Content

9 The Energy-Momentum Tensor ... 1
9.1 Conservation of energy ... 1
9.2 The Einstein equation with source ... 2
9.3 Perfect Fluids ... 2
9.4 More examples on stress-energy tensors ... 3
9.4.1 Pure Matter ... 3
9.4.2 More complicated fluids ... 4
9.4.3 The electromagnetic field ... 4
9.5 The Gödel Spacetime ... 6
9.5.1 The Basis one forms ... 7
9.5.2 Cartan's First Structure equation ... 7
9.5.3 The curvature two forms and the Riemann tensor ... 8
9.5.4 The Ricci Tensor and Ricci Scalar ... 8
9.5.5 Other formulations of the Gödel spacetime ... 9
9.6 The Einstein Cylinder ... 10
9.6.1 The line element ... 10
9.6.2 The Ricci tensor ... 10
9.6.3 The Einstein Equations ... 11
9.6.4 The Stress Energy Tensor ... 12
9.6.5 The Einstein tensor with a cosmological constant ... 12
9.7 The Newtonian Approximation – The right hand side! ... 13
9.7.1 The Ricci tensor ... 13
9.7.2 The Stress-energy Tensor ... 14
9.7.3 The Equation of Motion ... 14
Bibliografi ... 15

Space-time		Line-element	Chapter
Einstein cylinder	ds^2	$= -dt^2 + (a_0)^2\big(d\theta^2 + \sin^2\theta\,(d\phi^2 + \sin^2\phi\, d\psi^2)\big)$	9 (4), 13
Gödel metric	ds^2	$= \frac{1}{2\omega^2}\left((dt + e^x dz)^2 - dx^2 - dy^2 - \frac{1}{2}e^{2x}dz^2\right)$	2,9
Linearized metric	ds^2	$= (\eta_{ab} + \epsilon h_{ab})dx^a dx^b$	9, 12, 14

9 The Energy-Momentum Tensor

9.1 [a]Conservation of energy

In GR the conservation of energy is expressed in the conservation equation

$$\nabla^a T_{ab} = 0$$

Where T_{ab} is the stress energy tensor – the right hand side of the Einstein Equation. Notice that only the index a is summed over here. This indicates that the conserved quantity is not a scalar but a vector, namely the well known energy- momentum 4-vector, p_a.
The T_{ab} tensor includes all the different densities and fluxes of energy and momentum, and we will in the following chapters look at some examples. You might also want to look at the chapter on cosmology where the Robertson-Walker spacetime is analysed for some different T_{ab} (The Friedmann Equations).

9.2 b The Einstein equation with source

You can show that in a spacetime of dimension $n+1>2$, the Einstein equation with sources is equivalent to $R_{ab} \equiv \kappa\left(T_{ab} - \frac{T_c^c}{n-1} g_{ab}\right)$
The Einstein equations with sources and a cosmological constant are

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab} = \kappa T_{ab}$$

Contracting with g^{ab}

$$\Rightarrow \quad g^{ab} R_{ab} - \frac{1}{2} g^{ab} g_{ab} R + \Lambda g^{ab} g_{ab} = \kappa g^{ab} T_{ab}$$

$$\Rightarrow \quad R - \frac{1}{2}(n+1)R + (n+1)\Lambda = {}^{1}\kappa T$$

$$\Rightarrow \quad -\frac{1}{2}(n-1)R + (n+1)\Lambda = \kappa T$$

$$\Rightarrow \quad \frac{1}{2} R = -\frac{1}{(n-1)}\left(\kappa T - (n+1)\Lambda\right)$$

$$\Rightarrow \quad \frac{1}{2} g_{ab} R = -\frac{1}{(n-1)}\left(\kappa T g_{ab} - (n+1)\Lambda g_{ab}\right)$$

