Content

Appendix B: Collection of results........ 1
Bibliografi........ 15

Appendix B: Collection of results.

1) [1]General Surfaces.
   a) The line element: $ds^2 = a^2(d\theta^2 + f^2(\theta)d\phi^2)$
      i) Since the line element is the same for all ϕ, it corresponds to a surface that is axi-symmetric about an axis.
      ii) The circumference $C(\theta)$ of a circle of constant θ is $C(\theta) = \oint ds = \int_0^{2\pi} af(\theta)d\phi = 2\pi af(\theta)$
      iii) The distance from pole to pole is $d = a\int_0^{\pi} d\theta = \pi a$
2) The Surface of a Sphere.
   a) The line element: $ds^2 = a^2(d\theta^2 + \sin^2\theta\, d\phi^2)$
      i) $f(\theta) = \sin\theta$
      ii) $C(\theta) = 2\pi a\sin\theta\,, d = \pi a$
3) The Surface of a peanut.
      i) $f(\theta) = \sin\theta\left(1 - \frac{3}{4}\sin^2\theta\right)$
      ii) $C(\theta) = 2\pi a\sin\theta\left(1 - \frac{3}{4}\sin^2\theta\right)$
4) The Surface of an ellipsoid.
   a) The line element: $ds^2 = [\cos^2\theta\,(a^2\cos^2\phi + b^2\sin^2\phi) + c^2\sin^2\theta]d\theta^2 + \sin^2\theta\,(a^2\sin^2\phi + b^2\cos^2\phi)d\phi^2 + 2(b^2 - a^2)(\cos\phi\cos\theta\sin\phi\sin\theta)d\theta d\phi$
5) [2]The Surface of an Egg.
   a) The line element: $ds^2 = a^2[(\cos^2\theta + 4\sin^2\theta)d\theta^2 + \sin^2\theta\, d\phi^2]\; if\;\; a = b = \frac{1}{2}c$
6) [3]Cylindindrical Coordinates.
   a) The line element: $ds^2 = dr^2 + r^2d\phi^2 + dz^2$
   b) Geodesic equations.
      i) $\ddot{r} - r\dot{\phi}^2 = 0$
      ii) $\ddot{\phi} + \frac{1}{r}\dot{r}\dot{\phi} + \frac{1}{r}\dot{\phi}\dot{r} = 0$
      iii) $\ddot{z} = 0$
7) General non-diagonal space-time.
   a) The metric tensor: $g_{ab} = \begin{Bmatrix} & g_{12} & & \\ g_{12} & & & \\ & & g_{33} & \\ & & & g_{44} \end{Bmatrix}$
   b) Ricci constant: $R = 4g^{12}R^3{}_{132} + 4g^{12}R^4{}_{142} + 2g^{33}R^4{}_{343}$
8) [4]Metric Example 1.
   a) Line element: $ds^2 = y^2\sin x\, dx^2 + x^2\tan y\, dy^2$
   b) Ricci scalar: $R = \left(\frac{x\cos x\tan^2 y + y\sin^2 x + y\sin^2 x\tan^2 y}{x^2y^2\sin^2 x\tan^2 y}\right)$

http://physicssusan.mono.net logik.susan@gmail.com

9) [5]Metric Example 2.
   a) Line element: $ds^2 = d\psi^2 + \sinh^2\psi\ d\theta^2 + \sinh^2\psi \sin^2\theta\, d\phi^2$
   b) The basis one-forms: $\omega^{\hat{\psi}} = d\psi$, $\omega^{\hat{\theta}} = \sinh\psi\, d\theta$, $\omega^{\hat{\phi}} = \sinh\psi \sin\theta\, d\phi$
   c) Ricci rotation coefficients: $\Gamma^{\hat{\psi}}{}_{\hat{\theta}\hat{\theta}} = -\coth\psi$, $\Gamma^{\hat{\theta}}{}_{\hat{\psi}\hat{\theta}} = \coth\psi$, $\Gamma^{\hat{\phi}}{}_{\hat{\psi}\hat{\phi}} = \frac{\coth\psi}{\sin\theta}$, $\Gamma^{\hat{\psi}}{}_{\hat{\phi}\hat{\phi}} = -\frac{\coth\psi}{\sin\theta}$, $\Gamma^{\hat{\theta}}{}_{\hat{\phi}\hat{\phi}} = -\frac{\cot\theta}{\sinh^2\psi}$, $\Gamma^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}} = \frac{\cot\theta}{\sinh^2\psi}$

10) [6]Metric Example 3.
   a) The line element: $ds^2 = (u^2+v^2)du^2 + (u^2+v^2)dv^2 + u^2v^2d\theta^2$
   b) The Riemann tensor: $R_{abcd} = 0$

11) [7]General 4-dimensional space-time.
   a) The line element: $ds^2 = -dt^2 + L^2(t,r)dr^2 + B^2(t,r)d\phi^2 + M^2(t,r)dz^2$
   b) The basis one-forms: $\omega^{\hat{t}} = dt$, $\omega^{\hat{r}} = L(t,r)dr$, $\omega^{\hat{\phi}} = B(t,r)d\phi$, $\omega^{\hat{z}} = M(t,r)dz$
   c) The Einstein tensor: $G_{\hat{t}\hat{t}} = \frac{B'L'-B''L}{BL^3} + \frac{\dot{B}\dot{L}}{BL} + \frac{M'L'-M''L}{ML^3} + \frac{\dot{M}\dot{L}}{ML} + \frac{\dot{B}\dot{M}}{BM} - \frac{B'M'}{L^2BM}$, $G_{\hat{r}\hat{t}} = \frac{B'\dot{L}}{BL^2} - \frac{\dot{B}'}{BL} + \frac{M'\dot{L}}{ML^2} - \frac{\dot{M}'}{ML}$, $G_{\hat{r}\hat{r}} = -\frac{\ddot{B}}{B} - \frac{\ddot{M}}{M} + \frac{\dot{B}\dot{M}}{BM} - \frac{B'M'}{L^2BM}$, $G_{\hat{\phi}\hat{\phi}} = -\frac{\ddot{L}}{L} - \frac{\ddot{M}}{M}\frac{M'L'-M''L}{ML^3} + \frac{\dot{M}\dot{L}}{ML}$, $G_{\hat{z}\hat{z}} = -\frac{\ddot{L}}{L} - \frac{\ddot{B}}{B} + \frac{B'L'-B''L}{BL^3} + \frac{\dot{B}\dot{L}}{BL}$

12) [8]The Plane in polar coordinates.
   a) The line element: $dS^2 = dr^2 + r^2d\phi^2$
   b) The geodesic equations:
      i) $\ddot{r} = r\dot{\phi}^2$
      ii) $\ddot{\phi} = -\frac{2}{r}\dot{r}\dot{\phi}$

13) [9]Mathematical singularity: The two-dimensional plane in polar coordinates $dS^2 = dr^2 + r^2d\phi^2$ can blow up in a singularity by making the transformation $r = a^2/r'$ for some constant a $dS^2 = \frac{a^4}{r'^4}(dr'^2 + r'^2d\phi^2)$. This singularity arises because the transformation $r = a^2/r'$ map all the points at $r = \infty$ into $r = 0$.

14) [10]The Hyperbolic Plane.
   a) The line element: $dS^2 = y^{-2}(dx^2 + dy^2)$
   b) Conservation equations:
      i) $(x - x_0)^2 + y^2 = \frac{1}{K_1^2}$
      ii) $x = \frac{1}{K_1}\tanh(S)$
      iii) $y = \frac{1}{K_1\cosh(S)}$

15) [11]2-sphere With Radius a.
   a) Line element: $ds^2 = a^2d\theta^2 + a^2\sin^2\theta\, d\phi^2$.
   b) Basis one-forms: $\omega^{\hat{\theta}} = d$, $\omega^{\hat{\phi}} = \sin\theta\, d\phi$
   c) Killing vector: $X = \begin{pmatrix} X^\theta \\ X^\phi \end{pmatrix} = \begin{pmatrix} A'\sin\phi + B'\cos\phi \\ \cot\theta\,(A'\cos\phi - B'\sin\phi) + C' \end{pmatrix}$; $X = X^a\partial_a = AL_x + BL_y + CL_z$
      i) $L_x = \cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}$
      ii) $L_y = \sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}$
      iii) $L_z = \frac{\partial}{\partial\phi}$
   d) Ricci scalar: $R = 2$

http://physicssusan.mono.net logik.susan@gmail.com

16) [12]Three-dimensional Flat Space in Spherical Polar Coordinates.
   a) Line element: $ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$
   b) Christoffel symbols: $\Gamma^r{}_{\theta\theta} = -r$, $\Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = \frac{1}{r}$, $\Gamma^r{}_{\phi\phi} = -r\sin^2\theta$, $\Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} = \frac{1}{r}$, $\Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta$, $\Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta} = \cot\theta$

