\pi \, a^2$$

2.15.4 Distance, Area, Volume and four-volume of a metric

$$\begin{aligned} ds^2 &= -(1-Ar^2)^2dt^2 + (1-Ar^2)^2dr^2 + r^2d\theta^2 + r^2\sin^2\theta\, d\phi^2 \\ \Rightarrow \quad dl^0 &= (1-Ar^2)dt \\ dl^1 &= (1-Ar^2)dr \\ dl^2 &= rd\theta \\ dl^3 &= r\sin\theta\, d\phi \end{aligned}$$

The proper distance along a radial line from the center $r = 0$ to a coordinate radius $r = R$

$$l = \int dl^1 = \int_0^R (1-Ar^2)dr = \left[r - \frac{1}{3}Ar^3\right]_0^R = R\left(1 - \frac{1}{3}AR^2\right)$$

The area of a sphere of coordinate radius $r = R$

$$A = \int\int dl^2dl^3 = R^2\int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi = 4\pi R^2$$

The three-volume of a sphere of coordinate radius $r = R$

$$\begin{aligned} \mathcal{V} &= \int\int\int dl^1dl^2dl^3 = \int_0^R r^2(1-Ar^2)dr \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi = 4\pi\left[\frac{1}{3}r^3 - \frac{1}{5}Ar^5\right]_0^R \\ &= \frac{4\pi}{3}R^3\left(1 - \frac{3}{5}AR^2\right) \end{aligned}$$

The four-volume of a four-dimensional tube bounded by a sphere of coordinate radius R and two $t = constant$ planes separated by a time T

$$\begin{aligned} v &= \int\int\int\int dl^0dl^1dl^2dl^3 = \int_0^T dt \int_0^R r^2(1-Ar^2)^2dr \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi \\ &= 4\pi T\int_0^R r^2(1 + A^2r^4 - 2Ar^2)dr = 4\pi T\left[\frac{1}{3}r^3 + \frac{A^2}{7}r^7 - \frac{2A}{5}r^5\right]_0^R \\ &= \frac{4\pi}{3}TR^3\left(1 - \frac{6A}{5}R^2 + \frac{3A^2}{7}R^4\right) \end{aligned}$$

2.15.5 Distance, Area and Volume in the Curved Space of a Constant Density Spherical Star or a Homogenous Closed Universe

The spherical line element is (a is a constant related to the density of matter)

$$dS^2 = \frac{1}{1-\left(\frac{r}{a}\right)^2}dr^2 + r^2d\theta^2 + r^2\sin^2\theta\, d\phi^2$$

The proper length elements

$$\begin{aligned} dl^1 &= \frac{1}{\sqrt{1-\left(\frac{r}{a}\right)^2}}dr \\ dl^2 &= rd\theta \\ dl^3 &= r\sin\theta\, d\phi \end{aligned}$$

The circumference around equator where $r = R$ and $\theta = \frac{\pi}{2}$

$$C = \int dl^3 = \int_0^{2\pi} r\sin\theta\, d\phi = 2\pi R$$

The distance from the center ($r = 0$) to surface ($r = R$) along a line where $\theta = const.$ and $\phi = const.$

$$S = \int dl^1 = \int_0^R \frac{1}{\sqrt{1-\left(\frac{r}{a}\right)^2}}dr = \int_0^R \frac{a}{\sqrt{a^2-r^2}}dr \overset{[35]}{=} a\left[\sin^{-1}\frac{r}{a}\right]_0^R = a\sin^{-1}\frac{R}{a}$$

[35] (Spiegel, 1990) (14.237) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$

The area of the two-surface at $r = R$

$$A = \int dA = \int dl^2 \int dl^3 = R^2 \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi = 4\pi R^2$$

The volume inside $r = R$ is

$$\mathcal{V} = \int d\mathcal{V} = \int\int\int dl^1 dl^2 dl^3 = \int_0^R \frac{r^2}{\sqrt{1-\left(\frac{r}{a}\right)^2}} dr \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi = 4\pi a \int_0^R \frac{r^2}{\sqrt{a^2-r^2}} dr$$

$$= {}^{36} 4\pi a \left[\frac{a^2}{2}\sin^{-1}\left(\frac{r}{a}\right) - \frac{r}{2}\sqrt{a^2-r^2}\right]_0^R = 4\pi a^3 \left[\frac{1}{2}\sin^{-1}\left(\frac{r}{a}\right) - \frac{r}{2a}\sqrt{1-\left(\frac{r}{a}\right)^2}\right]_0^R$$