Substituting this into the Einstein equation we find

$$\Rightarrow \quad R_{ab} = \kappa T_{ab} + \frac{1}{2} g_{ab} R - \Lambda g_{ab}$$

$$= \kappa T_{ab} - \frac{1}{(n-1)}\left(\kappa T g_{ab} - (n+1)\Lambda g_{ab}\right) - \Lambda g_{ab}$$

$$= \kappa\left(T_{ab} - \frac{T}{(n-1)} g_{ab}\right) + \frac{2}{(n-1)}\Lambda g_{ab}$$

If $\Lambda = 0$ we have

$$R_{ab} = \kappa\left(T_{ab} - \frac{T_c^c}{(n-1)} g_{ab}\right) \qquad (9.1.)$$

In four dimensions $(n=3)$ these will have the more familiar looks

$$R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab} = -8\pi G T_{ab}$$

$$R_{ab} = -8\pi G\left(T_{ab} - \frac{1}{2} T g_{ab}\right) + \Lambda g_{ab}$$

9.3 c Perfect Fluids

The most general form of the stress energy tensor is

$$T^{ab} = A u^a u^b + B g^{ab}$$

In the local frame we know that

$$T^{\hat{a}\hat{b}} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

[1] $g^{ab} R_{ab} = R$, $g^{ab} g_{ab} = n+1$, $g^{ab} T_{ab} = T = T_c^c$

where [2]ρ is the mass density and P is the pressure.
The four velocity of a point particle in a local frame is
$$u^{\hat{a}} = (1,0,0,0)$$
Then we choose the metric with negative signature
$$\eta^{\hat{a}\hat{b}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$
This we can use to find the constants A and B
$$T^{\hat{0}\hat{0}} = Au^{\hat{0}}u^{\hat{0}} + B\eta^{\hat{0}\hat{0}} = A + B = \rho$$
$$T^{\hat{\imath}\hat{\jmath}} = Au^{\hat{\imath}}u^{\hat{\jmath}} + B\eta^{\hat{\imath}\hat{\jmath}} = -B = \begin{cases} P \; if \; i = j \\ 0 \; if \; i \neq j \end{cases}$$
$$\Rightarrow \quad B = -P \text{ and } A = \rho - B = \rho + P$$
Which leaves us with the most general form of the stress energy tensor for a perfect fluid for a metric with negative signature
$$T^{ab} = (\rho + P)u^a u^b - Pg^{ab}$$
If we instead choose the metric with positive signature
$$\eta^{\hat{a}\hat{b}} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$T^{\hat{0}\hat{0}} = Au^{\hat{0}}u^{\hat{0}} + B\eta^{\hat{0}\hat{0}} = A - B = \rho$$
$$T^{\hat{\imath}\hat{\jmath}} = Au^{\hat{\imath}}u^{\hat{\jmath}} + B\eta^{\hat{\imath}\hat{\jmath}} = B = \begin{cases} P \; if \; i = j \\ 0 \; if \; i \neq j \end{cases}$$
$$\Rightarrow \quad B = P \text{ and } A = \rho + B = \rho + P$$
Which leaves us with the most general form of the stress energy tensor for a perfect fluid for a metric with positive signature
$$T^{ab} = (\rho + P)u^a u^b + Pg^{ab}$$

9.4 More examples on stress-energy tensors

9.4.1 [d]Pure Matter

In the case of pure matter with no pressure the stress-energy tensor is
$$T^{ab} = \rho u^a u^b \qquad (9.2.)$$
Where u is the unit flow velocity and ρ is the rest-mass density.
[e]For a co-moving observer the four velocity reduces to
$$u^a = (1,0,0,0) \qquad (9.3.)$$
And the energy-momentum tensor reduces to
$$T^{ab} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
[f]In the case of a stationary observer the dust particles have a four velocity
$$u^a = (\gamma, \gamma u^x, \gamma u^y, \gamma u^z)$$
And the energy-momentum tensor is
$$T^{ab} = \rho\gamma^2 \begin{pmatrix} 1 & u^x & u^y & u^z \\ u^x & (u^x)^2 & u^x u^y & u^x u^z \\ u^y & u^y u^x & (u^y)^2 & u^y u^z \\ u^z & u^z u^x & u^z u^y & (u^z)^2 \end{pmatrix}$$