17) [13]Flat Minkowski space-time: Flat Minkowski space-time is the mathematical setting in which Einstein's special theory of relativity is most conveniently formulated.
   a) Line element:
      i) $ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$ (Spherical polar coordinates [14])
      ii) $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$
         (1) $ds^2 < 0$, time-like, inside the light cone
         (2) $ds = 0$: null-vector, on the light cone
         (3) $ds^2 > 0$, space-like, outside the light cone.
   b) Basis one-forms: $\omega^{\hat{t}} = dt$, $\omega^{\hat{r}} = dr$, $\omega^{\hat{\theta}} = r d\theta$, $\omega^{\hat{\phi}} = r\sin\theta\, d\phi$
   c) Orthonormal tetrad
      i) $l_a = \frac{1}{\sqrt{2}}(1, \quad 1, \quad 0, \quad 0)$, $l^a = \frac{1}{\sqrt{2}}(1, \quad -1, \quad 0, \quad 0)$
      ii) $n_a = \frac{1}{\sqrt{2}}(1, \quad -1, \quad 0, \quad 0)$, $n^a = \frac{1}{\sqrt{2}}(1, \quad 1, \quad 0, \quad 0)$
      iii) $m_a = \frac{1}{\sqrt{2}}(0, \quad 0, \quad r, \quad ir\sin\theta)$, $m^a = \frac{1}{\sqrt{2}}\left(0, \quad 0, \quad -\frac{1}{r}, \quad -\frac{i}{r\sin\theta}\right)$
      iv) $\overline{m_a} = \frac{1}{\sqrt{2}}(0, \quad 0, \quad r, \quad -ir\sin\theta)$, $\overline{m^a} = \frac{1}{\sqrt{2}}\left(0, \quad 0, \quad -\frac{1}{r}, \quad \frac{i}{r\sin\theta}\right)$
   d) Spin coefficients: $\pi = \nu = \lambda = \mu = \kappa = \tau = \rho = \sigma = \varepsilon = \gamma = 0$, $\alpha = \frac{\cot\theta}{r}$, $\beta = -\frac{\cot\theta}{r}$
   e) [15]Rotating the axes (t, r) by 45°:
      i) Coordinate transformation:
         (1) $3u \equiv t - r$
         (2) $v \equiv t + r$
      ii) Line element: $ds^2 = -dudv + \frac{1}{4}(u - v)^2(d\theta^2 + \sin^2\theta\, d\phi^2)$
   f) Make a further transformation, that map the coordinates from $-\infty < t < +\infty$ and $0 < r < +\infty$ into a finite range: $u' \equiv \tan^{-} u \equiv t' - r'$, $v' \equiv \tan^{-1} v \equiv t' + r'$

18) [16]Three-dimensional Flat Space-time.
   a) Line element: $ds^2 = -dt^2 + dr^2 + r^2 d\phi^2$
   b) Conservation equations: Light rays moves on straight lines in curved space. From our point of view the tip of the light cone (t, r) moves along a hyperbolic path.
      i) $\left(\frac{K_2}{K_1}\right)^2 = r^2 - (t - t_0)^2$
      ii) $t - t_0 = \frac{K_2}{K_1}\tan(\phi - \phi_0)$
      iii) $r = \pm\frac{K_2}{K_1}\sqrt{\tan^2(\phi - \phi_0) + 1}$
   c) Coordinate transformation:
      i) $t = t' - x'$
      ii) $x = t'$
         (1) Line element:[17] $ds^2 = -dt^2 + 2dxdt + dy^2 + dz^2$

19) [18]Two-dimensional Flat Space-time.
   a) Coordinate transformation:

   i) $t = X\sinh(T)$
   ii) $x = X\cosh(T)$
b) Line element: $ds^2 = -X^2 dT^2 + dX^2$
c) Conservation equation: $X^2 \frac{dT}{dS} = K$
d) Geodesic equation: $X(T) = -\frac{K}{\sinh(T-T^*)}$

20) Conformally Flat Space:
a) Line element: $g_{ab} = f(x)\eta_{ab}$
b) Weyl tensor: $C_{abcd} = 0$

21) [19]Static Weak Field Metric: In this model the flat spacetime geometry of special relativity is modified to introduce a slight curvature that will explain geometrically the behavior of clocks. Further, the world lines of extremal proper time in this modified geometry will reproduce the predictions of Newtonian mechanics for motion in a gravitational potential for nonrelativistic velocities. $\Phi(x^i)$ is a function of position satisfying the Newtonian field equation[20] $\nabla^2\Phi(\vec{x}) = 4\pi G\mu(\vec{x})$and assumed to vanish at infinity. For example outside Earth $\Phi(r) = -\frac{GM_{\oplus}}{r}$. This line element is predicted by general relativity for small curvatures produced by time-independent weak sources, and it is a good approximation to the curved spacetime geometry produced by the Sun.
a) Line element: $ds^2 = -\left(1 + \frac{2\Phi(x^i)}{c^2}\right)(cdt)^2 + \left(1 - \frac{2\Phi(x^i)}{c^2}\right)(dx^2 + dy^2 + dz^2)$

[21]$\Delta\tau_B \sim \left(1 + \frac{\Phi_B - \Phi_A}{c^2}\right)\Delta\tau_A$ tells us the observed fact, that when the receiver B is at a higher gravitational potential that the emitter A, the signals will be received more slowly than they were emitted and vice versa.

22) [22]Rindler metric: The Rindler coordinate system or frame describes a uniformly accelerating frame of reference in Minkowski space.
a) Line element: $ds^2 = \xi^2 d\tau^2 - d\xi^2$
b) Basis one-forms: $\omega^{\hat{\xi}} = d\xi, \omega^{\hat{\tau}} = \xi d\tau$
c) Geodesic equations:
   i) $\ddot{\xi} + \xi\dot{\tau}^2 = 0$
   ii) $\ddot{\tau} + \frac{2}{\xi}\dot{\xi}\dot{\tau} = 0$

23) [23]The Einstein Cylinder:
a) Line element: $ds^2 = -dt^2 + (a_0)^2\left(d\theta^2 + \sin^2\theta\,(d\phi^2 + \sin^2\phi\, d\psi^2)\right)$
b) Coordinate transformation: $r = \sin\theta$
   i) Line element: $ds^2 = -dt^2 + (a_0)^2\left(\frac{dr^2}{1-r^2} + r^2(d\phi^2 + \sin^2\phi\, d\psi^2)\right)$
c) Ricci scalar: $R = 3 \cdot \frac{2}{a_0^2}$
d) Einstein equations:
   i) $\mu = \frac{3}{a_0^2} - \Lambda$
   ii) $p = -\frac{1}{a_0^2} + \Lambda$

24) [24]Classical Anti-de Sitter Space-time
a) Line element: $ds^2 = -\cosh^2(r)\, dt^2 + dr^2 + \sinh^2(r)\, d\theta^2 + \sinh^2(r)\sin^2\theta\, d\phi^2$

b) Coordinate transformation: $\cosh(r) = \frac{1}{\cos\psi}$
   i) Line element: $ds^2 = -\frac{1}{\cos^2\psi}(dt^2 + d\psi^2 + \sin^2\psi\, d\theta^2 + \sin^2\psi \sin^2\theta\, d\phi^2)$ - which is conformally related to the Einstein cylinder.

c) Killing vectors and conservation equations
   i) $\boldsymbol{\xi} = (\xi^t, \xi^r, \xi^\theta, \xi^\phi) = (1,0,0,0) \Rightarrow \dot{t} = \frac{K_1}{\cosh^2(r)}$
   ii) $\boldsymbol{\zeta} = (\zeta^t, \zeta^r, \zeta^\theta, \zeta^\phi) = (0,0,0,1) \Rightarrow \dot{\phi} = \frac{K_2}{\sinh^2(r)\sin^2\theta}$

d) Geodesic equations:
   i) $0 = \cosh^2(r)\,\ddot{t} + 2\sinh(r)\,\dot{t}\dot{r}$
   ii) $0 = \ddot{r} + \sinh(r)\,\dot{t}^2 - \cosh(r)\,\dot{\theta}^2 - \cosh(r)\sin^2\theta\,\dot{\phi}^2$
   iii) $0 = 2\cosh(r)\,\dot{r}\dot{\theta} + \sinh^2(r)\,\ddot{\theta} - \sinh^2(r)\cos\theta\,\dot{\phi}^2$
   iv) $0 = 2\cosh(r)\sin^2\theta\,\dot{\phi} + 2\sinh^2(r)\cos\theta\,\dot{\phi} + \sinh^2(r)\sin^2\theta\,\ddot{\phi}$

e) <u>The coordinates t and r</u>: $\dot{r} = \cosh(r) \Rightarrow t - t_0 = \frac{\sinh(r)}{2\cosh^2(r)} + \tan^{-1}(e^r) - \left(\frac{\sinh(r_0)}{2\cosh^2(r_0)} + \tan^{-1}(e^{r_0})\right) \to \frac{\sinh(r)}{2\cosh^2(r)} + \tan^{-1}(e^r) - K_3$ (if $r_0 \to 0$) $\to K_4$ (if $r \to \infty$). Interpreting this means, that no matter how far the light travels in this space-time from $r = 0$ to $r \to \infty$ this happens within a limited time.