$$= 4\pi a^3 \left(\frac{1}{2}\sin^{-1}\left(\frac{R}{a}\right) - \frac{R}{2a}\sqrt{1-\left(\frac{R}{a}\right)^2}\right)$$

$$= {}^{37\ 38\ 39} 4\pi a^3 \frac{1}{2}\left(\frac{R}{a} + \frac{1}{2}\frac{1}{3}\left(\frac{R}{a}\right)^3 - \frac{R}{a}\left(1 - \frac{1}{2}\left(\frac{R}{a}\right)^2\right)\right) = 4\pi a^3 \frac{1}{2}\left(\frac{1}{2}\frac{1}{3}\left(\frac{R}{a}\right)^3 + \frac{1}{2}\left(\frac{R}{a}\right)^3\right)$$

$$= \frac{4\pi}{3} R^3$$

<u>Collecting the results:</u>

The circumference around equator where $r = R$ and $\theta = \frac{\pi}{2}$

$$C = 2\pi R$$

The distance from the center $(r = 0)$ to surface $(r = R)$ along a line where $\theta = const.$ and $\phi = const.$

$$S = a \sin^{-1}\frac{R}{a}$$

The area of the two-surface at $r = R$

$$A = 4\pi R^2$$

The volume inside $r = R$ is

$$\mathcal{V} = 4\pi a^3 \left(\frac{1}{2}\sin^{-1}\left(\frac{R}{a}\right) - \frac{R}{2a}\sqrt{1-\left(\frac{R}{a}\right)^2}\right)$$

2.16 [aa] Inverting a metric tensor.

2.16.1 Inverting a Diagonal tensor.

$$g_{ab} = \begin{Bmatrix} g_{00} & & & \\ & g_{11} & & \\ & & g_{22} & \\ & & & g_{33} \end{Bmatrix}$$

$$|g_{ab}| = g_{00} g_{11} g_{22} g_{33}$$

---

[36] (Spiegel, 1990) (14.239) $\int \frac{x^2 dx}{\sqrt{a^2-x^2}} = -\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$

[37] (Spiegel, 1990) (20.27) $\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} \ldots$

[38] (Spiegel, 1990) (20.12) $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{2\cdot 4}x^2 + \frac{1\cdot 3}{2\cdot 4\cdot 6}x^3 \ldots$

[39] $if\ \frac{R}{a} \ll 1$

$$\Rightarrow \quad g^{ab} = \frac{1}{|g_{ab}|}\begin{Bmatrix} g_{11}g_{22}g_{33} & & & \\ & g_{00}g_{22}g_{33} & & \\ & & g_{00}g_{11}g_{33} & \\ & & & g_{00}g_{11}g_{22} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{g_{00}} & & & \\ & \frac{1}{g_{11}} & & \\ & & \frac{1}{g_{22}} & \\ & & & \frac{1}{g_{33}} \end{Bmatrix}$$

2.16.2 Inverting a non-diagonal two-dimensional tensor.

$$g_{ab} = \begin{Bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{Bmatrix}$$

$$|g_{ab}| = g_{11}g_{22} - g_{12}g_{21}$$

$$\Rightarrow \quad g^{ab} = \frac{1}{|g_{ab}|}\begin{Bmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{Bmatrix}$$

2.16.2.1 [bb]*Example*

$$g_{ab} = \begin{Bmatrix} x^2 & 1 \\ 1 & -1 \end{Bmatrix}$$

$$|g_{ab}| = -x^2 - 1$$

$$\Rightarrow \quad g^{ab} = \frac{-1}{x^2+1}\begin{Bmatrix} -1 & -1 \\ -1 & x^2 \end{Bmatrix} =^{40} \begin{Bmatrix} \frac{1}{x^2+1} & \frac{1}{x^2+1} \\ \frac{1}{x^2+1} & \frac{-x^2}{x^2+1} \end{Bmatrix}$$

2.16.3 Inverting a non-diagonal four-dimensional tensor.

$$g_{ab} = \begin{Bmatrix} g_{00} & & & g_{03} \\ & g_{11} & & \\ & & g_{22} & \\ g_{30} & & & g_{33} \end{Bmatrix}$$

$$\Rightarrow \quad g^{ab} = \begin{Bmatrix} \frac{g_{33}}{g_{00}g_{33} - g_{03}g_{30}} & & & -\frac{g_{03}}{g_{00}g_{33} - g_{03}g_{30}} \\ & \frac{1}{g_{11}} & & \\ & & \frac{1}{g_{22}} & \\ -\frac{g_{30}}{g_{00}g_{33} - g_{03}g_{30}} & & & \frac{g_{00}}{g_{00}g_{33} - g_{03}g_{30}} \end{Bmatrix}$$