[2] This is not to be confused with the tensor ρ_{ab} mentioned above

9.4.2 [g]More complicated fluids

The most general form of a matter-stress-energy-tensor is the non-perfect fluid with viscosity and shear.

$$T^{ab} = \rho(1+\varepsilon)u^a u^b + (P - \zeta\theta)h^{ab} - 2\eta\sigma^{ab} + q^a u^b + q^b u^a$$

Where the various quantities are defined as

- ε - Specific energy density of the fluid in its rest frame
- P - Pressure
- $h^{ab} = u^a u^b + g^{ab}$ – The spatial projection tensor
- η - shear viscosity
- ζ - bulk viscosity
- $\theta = \nabla_a u^a$ – expansion
- $\sigma^{ab} = \frac{1}{2}\left(\nabla_c u^a h^{cb} + \nabla_c u^b h^{ca}\right) - \frac{1}{3}\theta h^{ab}$ – shear tensor
- q^a - energy flux tensor

9.4.3 [h]The electromagnetic field

The stress energy tensor of an electromagnetic field is the Maxwell tensor

$$T_{\alpha\beta} = F_\alpha{}^\lambda F_{\beta\lambda} - \frac{1}{4}\eta_{\alpha\beta}F^{\mu\nu}F_{\mu\nu}$$

9.4.3.1 [i]The Maxwell equations

The electromagnetic field tensor $F^{\mu\nu}$ is defined by

$$F^{\mu\nu} = \begin{Bmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -\epsilon^{123}B^3 & -\epsilon^{132}B^2 \\ E^2 & -\epsilon^{213}B^3 & 0 & -\epsilon^{231}B^1 \\ E^3 & -\epsilon^{312}B^2 & -\epsilon^{321}B^1 & 0 \end{Bmatrix} = \begin{Bmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{Bmatrix}$$

i.e.

$$F^{0i} = -E^i$$
$$F^{ij} = -\epsilon^{ijk}B^k$$

Expressed by the vector potential[3] A^μ

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

Or

$$\bar{\mathbf{B}} = {}^4 - \bar{\nabla} \times \bar{\mathbf{A}}$$

This we can use to find the four Maxwell equations

a) Notice that

$$\begin{aligned} \nabla\cdot\bar{\mathbf{B}} &= -\nabla\cdot(\bar{\nabla}\times\bar{\mathbf{A}}) \\ &= -\nabla\cdot\begin{Bmatrix} \partial_2 A^3 - \partial_3 A^2 \\ \partial_3 A^1 - \partial_1 A^3 \\ \partial_1 A^2 - \partial_2 A^1 \end{Bmatrix} \\ &= \partial_1(\partial_2 A^3 - \partial_3 A^2) + \partial_2(\partial_3 A^1 - \partial_1 A^3) + \partial_3(\partial_1 A^2 - \partial_2 A^1) \\ &= 0 \end{aligned} \qquad (9.4.)$$

b) Also notice

---

[3] Definition: Vector potential: A function $\bar{\mathbf{A}}$ such that $\bar{\mathbf{B}} \equiv \nabla \times \bar{\mathbf{A}}$. The most common use of a vector potential is the representation of a magnetic field. If a vector field has zero divergence, it may be represented by a vector potential http://mathworld.wolfram.com/VectorPotential.html