25) [25]The de Sitter Space-time: The de Sitter space-time is an example of the Robertson Walker metric in vacuum, positive curvature and a cosmological constant.

a) Line-element: $ds^2 = -dt^2 + a(t)^2\left(d\theta^2 + \sin^2\theta\,(d\phi^2 + \sin^2\phi\, d\psi^2)\right) = -dt^2 + \frac{1}{k^2}\cosh^2(kt)\left(d\theta^2 + \sin^2\theta\,(d\phi^2 + \sin^2\phi\, d\psi^2)\right)$

b) Einstein equations:
   i) $0 = \frac{3}{a^2}(1 + \dot{a}^2) - \Lambda$
   ii) $0 = 2\frac{\ddot{a}}{a} + \frac{1}{a^2}(1 + \dot{a}^2) - \Lambda$
   iii) $a^2 = \frac{1}{k^2}\cosh^2(kt)$

26) [26]Tolman-Bondi de Sitter: Spherical dust with a cosmological constant

a) Line element: $ds^2 = dt^2 - e^{-2\psi(t,r)}dr^2 - R^2(t,r)d\theta^2 - R^2(t,r)\sin^2\theta\, d\phi^2$

b) Basis one-forms: $\omega^{\hat{t}} = dt$, $\omega^{\hat{r}} = e^{-\psi(t,r)}dr$, $\omega^{\hat{\theta}} = R(t,r)d\theta$, $\omega^{\hat{\phi}} = R(t,r)\sin\theta\, d\phi$

c) The Einstein tensor.
   i) $G_{tt} = \frac{1}{R^2}\left[1 - 2R\dot{R}\dot{\psi} + (\dot{R})^2 - (2RR'' + 2RR'\psi' + (R')^2)e^{2\psi}\right]$
   ii) $G_{rt} = -2\left[\frac{(\dot{R})'}{R} + \frac{R'\dot{\psi}}{R}\right]$
   iii) $G_{rr} = \frac{1}{R^2}\left[(R')^2 - \left(2R\ddot{R} + 1 + (\dot{R})^2\right)e^{-2\psi}\right]$
   iv) $G_{\theta\theta} = R^2\left(\left[\ddot{\psi} - (\dot{\psi})^2\right] + \frac{1}{R}\left[(R'' + R'\psi')e^{2\psi} + \dot{R}\dot{\psi} - \ddot{R}\right]\right)$
   v) $G_{\phi\phi} = R^2\sin^2\theta\left(\left[\ddot{\psi} - (\dot{\psi})^2\right] + \frac{1}{R}\left[(R'' + R'\psi')e^{2\psi} + \dot{R}\dot{\psi} - \ddot{R}\right]\right)$

27) [27]The Anti-de Sitter Space-time: This spacetime is a solution to the Einstein vacuum equation with cosmological constant $\Lambda = -3$.

a) Line element: $ds^2 = -dt^2 + \cos^2(t)\,dr^2 + \cos^2(t)\sinh^2(r)\,d\theta^2 + \cos^2(t)\sinh^2(r)\sin^2\theta\,d\phi^2$

b) Ricci scalar: $R = 12$

28) [28]2+1 dimensions: Gravitational Collapse of an Inhomogenous Spherically Symmetric Dust Cloud with Nonzero Cosmological Constant.

a) Line element: $ds^2 = -dt^2 + e^{2b(t,r)}dr^2 + R^2(t,r)d\phi^2$

b) Basis one-forms: $\omega^{\hat{t}} = dt,\ \omega^{\hat{r}} = e^{b(t,r)}dr,\ \omega^{\hat{\phi}} = R(t,r)d\phi$

c) The stress tensor: $T_{ab} = \rho u_a u_b,\ u^a = (u^t, u^r, u^\phi) = (1,0,0)$

d) The Einstein equations.

i) $-\left[(R'' - R'b')\frac{e^{-2b}}{R} - \frac{\dot{R}\dot{b}}{R}\right] + \lambda^2 = \kappa\rho$

ii) $(\dot{R})' - R'\dot{b} = 0$

iii) $\ddot{R} + \lambda^2 R = 0$

iv) $-\left[\ddot{b} + (\dot{b})^2\right] - \lambda^2 = 0$

29) [29]Robertson-Walker: Homogenous, isotropic and expanding universe. The constant $k = -1,0,1$ depending on whether the universe is open, flat or closed.

a) Line element: $ds^2 = -dt^2 + \frac{a^2(t)}{1-kr^2}dr^2 + a^2(t)r^2d\theta^2 + a^2(t)r^2\sin^2\theta\, d\phi^2$

b) Basis one-forms: $\omega^{\hat{t}} = dt, \omega^{\hat{r}} = \frac{a(t)}{\sqrt{1-kr^2}}dr, \omega^{\hat{\theta}} = a(t)rd\theta, \omega^{\hat{\phi}} = a(t)r\sin\theta\, d\phi$

c) Einstein tensor.

i) $G_{\hat{t}\hat{t}} = 3\left(\frac{(\dot{a}^2+\mathrm{k})}{\mathrm{a}^2}\right)$

ii) $G_{\hat{r}\hat{r}} = G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}} = -\left(2\frac{\ddot{a}(t)}{a(t)} + \frac{\dot{a}(t)^2+k}{a(t)^2}\right)$

d) Stress tensor: $T_{\hat{a}\hat{b}} = diag(\rho, P, P, P)$

e) Einstein equations (The Friedmann equations).

i) $8\pi\rho = \frac{3}{a^2}(k + \dot{a}^2) + \Lambda$

ii) $8\pi P = 2\frac{\ddot{a}}{a} + \frac{1}{a^2}(k + \dot{a}^2) + \Lambda$

f) Solutions.

i) [30]Matter, no radiation $P = 0$: $\rho \propto a^{-3}$

ii) [31]No curvature,$k = 0$, no cosmological constant, $\Lambda = 0$: $a(t) \propto t^{\frac{2}{3}}$

iii) [32]Matter and radiation, late universe $P = \frac{1}{3}\rho$: $\rho \propto a^{-4}$

iv) [33]No matter, radiation, early universe: $a(t) \propto \sqrt{t}$

v) [34]No matter, no radiation, no curvature $k = 0$, cosmological constant: $a(t) = Ce^{\sqrt{\frac{\Lambda}{3}}t}$

vi) [35] A particle of mass m, sitting on a surface of a ball of radius R and mass density ρ, experiences an acceleration, $\frac{d^2R}{dt^2}$ given by $\frac{\frac{4\pi}{3}R^3G\rho}{R^2}$, and so $\frac{1}{R}\frac{d^2R}{dt^2} = \frac{4\pi}{3}G\rho$. If we formally identify R with the radius of the Universe, and ρ with the mass density of the Universe, this is Einstein's equation for how the size of the Universe evolves, assuming the absence of pressure.

vii) [36] It is not difficult to see how accelerated expansion arises. One of Einstein's equations is: $\frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi}{3}(\rho + 3p)$. Notice that if the right-hand side of this equation is positive, the scale factor will grow at an increasing rate: the Universe's rate of growth will accelerate with time.

30) [37]Light Travelling in the Universe: $ds^2 = dt^2 - a^2(t)dx^2$, where $dx^2 = dx_1{}^2 + dx_2{}^2 + dx_3{}^2$ and the x_i are commoving coordinates.Light travel along null geodesic i.e. $ds^2 = 0$. We can now write $\int_t^{t_0}\frac{dt}{a(t)}$

http://physicssusan.mono.net logik.susan@gmail.com

for the total commoving distance light emitted at time t can travel by time t_0. If we multiply this by the value of the scale factor $a(t_0)$ at time t_0, then we will have calculated the physical distance that the light has traveled in this time interval. This algorithm can be widely used to calculate how far light can travel in any given time interval, revealing whether to points in space , for example are in causal contact. As you can see, for accelerated expansion, even for arbitrarily large t_0 the integral is bounded, showing that the light will never reach arbitrarily distant commoving locations. Thus, in a universe with accelerated expansion, there are locations with which we can never communicate.