2.16.4 [cc]Example: The Gödel Metric tensor.

$$g_{ab} = \frac{1}{2\omega^2}\begin{Bmatrix} 1 & 0 & 0 & e^x \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ e^x & 0 & 0 & \frac{1}{2}e^{2x} \end{Bmatrix}$$

$$g_{00}g_{33} - g_{03}g_{30} = \frac{1}{2}e^{2x} - e^{2x} = -\frac{1}{2}e^{2x}$$

---

[40] Checking: $\begin{Bmatrix} x^2 & 1 \\ 1 & -1 \end{Bmatrix}\begin{Bmatrix} \frac{1}{x^2+1} & \frac{1}{x^2+1} \\ \frac{1}{x^2+1} & \frac{-x^2}{x^2+1} \end{Bmatrix} = \begin{Bmatrix} x^2\frac{1}{x^2+1} + \frac{1}{x^2+1} & x^2\frac{1}{x^2+1} + \frac{-x^2}{x^2+1} \\ \frac{1}{x^2+1} - \frac{1}{x^2+1} & \frac{1}{x^2+1} + (-1)\frac{-x^2}{x^2+1} \end{Bmatrix} = \begin{Bmatrix} 1 & 0 \\ 0 & 1 \end{Bmatrix}$

http://physicssusan.mono.net logik.susan@gmail.com

$$\Rightarrow \qquad g^{ab} = {}^{41}2\omega^2 \begin{Bmatrix} -1 & 0 & 0 & 2e^{-x} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 2e^{-x} & 0 & 0 & -2e^{-2x} \end{Bmatrix}$$

2.16.5 [dd]The inverse metric of the Kerr Spinning Black Hole

The Kerr metric of a spinning black hole with mass m and angular momentum S.

$$ds^2 = \left(1 - \frac{2mr}{\Sigma}\right)dt^2 + \frac{4amr\sin^2\theta}{\Sigma}dtd\phi - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2 - \left(r^2 + a^2 + \frac{2a^2mr\sin^2\theta}{\Sigma}\right)\sin^2\theta\, d\phi^2$$

$$\Delta = r^2 - 2mr + a^2$$

$$\Sigma = r^2 + a^2\cos^2\theta = r^2 + a^2 - a^2\sin^2\theta = \Delta + 2mr - a^2\sin^2\theta$$

$$a = \frac{S}{m}$$

The metric tensor

$$g_{ab} = \begin{Bmatrix} \left(1 - \frac{2mr}{\Sigma}\right) & & & \frac{2amr\sin^2\theta}{\Sigma} \\ & -\frac{\Sigma}{\Delta} & & \\ & & -\Sigma & \\ \frac{2amr\sin^2\theta}{\Sigma} & & & -\left(r^2 + a^2 + \frac{2a^2mr\sin^2\theta}{\Sigma}\right)\sin^2\theta \end{Bmatrix}$$

with the inverse

$$g^{ab} = \begin{Bmatrix} g^{tt} & & & g^{t\phi} \\ & -\frac{\Delta}{\Sigma} & & \\ & & -\frac{1}{\Sigma} & \\ g^{\phi t} & & & g^{\phi\phi} \end{Bmatrix}$$

$$= \begin{Bmatrix} \frac{1}{\Sigma\Delta}\left((r^2 + a^2)^2 - \Delta a^2\sin^2\theta\right) & & & \frac{2amr}{\Sigma\Delta} \\ & -\frac{\Delta}{\Sigma} & & \\ & & -\frac{1}{\Sigma} & \\ \frac{2amr}{\Sigma\Delta} & & & -\frac{(\Delta - a^2\sin^2\theta)}{\Sigma\Delta\sin^2\theta} \end{Bmatrix}$$

where we can calculate $g^{tt}, g^{t\phi}, g^{\phi\phi}$ from the inverse[42]

$$g^{ab} = \frac{1}{g_{tt}g_{\phi\phi} - (g_{\phi t})^2}\begin{Bmatrix} g_{\phi\phi} & -g_{t\phi} \\ -g_{\phi t} & g_{tt} \end{Bmatrix}$$