[4] $\epsilon^{ijk}B^k = -F^{ij} = -(\partial^i A^j - \partial^j A^i) \Rightarrow \bar{\mathbf{B}} = \begin{Bmatrix} B^1 \\ B^2 \\ B^3 \end{Bmatrix} = -\begin{Bmatrix} F^{23} \\ -F^{13} \\ F^{12} \end{Bmatrix} = -\begin{Bmatrix} \partial^2 A^3 - \partial^3 A^2 \\ -(\partial^1 A^3 - \partial^3 A^1) \\ \partial^1 A^2 - \partial^2 A^1 \end{Bmatrix} = -\begin{Bmatrix} \partial^2 A^3 - \partial^3 A^2 \\ \partial^3 A^1 - \partial^1 A^3 \\ \partial^1 A^2 - \partial^2 A^1 \end{Bmatrix} = -\begin{Bmatrix} -\partial_2 A^3 + \partial_3 A^2 \\ -\partial_3 A^1 + \partial_1 A^3 \\ -\partial_1 A^2 + \partial_2 A^1 \end{Bmatrix} = \begin{Bmatrix} \partial_2 A^3 - \partial_3 A^2 \\ \partial_3 A^1 - \partial_1 A^3 \\ \partial_1 A^2 - \partial_2 A^1 \end{Bmatrix} = \bar{\nabla} \times \bar{\mathbf{A}}$

$$
\begin{aligned}
\overline{\mathbf{E}} &=^{5} -\partial_0 \overline{\mathbf{A}} - \overline{\nabla} A_0 \\
\Rightarrow \quad \overline{\nabla} \times \overline{\mathbf{E}} &= \overline{\nabla} \times (-\partial_0 \overline{\mathbf{A}} - \overline{\nabla} A_0) \\
&= -\overline{\nabla} \times \begin{Bmatrix} \partial_0 A_1 + \partial_1 A_0 \\ \partial_0 A_2 + \partial_2 A_0 \\ \partial_0 A_3 + \partial_3 A_0 \end{Bmatrix} \\
&= -\begin{Bmatrix} \partial_2(\partial_0 A_3 + \partial_3 A_0) - \partial_3(\partial_0 A_2 + \partial_2 A_0) \\ \partial_3(\partial_0 A_1 + \partial_1 A_0) - \partial_1(\partial_0 A_3 + \partial_3 A_0) \\ \partial_1(\partial_0 A_2 + \partial_2 A_0) - \partial_2(\partial_0 A_1 + \partial_1 A_0) \end{Bmatrix} \\
&= -\begin{Bmatrix} \partial_2(\partial_0 A_3) - \partial_3(\partial_0 A_2) \\ \partial_3(\partial_0 A_1) - \partial_1(\partial_0 A_3) \\ \partial_1(\partial_0 A_2) - \partial_2(\partial_0 A_1) \end{Bmatrix} \\
&= -\partial_0 \begin{Bmatrix} \partial_2(A_3) - \partial_3(A_2) \\ \partial_3(A_1) - \partial_1(A_3) \\ \partial_1(A_2) - \partial_2(A_1) \end{Bmatrix} \\
&= -\partial_0(\overline{\nabla} \times \overline{\mathbf{A}}) \\
&= -\partial_0 \overline{\mathbf{B}} \qquad (9.5.)
\end{aligned}
$$

c) The Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

The Euler-Lagrange equation

$$
\begin{aligned}
0 &= \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) - \frac{\partial \mathcal{L}}{\partial A_\nu} \\
\frac{\partial \mathcal{L}}{\partial A_\nu} &= 0 \\
\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} &= \frac{\partial \left(-\frac{1}{4} F_{\rho\delta} F^{\rho\delta} \right)}{\partial(\partial_\mu A_\nu)} \\
&= -\frac{1}{4} \left[F^{\rho\delta} \frac{\partial(F_{\rho\delta})}{\partial(\partial_\mu A_\nu)} + F_{\rho\delta} \frac{\partial(F^{\rho\delta})}{\partial(\partial_\mu A_\nu)} \right] \\
&= -\frac{1}{4} \left[F^{\rho\delta} \frac{\partial(F_{\rho\delta})}{\partial(\partial_\mu A_\nu)} + F^{\rho\delta} \frac{\partial(F_{\rho\delta})}{\partial(\partial_\mu A_\nu)} \right] \\
&= -\frac{1}{2} F^{\rho\delta} \frac{\partial(F_{\rho\delta})}{\partial(\partial_\mu A_\nu)} \\
&= -\frac{1}{2} F^{\rho\delta} \left(\frac{\partial(\partial_\rho A_\delta - \partial_\delta A_\rho)}{\partial(\partial_\mu A_\nu)} \right) \\
&= -\frac{1}{2} F^{\mu\nu} \left(\frac{\partial(\partial_\mu A_\nu - \partial_\nu A_\mu)}{\partial(\partial_\mu A_\nu)} \right) - \frac{1}{2} F^{\nu\mu} \left(\frac{\partial(\partial_\nu A_\mu - \partial_\mu A_\nu)}{\partial(\partial_\mu A_\nu)} \right) \\
&=^{6} -\frac{1}{2} F^{\mu\nu} + \frac{1}{2} F^{\nu\mu} \\
&= -\frac{1}{2} F^{\mu\nu} - \frac{1}{2} F^{\mu\nu} \\
&= -F^{\mu\nu} \\
\Rightarrow \quad \partial_\mu(F^{\mu\nu}) &= 0
\end{aligned}
$$