31) [38]The Schwarzschild Space-time.

a) Line element: $ds^2 = \left(1-\frac{2m}{r}\right)dt^2 - \frac{dr^2}{1-\frac{2m}{r}} - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$

b) Basis one-forms: $\omega^{\hat{t}} = \sqrt{1-\frac{2m}{r}}dt$, $\omega^{\hat{r}} = \frac{1}{\sqrt{1-\frac{2m}{r}}}dr$, $\omega^{\hat{\theta}} = rd\theta$, $\omega^{\hat{\phi}} = r\sin\theta\, d\phi$

c) Killing vectors and conservation laws.

i) [39]The Killing vector that corresponds to the independence of the metric of t is $\xi = (1,0,0,0)$. The conserved energy per unit rest mass: $e = -\overline{\xi}\cdot\overline{u} = -g_{ab}\xi^a u^b = -g_{tt}\cdot 1\cdot\frac{dt}{d\tau} = -\left(1-\frac{2m}{r}\right)\frac{dt}{d\tau}$

ii) and of ϕ is $\eta = (0,0,0,1)$. The conserved angular momentum per unit rest mass: $l = \overline{\eta}\cdot\overline{u} = g_{ab}\eta^a u^b = g_{\phi\phi}\cdot 1\cdot\frac{d\phi}{d\tau} = -r^2\sin^2\theta\frac{d\phi}{d\tau} = -r^2\frac{d\phi}{d\tau}$ for $\theta = \frac{\pi}{2}$

d) Geodesic Equations.

i) $\ddot{t} + \frac{2m}{r(r-2m)}\dot{r}\dot{t} = 0$

ii) $\ddot{r} + \frac{m}{r^3}(r-2m)\dot{t}^2 - \frac{m}{r(r-2m)}\dot{r}^2 - (r-2m)\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right) = 0$

iii) $\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 = 0$

iv) $\ddot{\phi} + 2\cot\theta\,\dot{\theta}\dot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} = 0$

e) Ricci, Einstein and Stress Tensor: $R_{ab} = G_{ab} = T_{ab} = 0$

f) Ortonormal tetrad.

i) $l_a = \frac{1}{\sqrt{2}}\left(\sqrt{1-\frac{2m}{r}},\quad \frac{1}{\sqrt{1-\frac{2m}{r}}},\quad 0,\quad 0\right)$, $l^a = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{1-\frac{2m}{r}}},\quad -\sqrt{1-\frac{2m}{r}},\quad 0,\quad 0\right)$

ii) $n_a = \frac{1}{\sqrt{2}}\left(\sqrt{1-\frac{2m}{r}},\quad -\frac{1}{\sqrt{1-\frac{2m}{r}}},\quad 0,\quad 0\right)$, $n^a = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{1-\frac{2m}{r}}},\quad \sqrt{1-\frac{2m}{r}},\quad 0,\quad 0\right)$

iii) $m_a = \frac{1}{\sqrt{2}}(0,\quad 0,\quad r,\quad ir\sin\theta)$, $m^a = \frac{1}{\sqrt{2}}\left(0,\quad 0,\quad -\frac{1}{r},\quad -i\frac{1}{r\sin\theta}\right)$

iv) $\overline{m_a} = \frac{1}{\sqrt{2}}(0,\quad 0,\quad r,\quad -ir\sin\theta)$, $\overline{m}^a = \frac{1}{\sqrt{2}}\left(0,\quad 0,\quad -\frac{1}{r},\quad i\frac{1}{r\sin\theta}\right)$

g) Spin Coefficients.

i) $\pi = \nu = \lambda = \mu = \kappa = 0$

ii) Expansion: $\rho = \frac{1}{\sqrt{2}}\sqrt{\left(1-\frac{2m}{r}\right)}\frac{1}{r}$ (no twist)

iii) $\sigma = 0$ (no shear)

iv) $\tau = 0$ (the null rays defined by l_a are parallel.

v) $\varepsilon = -\frac{1}{2^{\frac{3}{2}}}\frac{1}{\sqrt{1-\frac{2m}{r}}}\frac{m}{r^2}$, $\gamma = -\frac{1}{2^{\frac{3}{2}}}\frac{1}{\sqrt{1-\frac{2m}{r}}}\frac{m}{r^2}$, $\alpha = \frac{1}{2^{\frac{3}{2}}}\frac{1}{r}\cot\theta$, $\beta = -\frac{1}{2^{\frac{3}{2}}}\frac{1}{r}\cot\theta$

h) Weyl scalars.
   i) $\Psi_0 = \Psi_1 = {}^{a}\Psi_3 = \Psi_4 = 0$
   ii) $\Psi_2 = \frac{1}{2r^2} - \frac{2m}{r^3}$
i) Petrov Classification: $\Psi_2 \neq 0$: This is a Petrov type D, which means there are two principal null directions. The Petrov type D is associated with the gravitational field of a star or a black hole. The two principal null directions correspond to ingoing and outgoing congruence of light rays.
j) Particle (planetary) orbits: Choose coordinates so that the particle moves initially in a plane $\theta = \frac{\pi}{2}$.
   i) Conservation equation: [40]These values correspond to circular orbits, and we can use the binomial expansion of the term under the square root to rewrite the term as. $r_{1,2} = \frac{l^2}{2m}\left(1 \pm \sqrt{1 - \frac{12m^2}{l^2}}\right) \cong \frac{l^2}{2m}\left[1 \pm \left(1 - \frac{6m^2}{l^2}\right)\right] \Rightarrow$ Stable orbit: $r_1 \cong \frac{l^2}{m}$ unstable orbit: $r_2 \cong 3m$
   ii) Geodesic equations.
      (1) [41]$\ddot{t} + 2\frac{d\nu}{dr}\dot{r}\dot{t} = 0$
      (2) $\ddot{r} + \frac{d\lambda}{dr}\dot{r}^2 + e^{2(\nu(r)-\lambda(r))}\frac{d\nu}{dr}\dot{t}^2 - re^{-2\lambda(r)}\dot{\phi}^2 = 0$
      (3) $\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} = 0$
      (4) $\Rightarrow \left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{k^2-1}{h^2} + \frac{2m}{h^2}u + 2mu^3$ Which can be interpreted in terms of elliptic functions, $u = \frac{1}{r}$, and *h* and *k* are constants of integration.
k) [42]The Advance of Perihelion: $u = \frac{m}{h^2}(1 + e\cos(\phi - \omega))$ corresponds to the perihelion ω advances a fraction of a revolution equal to $\frac{\delta\omega}{\phi} = \frac{12\pi^2 a^2}{c^2T^2(1-e^2)}$
l) [43]The Deflection of Light Rays: $\theta = \frac{\pi}{2}, ds^2 = 0 \Rightarrow \left(1 - \frac{2m}{r}\right)^2 \dot{t}^2 - \left(1 - \frac{2m}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 = 0$. [44]Deflection angle, in the weak field perturbation: $\alpha = \frac{4GM}{c^2 b}$, where *b* is closest approach.
m) [45]Time Delay: $\theta = \frac{\pi}{2}, ds^2 = 0, r_0 = r\sin\phi \Rightarrow dt^2 = \frac{dr^2\left(1-\frac{2mr_0^2}{r^3}\right)}{\left(1-\frac{r_0^2}{r^2}\right)\left(1-\frac{2m}{r}\right)^2}$. To consider the travel time of light between Earth and a planet in the solar system, we integrate between r_0 (distance of closets approach) to r_v (planet orbit radius) and r_0 to r_e (Earth orbit radius): $t_{delay} = \frac{mG}{c^3}\left[2\ln\left(\frac{\left(r_v+\sqrt{r_v{}^2-r_0{}^2}\right)\left(r_e+\sqrt{r_e{}^2-r_0{}^2}\right)}{r_0^2}\right) - \frac{\sqrt{r_v{}^2-r_0{}^2}}{r_v} - \frac{\sqrt{r_e{}^2-r_0{}^2}}{r_e}\right]$. The ordinary flat space term is given by $\sqrt{r_v^2 - r_0^2} + \sqrt{r_e^2 - r_0^2}$.
n) [46]Gravitational Red Shift: $d\tau = \sqrt{1 - \frac{2m}{r}}dt$. Light emitted upward in a gravitational field, from an observer located at some inner radius r_1 to an observer positioned at some outer radius r_2. $\alpha = \frac{\sqrt{1-\frac{2m}{r_2}}}{\sqrt{1-\frac{2m}{r_1}}}$
o) [47]The Path of a Radially Infalling Particle.