---

[41] Checking: $\frac{1}{2\omega^2}\begin{Bmatrix} 1 & 0 & 0 & e^x \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ e^x & 0 & 0 & \frac{1}{2}e^{2x} \end{Bmatrix} 2\omega^2 \begin{Bmatrix} -1 & 0 & 0 & 2e^{-x} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 2e^{-x} & 0 & 0 & -2e^{-2x} \end{Bmatrix} =$

$$\begin{Bmatrix} -1 + e^x 2e^{-x} & 0 & 0 & 2e^{-x} + e^x(-2e^{-2x}) \\ 0 & (-1)(-1) & 0 & 0 \\ 0 & 0 & (-1)(-1) & 0 \\ e^x(-1) + \frac{1}{2}e^{2x}2e^{-x} & 0 & 0 & e^x 2e^{-x} + \frac{1}{2}e^{2x}(-2e^{-2x}) \end{Bmatrix} = \begin{Bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Bmatrix}$$

[42] $\begin{Bmatrix} a & b \\ c & d \end{Bmatrix}^{-1} = \frac{1}{ad-bc}\begin{Bmatrix} d & -b \\ -c & a \end{Bmatrix}$

First we calculate the common factor

$$\frac{1}{g_{tt}g_{\phi\phi}-(g_{\phi t})^2} = \left(-\left(1-\frac{2mr}{\Sigma}\right)\left(r^2+a^2+\frac{2a^2mr\sin^2\theta}{\Sigma}\right)\sin^2\theta-\left(\frac{2amr\sin^2\theta}{\Sigma}\right)^2\right)^{-1}$$

$$= \left(-\left(1-\frac{2mr}{\Sigma}\right)\left(r^2+a^2+\frac{2a^2mr\sin^2\theta}{\Sigma}\right)-\left(\frac{2amr}{\Sigma}\right)^2\sin^2\theta\right)^{-1}\frac{1}{\sin^2\theta}$$

$$=^{43}\left(-\left(r^2+a^2+\frac{2a^2mr\sin^2\theta}{\Sigma}\right)+\frac{2mr}{\Sigma}(r^2+a^2)\right)^{-1}\frac{1}{\sin^2\theta}$$

$$= \left(-\left(r^2+a^2+\frac{2mr}{\Sigma}a^2\sin^2\theta\right)+\frac{2mr}{\Sigma}(r^2+a^2)\right)^{-1}\frac{1}{\sin^2\theta}$$

$$=^{44}\left(-(r^2+a^2)+\frac{2mr}{\Sigma}(r^2+a^2\cos^2\theta)\right)^{-1}\frac{1}{\sin^2\theta}$$

$$= \left(-(r^2+a^2)+\frac{2mr}{\Sigma}\Sigma\right)^{-1}\frac{1}{\sin^2\theta} = (-(r^2+a^2)+2mr)^{-1}\frac{1}{\sin^2\theta}$$

$$= -\frac{1}{\Delta\sin^2\theta}$$

Now we can calculate the inverse metric

$$g^{tt} = \frac{g_{\phi\phi}}{g_{tt}g_{\phi\phi}-(g_{\phi t})^2} = -\frac{g_{\phi\phi}}{\Delta\sin^2\theta} = -\frac{-\left(r^2+a^2+\frac{2a^2mr\sin^2\theta}{\Sigma}\right)\sin^2\theta}{\Delta\sin^2\theta}$$

$$= \frac{1}{\Delta}\left(r^2+a^2+\frac{2a^2mr\sin^2\theta}{\Sigma}\right) = \frac{1}{\Sigma\Delta}(\Sigma(r^2+a^2)+2a^2mr\sin^2\theta)$$

$$= \frac{1}{\Sigma\Delta}\left((r^2+a^2\cos^2\theta)(r^2+a^2)+(r^2+a^2-\Delta)a^2\sin^2\theta\right)$$

$$= \frac{1}{\Sigma\Delta}\left((r^2+a^2\cos^2\theta)(r^2+a^2)+(r^2+a^2)a^2(1-\cos^2\theta)-\Delta a^2\sin^2\theta\right)$$

$$= \frac{1}{\Sigma\Delta}((r^2+a^2)^2-\Delta a^2\sin^2\theta)$$

$$g^{t\phi} = -\frac{g_{t\phi}}{g_{tt}g_{\phi\phi}-(g_{\phi t})^2} = \frac{g_{t\phi}}{\Delta\sin^2\theta} = \frac{1}{\Delta\sin^2\theta}\frac{2amr\sin^2\theta}{\Sigma} = \frac{2amr}{\Sigma\Delta}$$

$$g^{\phi\phi} = \frac{g_{tt}}{g_{tt}g_{\phi\phi}-(g_{\phi t})^2} = -\frac{g_{tt}}{\Delta\sin^2\theta} = -\frac{1}{\Delta\sin^2\theta}\left(1-\frac{2mr}{\Sigma}\right)$$

$$= -\frac{1}{\Sigma\Delta\sin^2\theta}(\Sigma-2mr) = -\frac{1}{\Sigma\Delta\sin^2\theta}(\Delta+2mr-a^2\sin^2\theta-2mr)$$

$$= -\frac{(\Delta-a^2\sin^2\theta)}{\Sigma\Delta\sin^2\theta}$$

2.17 [ee]Conformal space-time

Two metrics are conformally related if

$$g'_{ab} = f^2(x)g_{ab}$$

A metric is conformally flat if

$$g_{ab} = f^2(x)\eta_{ab}$$

---

[43] $= \left(-\left(r^2+a^2+\frac{2a^2mr\sin^2\theta}{\Sigma}\right)+\frac{2mr}{\Sigma}\left(r^2+a^2+\frac{2a^2mr\sin^2\theta}{\Sigma}\right)-\left(\frac{2amr}{\Sigma}\right)^2\sin^2\theta\right)^{-1}\frac{1}{\sin^2\theta} =$

[44] $= \left(-\left(r^2+a^2+\frac{2mr}{\Sigma}a^2\sin^2\theta\right)+\frac{2mr}{\Sigma}(r^2+a^2\cos^2\theta+a^2\sin^2\theta)\right)^{-1}\frac{1}{\sin^2\theta} =$

Citerede værker

Charles Misner, K. T. (2017). *Gravitation.* Princeton University Press.
d'Inverno, R. (1992). *Introducing Einstein's Relativity.* Oxford: Clarendon Press.
Ellis, S. H. (1973). *The Large Scale Structure of Space-time.* Cambridge: Cambridge University Press.
Hartle, J. B. (2003). *An Introduction to Einstein's General Relativity.* San Francisco: Pearson Education.
Kay, D. C. (1988). *Tensor Calculus.* McGraw-Hill.
McMahon, D. (2006). *Relativity Demystified.* McGraw-Hill.
Penrose, R. (2004). *The Road to Reality.* New York: Vintage Books.
Spiegel, M. R. (1990). *SCHAUM'S OUTLINE SERIES: Mathematical Handbook of FORMULAS and TABLES.* McGraw-Hill Publishing Company.

---

[a] (d'Inverno, 1992, p. 82)
[b] (McMahon, 2006, s. 36)
[c] (Hartle, 2003, s. 57)
[d] (Hartle, 2003, s. 58)
[e] (McMahon, 2006, s. 186)
[f] (Hartle, 2003, s. 135)
[g] (Hartle, 2003) pp 66-68
[h] (McMahon, 2006, s. 84), (Hartle, 2003, s. 143, 165, 184), (Kay, 1988, s. 126)
[i] (Hartle, 2003, s. 164)
[j] (Hartle, 2003, s. 137)
[k] (Hartle, 2003, s. 140)
[l] (Hartle, 2003, s. 141)
[m] (Hartle, 2003, s. 181)
[n] (Hartle, 2003, s. 126), (Ellis, 1973, p. 118)
[o] (Hartle, 2003) eq. (3.18)
[p] (Hartle, 2003, s. 127)
[q] (Hartle, 2003, s. 128)
[r] (Hartle, 2003, s. 44)
[s] (Hartle, 2003, s. 157)
[t] https://en.wikipedia.org/wiki/Planck_length
[u] https://en.wikipedia.org/wiki/Planck_energy
[v] (Kay, 1988, s. 126)
[w] (Hartle, 2003, s. 136)
[x] (Hartle, 2003, s. 163)
[y] (McMahon, 2006, s. 46)
[z] (Hartle, 2003, s. 146)
[æ] (Hartle, 2003, s. 147)
[ø] (Hartle, 2003, s. 166)
[å] (Hartle, 2003, s. 29)
[aa] (McMahon, 2006, s. 37)
[bb] (McMahon, 2006, s. 39)
[cc] (McMahon, 2006, s. 326)
[dd] (McMahon, 2006, s. 246)
[ee] (McMahon, 2006, s. 90)