[a] $\Psi_3 = \frac{1}{4}\cot\theta\sqrt{1 - \frac{2m}{r}\frac{1}{r^2}} = 0$ if $\theta = \frac{\pi}{2}$

i) Line element: From infinity with vanishing initial velocity: $d\theta = d\phi = 0 \Rightarrow 1 - \frac{2m}{r} = \left(1 - \frac{2m}{r}\right)^2 \left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dr}{d\tau}\right)^2$

ii) Killing equation: From Killings equation we know that $\left(1 - \frac{2m}{r}\right)\frac{dt}{d\tau}$ is a constant $\Rightarrow \left(1 - \frac{2m}{r}\right)\frac{dt}{d\tau} = 1$ and $\left(\frac{dr}{d\tau}\right)^2 = \frac{2m}{r} \Rightarrow t - t_0 = \frac{2}{3\sqrt{2m}}\left(r_0^{\frac{3}{2}} - r^{\frac{3}{2}} + 6m\sqrt{r_0} - 6m\sqrt{r}\right) + 2m \ln \frac{\sqrt{r_0} - \sqrt{2m}}{\sqrt{r_0} + \sqrt{2m}} \frac{\sqrt{r} + \sqrt{2m}}{\sqrt{r} - \sqrt{2m}}$

32) [48]Schwarzschild Space-time with $\theta = \frac{\pi}{2}$.

a) Line element: $ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\phi^2$

b) Geodesic equations:

i) $\ddot{t} + \frac{2m}{r(r-2m)}\dot{r}\dot{t} = 0$

ii) $\ddot{r} + \frac{m}{r^3}(r - 2m)\dot{t}^2 - \frac{m}{r(r-2m)}\dot{r}^2 - (r - 2m)\dot{\phi}^2 = 0$

iii) $\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} = 0$

33) [49]The Schwarzschild Metric in Kruskal Coordinates.

a) Line element: $ds^2 = \frac{32m^3}{r} e^{-\frac{r}{2m}}(dv^2 - du^2) - r^2(d\theta + \sin^2\theta\, d\phi)$

i) <u>$r > 2m$:</u>

(1) $u = e^{\frac{r}{4m}} \sqrt{\frac{r}{2m} - 1} \cosh \frac{t}{4m}$

(2) $v = e^{\frac{r}{4m}} \sqrt{\frac{r}{2m} - 1} \sinh \frac{t}{4m}$

ii) <u>$r < 2m$:</u>

(1) $u = e^{\frac{r}{4m}} \sqrt{1 - \frac{r}{2m}} \sinh \frac{t}{4m}$

(2) $v = e^{\frac{r}{4m}} \sqrt{1 - \frac{r}{2m}} \cosh \frac{t}{4m}$

b) Ricci scalar: $R = 0$

34) [50]The General Schwarzschild Metric: a static, spherically symmetric space-time.

a) Line element: $ds^2 = e^{2\nu(r)}dt^2 - e^{2\lambda(r)}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2$

b) Basis one-forms: $\omega^{\hat{t}} = e^{\nu(r)}dt$, $\omega^{\hat{r}} = e^{\lambda(r)}dr$, $\omega^{\hat{\theta}} = r d\theta$, $\omega^{\hat{\phi}} = r \sin\theta\, d\phi$

c) Geodesic equations.

i) $\ddot{t} + 2\frac{d\nu}{dr}\dot{r}\dot{t} = 0$

ii) $\ddot{r} + \frac{d\lambda}{dr}\dot{r}^2 + e^{2(\nu(r) - \lambda(r))}\frac{d\nu}{dr}\dot{t}^2 - re^{-2\lambda(r)}\dot{\theta}^2 - r\sin^2\theta\, e^{-2\lambda(r)}\dot{\phi}^2 = 0$

iii) $\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \cos\theta \sin\theta\, \dot{\phi}^2 = 0$

iv) $\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot\theta\, \dot{\theta}\dot{\phi} = 0$

d) Ricci tensor.

i) $R_{\hat{t}\hat{t}} = \left(\nu'' + \nu'(\nu' - \lambda') + 2\frac{\nu'}{r}\right) e^{-2\lambda(r)}$

ii) $R_{\hat{r}\hat{r}} = -\left(\nu'' + \nu'(\nu' - \lambda') - 2\frac{\lambda'}{r}\right) e^{-2\lambda(r)}$

iii) $R_{\hat{\theta}\hat{\theta}} = R_{\hat{\phi}\hat{\phi}} = -\frac{\nu'}{r}e^{-2\lambda(r)} + \frac{\lambda'}{r}e^{-2\lambda(r)} + \frac{(1 - e^{-2\lambda(r)})}{r^2}$

35) [51]General Time Dependent Schwarzschild Metric.

a) Line element: $ds^2 = e^{2\nu(t,r)}dt^2 - e^{2\lambda(t,r)}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2$

b) Basis one-forms: $\omega^{\hat{t}} = e^{\nu(t,r)}dt$, $\omega^{\hat{r}} = e^{\lambda(t,r)}dr$, $\omega^{\hat{\theta}} = r d\theta$, $\omega^{\hat{\phi}} = r\sin\theta\, d\phi$

c) Ricci tensor.

i) $R_{tt} = -\left(\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\nu})\right) + \left(\nu'' + \nu'(\nu' - \lambda') + 2\frac{\nu'}{r}\right) e^{2\nu(t,r)-2\lambda(t,r)}$

ii) $R_{rt} = 2\frac{\dot{\lambda}}{r}$

iii) $R_{rr} = \left(\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\nu})\right) e^{-2\nu(t,r)+2\lambda(t,r)} - \left(\nu'' + \nu'(\nu' - \lambda') - 2\frac{\lambda'}{r}\right)$

iv) $R_{\theta\theta} = \left((-\nu' + \lambda')r - 1\right)e^{-2\lambda(t,r)} + 1$

v) $R_{\phi\phi} = \left(\left((-\nu' + \lambda')r - 1\right)e^{-2\lambda(t,r)} + 1\right)\sin^2\theta$

36) [52]The General Schwarzschild Metric: A Static, Spherically Symmetric Space-time With Cosmological Constant, the Spatial Line Element.

a) Line element: $d\sigma^2 = \frac{dr^2}{1 - \frac{2m}{r} + \frac{1}{3}\Omega r^2} + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$

b) Ricci scalar: $R = 4\Omega$

37) [53]The Reissner-Nordström Space-time: A static solution to the Einstein-Maxwell field equations, which corresponds to the gravitational field of a charged, non-rotating spherically symmetric body of mass M.

a) The line element: $ds^2 = \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)dt^2 - \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2\sin^2\theta\, d\phi^2$

b) Basis one-forms: $\omega^{\hat{t}} = \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{\frac{1}{2}} dt$, $\omega^{\hat{r}} = \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{-\frac{1}{2}} dr$, $\omega^{\hat{\theta}} = r d\theta$, $\omega^{\hat{\phi}} = r\sin\theta\, d\phi$

c) Geodesic equations.

i) $\ddot{t} + 2\left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{-1}\left(\frac{mr - e^2}{r^3}\right)\dot{r}\dot{t} = 0$

ii) $\ddot{r} - \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{-1}\left(\frac{mr - e^2}{r^3}\right)\dot{r}^2 + \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)\left(\frac{mr - e^2}{r^3}\right)\dot{t}^2 - r\left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)\dot{\theta}^2 -$
$r\left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)\sin^2\theta\, \dot{\phi}^2 = 0$

iii) $\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \cos\theta\sin\theta\, \dot{\phi}^2 = 0$

iv) $\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot\theta\, \dot{\theta}\dot{\phi} = 0$

d) Spin Coefficients: $\rho, \varepsilon, \gamma, \alpha, \beta \neq 0$

e) Petrov Classification: Ψ_2 (and Ψ_3) $\neq 0$

38) [54]The Kerr Black Hole (a Spinning Black Hole):

a) The line element: $ds^2 = \left(1 - \frac{2mr}{\Sigma}\right)dt^2 + \frac{4amr\sin^2\theta}{\Sigma}dt d\phi - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2 - \left(r^2 + a^2 + \frac{2a^2 mr\sin^2\theta}{\Sigma}\right)\sin^2\theta\, d\phi^2$

i) $\Delta = r^2 - 2mr + a^2$

ii) $\Sigma = r^2 + a^2\cos^2\theta$

b) Killing vectors and conservation laws:

i) The Killing vector that corresponds to the independence of the metric of t is $X = (1,0,0,0) \Rightarrow$
$X \cdot u = g_{ab}X^a u^b = g_{tt}X^t u^t + g_{t\phi}X^t u^\phi = \left(1 - \frac{2mr}{\Sigma}\right)\frac{dt}{d\tau} + \frac{2amr\sin^2\theta}{\Sigma}\frac{d\phi}{d\tau} = const$

ii) The Killing vector that corresponds to the independence of the metric of ϕ is $Y = (0,0,0,1) \Rightarrow$ $Y \cdot u = g_{ab}Y^a u^b = g_{\phi\phi}Y^\phi u^\phi + g_{\phi t}Y^\phi u^t = \left(r^2 + a^2 + \frac{2a^2 mr\sin^2\theta}{\Sigma}\right)\sin^2\theta \frac{d\phi}{d\tau} + \frac{2amr\sin^2\theta}{\Sigma}\frac{dt}{d\tau} = const$

c) Ricci scalar: $R = 0$

d) The Static limit: $r = r_s = 2m, R^2 = 6m^2$

i) $\frac{d\phi}{dt} = \frac{a}{3m^2}$ Light that is emitted in the same direction in which the black hole is spinning

ii) $\frac{d\phi}{dt} = 0$ The light is emitted in the direction opposite to that of a black hole's rotation; that is, the light is completely stationary.

39) [55]The Spatial Part of a Homogenous, Isotropic Metric.

a) Line element: $d\sigma^2 = \frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi$

i) $k = 0$: $d\sigma^2 = d\chi^2 + \chi^2 d\theta^2 + \chi^2\sin^2\theta\, d\phi^2$

ii) $k = 1$: $d\sigma^2 = d\chi^2 + \sin^2(\chi)\, d\theta^2 + \sin^2(\chi)\sin^2\theta\, d\phi^2$

iii) $k = -1$: $d\sigma^2 = d\chi^2 + \sinh^2(\chi)\, d\theta^2 + \sinh^2(\chi)\sin^2\theta\, d\phi^2$

40) [56]Linearized Theory.

a) Line element: $ds^2 = g_{ab}dx^a dx^b$, $g_{ab} = \eta_{ab} + \epsilon h_{ab}$, $\epsilon \ll 1$

b) $\Gamma^a{}_{bc} = \frac{\epsilon}{2}\eta^{ad}\left(\frac{\partial h_{bd}}{\partial x^c} + \frac{\partial h_{cd}}{\partial x^b} - \frac{\partial h_{bc}}{\partial x^d}\right)$

c) $R^a{}_{bcd} = \frac{\epsilon}{2}\eta^{ae}\left(\frac{\partial h_{de}}{\partial x^b\partial x^c} - \frac{\partial h_{bd}}{\partial x^c\partial x^e} - \frac{\partial h_{ce}}{\partial x^b\partial x^d} + \frac{\partial h_{bc}}{\partial x^d\partial x^e}\right)$

d) $R_{ab} = \frac{\epsilon}{2}\left(\frac{\partial h^c{}_b}{\partial x^a\partial x^c} - Wh_{ab} - \frac{\partial h}{\partial x^a\partial x^b} + \frac{\partial h^c{}_a}{\partial x^b\partial x^c}\right)$

e) $R = \epsilon\left(\frac{\partial h^{ab}}{\partial x^a\partial x^b} - Wh\right)$

f) $G_{ab} = \frac{\epsilon}{2}\left(\frac{\partial h^c{}_b}{\partial x^a\partial x^c} - Wh_{ab} - \frac{\partial h}{\partial x^a\partial x^b} + \frac{\partial h^c{}_a}{\partial x^b\partial x^c} - \eta_{ab}\frac{\partial h^{cd}}{\partial x^c\partial x^c} + \eta_{ab}Wh\right)$

g) [57]The Newtonian Approximation: Weak gravitational field and bodies of low masses and velocities.

$v = \frac{dx^1}{dx^0} \ll c \Rightarrow \left(\frac{ds}{dx^0}\right)^2 \to -1 + \epsilon h_{00}$

i) Christoffel symbol: $\Gamma^i{}_{00} = -\frac{1}{2}\epsilon\frac{\partial h_{00}}{\partial x^i}$

ii) Geodesic equations: $\frac{d^2x^i}{(dx^0)^2} = \frac{1}{2}\epsilon\frac{\partial h_{00}}{\partial x^i}$

iii) Ricci tensor: $R_{00} = -\frac{1}{2}\epsilon\nabla^2 h_{00}$

iv) Stress tensor and Einstein equation: Pure matter no pressure: $T_{00} = \mu \Rightarrow \epsilon\nabla^2 h_{00} = -\frac{1}{2}G_E\mu$

41) [58]Plane Gravitational Waves.

a) Line element: $ds^2 = g_{ab}dx^a dx^b = (\eta_{ab} + h_{ab})dx^a dx^b$

b) Plane waves: $h_{ab} = h_{ab}(t - z) \Rightarrow$

i) $\frac{\partial h_{ab}}{\partial x} = \frac{\partial h_{ab}}{\partial y} = 0$

ii) $\frac{\partial h_{ab}}{\partial z} = -\frac{\partial h_{ab}}{\partial t}$

iii) $\frac{\partial^2 h_{ab}}{\partial z^2} = \frac{\partial^2 h_{ab}}{\partial t^2}$

c) Riemann tensor: $R^a{}_{bcd} = \frac{1}{2}\varepsilon\eta^{af}\left(\frac{\partial^2 h_{df}}{\partial x^c\partial x^b} - \frac{\partial^2 h_{bd}}{\partial x^c\partial x^f} + \frac{\partial^2 h_{bc}}{\partial x^d\partial x^f} - \frac{\partial^2 h_{cf}}{\partial x^d\partial x^b}\right)$

42) The Einstein Gauge.

a) Line element: $ds^2 = dt^2 - (1 - \varepsilon h_{xx})dx^2 - (1 + \varepsilon h_{xx})dy^2 - dz^2 + 2\varepsilon h_{xy}dxdy$

i) [59]+-polarization, $h_{xy}=0$: $ds^2=dt^2-(1-\varepsilon h_{xx})dx^2-(1+\varepsilon h_{xx})dy^2-dz^2$
ii) [60]x-polarization,$h_{xx}=0$: $ds^2=dt^2-dx^2-dy^2-dz^2+2\varepsilon h_{xy}dxdy$

b) Coordinate Transformation:
i) $dx=\frac{1}{\sqrt{2}}(dx'+dy')$
ii) $dy=\frac{-1}{\sqrt{2}}(dx'-dy')$
iii) Line element: $ds^2=dt^2-\left(1+\varepsilon h_{xy}\right)dx'^2-\left(1-\varepsilon h_{xy}\right)dy'^2-dz^2$

43) [61]The Rosen Line Element.
a) Line Element: $ds^2=dUdV-a^2(U)dx^2-b^2(U)dy^2, U=t-z, a^2(U)=1-h_{xx}(U), b(U)=1+h_{xx}(U)$
b) Basis one-forms: $\omega^{\hat{0}}=\frac{1}{2}(dU+dV), \omega^{\hat{1}}=\frac{1}{2}(dU-dV), \omega^{\hat{2}}=a(U)dx, \omega^{\hat{3}}=b(U)dy$
c) Einstein Tensor: $G_{00}=-\frac{1}{4}\left(\frac{1}{a}\frac{d^2a}{dU^2}+\frac{1}{b}\frac{d^2b}{dU^2}\right)$

44) [62]Kahn- Penrose Metric: Colliding gravitational waves.
a) Line element: $ds^2=2dudv-(1-u)^2dx^2-(1+u)^2dy^2$
b) Ricci tensor: $R=0$

45) [63]Brinkmann Metric: Plane gravitational waves.
a) Line element: $ds^2=H(u,x,y)du^2+2dudv-dx^2-dy^2(u,v,x,y)$
b) Principal null direction: $u, l^a=(0,1,0,0)$
c) Basis one-forms: $\omega^{\hat{u}}=\frac{1}{2}(H+1)du+dv, \omega^{\hat{v}}=\frac{1}{2}(1-H)du-dv, \omega^{\hat{x}}=dx, \omega^{\hat{y}}=dy$
d) Ricci and Einstein tensor: [b]$R_{uu}=G_{uu}=\frac{1}{2}\frac{\partial^2H}{\partial x^2}+\frac{1}{2}\frac{\partial^2H}{\partial y^2}$
e) Ortonormal tetrad.
i) $l_a=\frac{1}{\sqrt{2}}(1,\quad 0,\quad 0,\quad 0), l^a=\frac{1}{\sqrt{2}}(0,\quad 1,\quad 0,\quad 0)$
ii) $n_a=\frac{1}{\sqrt{2}}(H,\quad 2,\quad 0,\quad 0), n^a=\frac{1}{\sqrt{2}}(2,\quad -H,\quad 0,\quad 0)$
iii) $m_a=\frac{1}{\sqrt{2}}(0,\quad 0,\quad 1,\quad i), m^a=\frac{1}{\sqrt{2}}(0,\quad 0,\quad -1,\quad -i)$
iv) $\overline{m_a}=\frac{1}{\sqrt{2}}(0,\quad 0,\quad 1,\quad -i), \overline{m}^a=\frac{1}{\sqrt{2}}(0,\quad 0,\quad -1,\quad i)$
f) Spin coefficients.
i) $\pi=\lambda=\mu=\kappa=\varepsilon=\gamma=\alpha=\beta=0$
ii) $\rho=0$ (no expansion, no twist)
iii) $\sigma=0$ (no shear)
iv) $\tau=0$ (the null rays defined by l_a are parallel.
v) $\nu=\frac{1}{\sqrt{2}}\left(-\frac{\partial H}{\partial x}+i\frac{\partial H}{\partial y}\right)$
g) Weyl scalars.
i) $\Psi_0=\Psi_1=\Psi_2=\Psi_3$
ii) $\Psi_4=\frac{1}{2}\left[\frac{\partial^2H}{\partial x^2}-\frac{\partial^2H}{\partial y^2}-2i\frac{\partial^2H}{\partial x\partial y}\right]$
h) Petrov classification: $\Psi_4\neq 0$. This is a Petrov type N, which means there is a single principal null direction of multiplicity 4. This corresponds to transverse gravity waves.

[b] According to the Weyl scalar calculation the sign is wrong

i) [64]General Plane wave: $H(u,x,y) = a(u)(x^2 - y^2) + 2b(u)xy + c(u)(x^2 + y^2)$, where a and b describe the polarization states, and c represents waves of other types of radiation.
   i) Einstein tensor: $G_{uu} = 2c(u)$
j) [65]Plane Wave in vacuum: $H(u,x,y) = a(u)(x^2 - y^2) + 2b(u)xy$
   i) Einstein tensor: $G_{uu} = 0$
k) [66]Constant Linear Polarization in Vacuum: $H(u,x,y) = h(u)[\cos\alpha\,(x^2 - y^2) + 2\sin\alpha\, xy$, where α is the angle between the polarization vector and the x-axis.
   i) Einstein tensor: $G_{uu} = 0$

46) [67]The Aichelburg-Sexl Solution: A black hole passing near by.
   a) Line element: $ds^2 = 4\mu\log(x^2 + y^2)\,du^2 + dudr - dx^2 - dy^2$, $H(u,x,y) = 4\mu\log(x^2 + y^2)$

47) [68]Colliding Gravity Waves.
   a) Line element: $ds^2 = 2dudv - [1 - u\Theta(u)]^2dx^2 - [1 + u\Theta(u)]^2dy^2$

48) [69]Impulsive Gravitational wave region III.
   a) Line element: $ds^2 = 2dudv - [1 - v\Theta(v)]^2dx^2 - [1 + v\Theta(v)]^2dy^2$
   b) principal null direction $v, n^a = (0,1,0,0)$
   c) Basis one-forms: $\omega^{\hat{u}} = \frac{1}{\sqrt{2}}(du + dv)$, $\omega^{\hat{v}} = \frac{1}{\sqrt{2}}(du - dv)$, $\omega^{\hat{x}} = \left(1 - v\Theta(v)\right)dx$, $\omega^{\hat{y}} = \left(1 + v\Theta(v)\right)dy$
   d) Geodesic equations.
      i) $\ddot{v} = 0$
      ii) $\ddot{u} - \Theta(v)[1 - v\Theta(v)]\dot{x}^2 + \Theta(v)[1 + v\Theta(v)]\dot{y}^2 = 0$
      iii) $\ddot{x} - \frac{2\Theta(v)}{[1 - v\Theta(v)]}\dot{x}\dot{v} = 0$
      iv) $\ddot{y} + \frac{2\Theta(v)}{[1 + v\Theta(v)]}\dot{y}\dot{v} = 0$
   e) orthonormal tetrad (u,v,x,y)
      i) $l_a = (1,\quad 0,\quad 0,\quad 0)$, $l^a = (0,\quad 1,\quad 0,\quad 0)$
      ii) $n_a = (0,\quad 1,\quad 0,\quad 0)$, $n^a = (1,\quad 0,\quad 0,\quad 0)$
      iii) $m_a = \frac{1}{\sqrt{2}}\left(0,\quad 0,\quad \left(1 - v\Theta(v)\right),\quad i\left(1 + v\Theta(v)\right)\right)$, $m^a = \frac{1}{\sqrt{2}}\left(0,\quad 0,\quad -\frac{1}{(1 - v\Theta(v))},\quad -i\frac{1}{(1 + v\Theta(v))}\right)$
      iv) $\overline{m_a} = \frac{1}{\sqrt{2}}\left(0,\quad 0,\quad \left(1 - v\Theta(v)\right),\quad -i\left(1 + v\Theta(v)\right)\right)$, $\overline{m}^a = \frac{1}{\sqrt{2}}\left(0,\quad 0,\quad -\frac{1}{(1 - v\Theta(v))},\quad i\frac{1}{(1 + v\Theta(v))}\right)$
   f) Spin coefficients.
      i) $\pi = \nu = \lambda = \mu = \kappa = \varepsilon = \gamma = \alpha = \beta = 0$
      ii) $\rho = \frac{v\Theta(v)}{(1 + v\Theta(v))(1 - v\Theta(v))} \neq 0$ (expansion)
      iii) $\sigma = \frac{\Theta(v)}{(1 + v\Theta(v))(1 - v\Theta(v))} \neq 0$ (shear)
      iv) $\tau = 0$, the null rays defined by n_a are parallel
   g) Weyl scalars.
      i) $\Psi_1 = \Psi_2 = \Psi_3 = \Psi_4$
      ii) $\Psi_0 = \delta(v) - 4\frac{v\Theta(v)}{\left(1 - v^2\Theta(v)\right)^2}$

h) Petrov classification: [70]$\Psi_0 \neq 0$: This is a Petrov type N, which means there is a single principal null direction of multiplicity 4. This corresponds to transverse gravity waves in region III.

49) [71]Two Interacting Waves: A null congruence begins in vacuum. $v < 0$: flat region of spacetime. Defining a plane wave by $v = const$, we choose the null vector l^a to point along v. The null hypersurface is given by $v = 0$. In the region past $v = 0$, an opposing wave is encountered.

a) Line element: $ds^2 = 2dudv - \cos^2 av\, dx^2 - \cosh^2 av\, dy^2\ (u, v, x, y)$

b) Basis one-forms: $\omega^{\hat{u}} = \frac{1}{\sqrt{2}}(du + dv)$, $\omega^{\hat{v}} = \frac{1}{\sqrt{2}}(du - dv)$, $\omega^{\hat{x}} = \cos av\, dx$, $\omega^{\hat{y}} = \cosh av\, dy$

c) Geodesic equations:

i) $0 = \ddot{u} - a\cos av \sin av\, \dot{x}^2 + a\cosh av \sinh av\, \dot{y}^2$

ii) $0 = \ddot{v}$

iii) $0 = \ddot{x} - 2a\tan av\, \dot{v}\dot{x}$

iv) $0 = \ddot{y} + 2a\tanh av\, \dot{v}\dot{y}$

d) Orthonormal tetrad (u, v, x, y)

i) $l_a = (1,\ \ 0,\ \ 0,\ \ 0)$, $l^a = (0,\ \ 1,\ \ 0,\ \ 0)$

ii) $n_a = (0,\ \ 1,\ \ 0,\ \ 0)$, $n^a = (1,\ \ 0,\ \ 0,\ \ 0)$

iii) $m_a = \frac{1}{\sqrt{2}}(0,\ \ 0,\ \ \cos av,\ \ i\cosh av)$, $m^a = \frac{1}{\sqrt{2}}\left(0,\ \ 0,\ \ -\frac{1}{\cos av},\ \ -i\frac{1}{\cosh av}\right)$

iv) $\overline{m_a} = \frac{1}{\sqrt{2}}(0,\ \ 0,\ \ \cos av,\ \ -i\cosh av)$, $\overline{m}^a = \frac{1}{\sqrt{2}}\left(0,\ \ 0,\ \ -\frac{1}{\cos av},\ \ i\frac{1}{\cosh av}\right)$

e) Spin coefficients.

i) $\pi = \nu = \lambda = \mu = \kappa = \tau = \varepsilon = \gamma = \alpha = \beta = 0$

ii) $\rho = \frac{a}{2}(\tan av - \tanh av)$,

iii) $\sigma = \frac{a}{2}(\tan av + \tanh av)$

f) Petrov classification: [72]$\Psi_0 \neq 0$: This is a Petrov type N, which means there is a single principal null direction of multiplicity 4. This corresponds to transverse gravity waves in region III

50) [73]The Narai Space-time: A solution to the vacuum field equations with positive cosmological constant.

a) Line element: $ds^2 = -\Lambda v^2 du^2 + 2dudv - \frac{1}{\Omega^2}(dx^2 + dy^2)$, $\Omega = 1 + \frac{\Lambda}{4}(x^2 + y^2)$

b) Basis one-forms: $\omega^{\hat{u}} = \frac{1}{2}(-\Lambda v^2 + 1)du + dv$, $\omega^{\hat{v}} = \frac{1}{2}(\Lambda v^2 + 1)du - dv$, $\omega^{\hat{x}} = \frac{1}{\Omega}dx$, $\omega^{\hat{y}} = \frac{1}{\Omega}dy$, $\eta^{ii} = (1, -1, -1, -1)$

c) Ricci tensor: $R_{uu} = \Lambda^2 v^2$, $R_{uv} = -\Lambda$, $R_{xx} = R_{yy} = \frac{\Lambda}{\Omega^2}$

d) Ricci scalar: $R = -4\Lambda$

e) Spin coefficients: $\gamma = -\frac{1}{\sqrt{2}}\Lambda v$, $\alpha = -\frac{\Lambda}{4\sqrt{2}}(x - iy)$, $\beta = \frac{\Lambda}{4\sqrt{2}}(x + iy)$

f) Weyl scalars: [74]$\Psi_2 = -\frac{1}{3}\Lambda$, $\Lambda_{\mathrm{NP}} = \frac{1}{6}\Lambda$, $\Phi_{11} = 0$

g) Petrov classification: With $\Psi_2 \neq 0$ we can conclude that this is a Petrov type D. This means there are two principal null directions, each doubly repeated. The fact that this spacetime contain Ψ_2 and not Ψ_4 or Ψ_0 indicates that this spacetime describes electromagnetic fields and not gravitational radiation. This spacetime represents a vacuum universe that contains electromagnetic fields with no matter.

51) [75]Collision of a Gravitational Wave with an Electromagnetic Wave.

a) Line element: $ds^2 = 2dudv - \cos^2 av\,(dx^2 + dy^2)$

b) Basis one-forms: $\omega^{\hat{u}} = \frac{1}{\sqrt{2}}(du + dv)$, $\omega^{\hat{v}} = \frac{1}{\sqrt{2}}(du - dv)$, $\omega^{\hat{x}} = \cos av\, dx$, $\omega^{\hat{y}} = \cos av\, dy$

http://physicssusan.mono.net logik.susan@gmail.com

c) Spin coefficient: $\rho = a \tan av$

52) [76]The Gödel Metric: The Gödel metric is an exact solution of the Einstein field equations in which the stress-energy tensor contains two terms, the first representing the matter density of a homogeneous distribution of swirling dust particles, and the second associated with a nonzero cosmological constant.[77]

a) Line element: $ds^2 = \frac{1}{2\omega^2}\left((dt + e^x dz)^2 - dx^2 - dy^2 - \frac{1}{2}e^{2x}dz^2\right)$

b) Ricci scalar: $R = 2\omega^2$

53) [78]Warp-Drive Space-time.

a) Line element: $ds^2 = -dt^2 + [dx - V_s(t)f(r_s)dt]^2 + dy^2 + dz^2$

54) [79]Worm Hole Geometry.

a) Line element: $ds^2 = -dt^2 + dr^2 + (b^2 + r^2)(d\theta^2 + \sin^2\theta\, d\phi^2)$

b) Conservation equation: The travel time through a worm-hole: $\Delta\tau = \frac{2R}{U}$

c) Geodesic equations.

i) $\ddot{t} = 0$

ii) $\ddot{r} = r\dot{\theta}^2 + r\sin^2\theta\,\dot{\phi}^2$

iii) $\ddot{\theta} = \sin\theta\cos\theta\,\dot{\phi}^2 - \frac{2r}{(b^2+r^2)}\dot{r}\dot{\theta}$

iv) $\ddot{\phi} = -\frac{2r}{(b^2+r^2)}\dot{r}\dot{\phi} - 2\cot\theta\,\dot{\theta}\dot{\phi}$

Bibliografi

A.S.Eddington. (1924). *The Mathematical Theory of Relativity.* Cambridge: At the University Press.

Carroll, S. M. (2004). *An Introduction to General Relativity, Spacetime and Geometry.* San Fransisco, CA: Addison Wesley.

Choquet-Bruhat, Y. (2015). *Introduction to General Relativity, Black Holes and Cosmology.* Oxford: Oxford University Press.

Greene, B. (2004). *The Fabric of the Cosmos.* Penguin books.

Hartle, J. B. (2003). *Gravity - An introduction to Einstein's General Relativity.* Addison Wesley.

McMahon, D. (2006). *Relativity Demystified.* McGraw-Hill Companies, Inc.

---

[1] (Hartle, 2003, p. 26)
[2] (Hartle, 2003, p. 29)
[3] (McMahon, p. 83)
[4] (McMahon, p. 92)
[5] (McMahon, 2006, p. 325)
[6] (McMahon, 2006, p. 324)
[7] (McMahon, p. 153)
[8] (Hartle, 2003, p. 171)
[9] (Hartle, 2003, p. 136)
[10] (Hartle, 2003, p. 184)
[11] (McMahon, p. 170)
[12] (McMahon, 2006, p. 91)
[13] (McMahon, p. 186)
[14] (McMahon, p. 91)
[15] (Hartle, 2003, p. 137)
[16] (Hartle, 2003, p. 184)
[17] (Hartle, 2003, p. 164)

[18] (Hartle, 2003, p. 143)
[19] (Hartle, 2003, p. 126)
[20] (Hartle, 2003, p. 40)
[21] (Hartle, 2003, p. 127)
[22] http://en.wikipedia.org/wiki/Rindler_coordinates, (McMahon, p. 84)
[23] (Choquet-Bruhat, 2015, s. 95)
[24] (Choquet-Bruhat, 2015, s. 97)
[25] (Choquet-Bruhat, 2015, s. 96)
[26] (McMahon, p. 121)
[27] (Choquet-Bruhat, 2015, s. 97)
[28] (McMahon, p. 150)
[29] (McMahon, p. 161)
[30] (McMahon, p. 270)
[31] (McMahon, p. 274)
[32] (McMahon, p. 271)
[33] (McMahon, p. 272)
[34] (McMahon, p. 273)
[35] (Greene, 2004, s. 515) note 6
[36] (Greene, s. 520) note 9
[37] (Greene, s. 516) note 10
[38] (McMahon, p. 215)
[39] (McMahon, p. 220)
[40] (McMahon, 2006, p. 222)
[41] (A.S.Eddington, pp. 85-86)
[42] (A.S.Eddington, p. 88
[43] (McMahon, p. 224)
[44] (Carroll, p. 291)
[45] (McMahon, p. 229)
[46] (McMahon, p. 234)
[47] (McMahon, p. 238)
[48] (Hartle, 2003, p. 183)
[49] (McMahon, 2006, p. 242)
[50] (McMahon, p. 211)
[51] (McMahon, p. 231)
[52] (McMahon, p. 277)
[53] (McMahon, p. 325)
[54] (McMahon, p. 244)
[55] (McMahon, p. 262)
[56] (McMahon, 2006, p. 280)
[57] (Choquet-Bruhat, 2015, s. 75)
[58] (McMahon, p. 288)
[59] (McMahon, p. 291)
[60] (McMahon, p. 293)
[61] (McMahon, p. 298)
[62] (McMahon, 2006, p. 92)
[63] (McMahon, p. 195)
[64] (McMahon, p. 301)
[65] (McMahon, p. 301)
[66] (McMahon, p. 303)
[67] (McMahon, p. 303)
[68] (McMahon, p. 304)
[69] (McMahon, p. 305)
[70] http://www-staff.lboro.ac.uk/~majbg/jbg/book/chap3.pdf
[71] (McMahon, p. 313)
[72] http://www-staff.lboro.ac.uk/~majbg/jbg/book/chap3.pdf

[73] (McMahon, p. 318)
[74] (McMahon, p. 321)
[75] (McMahon, p. 322)
[76] (McMahon, 2006, p. 326)
[77] http://en.wikipedia.org/wiki/G%C3%B6del_metric
[78] (Hartle, 2003, p. 144)
[79] (Hartle, 2003, p. 148)