if ν is space-like we get

---

[5] $E^i = -F^{0i} = -(\partial^0 A^i - \partial^i A^0) = -(\partial_0 A^i + \partial_i A_0) \Rightarrow \overline{\mathbf{E}} = -\partial_0 \overline{\mathbf{A}} - \overline{\nabla} A_0$

[6] $F_{\nu\mu} = \partial_\nu A_\mu - \partial_\mu A_\nu = -(\partial_\mu A_\nu - \partial_\nu A_\mu) = -F_{\mu\nu}$

$$\partial_\mu(F^{\mu\nu}) = \partial_\mu(F^{\mu i}) = \partial_0(F^{0i}) + \partial_j(F^{ji}) = -\partial_0 E^i + \partial_j(-\epsilon^{jik}B^k) = 0$$

$$\Rightarrow \quad 0 = \begin{Bmatrix} -\partial_0 E^1 + \partial_j(-\epsilon^{j1k}B^k) \\ -\partial_0 E^2 + \partial_j(-\epsilon^{j2k}B^k) \\ -\partial_0 E^3 + \partial_j(-\epsilon^{j3k}B^k) \end{Bmatrix}$$

$$= \begin{Bmatrix} -\partial_0 E^1 + \partial_3(-\epsilon^{312}B^2) + \partial_2(-\epsilon^{213}B^3) \\ -\partial_0 E^2 + \partial_3(-\epsilon^{321}B^1) + \partial_1(-\epsilon^{123}B^3) \\ -\partial_0 E^3 + \partial_2(-\epsilon^{231}B^1) + \partial_1(-\epsilon^{132}B^2) \end{Bmatrix}$$

$$= -\partial_0 \bar{\mathbf{E}} + \begin{Bmatrix} -\partial_3(B^2) + \partial_2(B^3) \\ \partial_3(B^1) - \partial_1(B^3) \\ -\partial_2(B^1) + \partial_1(B^2) \end{Bmatrix}$$

$$= -\partial_0 \bar{\mathbf{E}} + \bar{\nabla} \times \bar{\mathbf{B}}$$

$$\Rightarrow \quad \bar{\nabla} \times \bar{\mathbf{B}} = -\partial_0 \bar{\mathbf{E}} \qquad (9.6.)$$

d) If μ is space-like we get and ν is space-like we get

$$\partial_\mu(F^{\mu\nu}) = \partial_i(F^{i0}) = -\partial_i E^i = -\text{div}\,\bar{\mathbf{E}} = 0 \qquad (9.7.)$$

Collecting the results

$$\nabla \cdot \bar{\mathbf{E}} = 0 \qquad (9.7.)$$ [7]

$$\nabla \cdot \bar{\mathbf{B}} = 0 \qquad (9.4.)$$

$$\nabla \times \bar{\mathbf{E}} = -\frac{\partial \bar{\mathbf{B}}}{\partial t} \qquad (9.5.)$$

$$\nabla \times \bar{\mathbf{B}} = \frac{\partial \bar{\mathbf{E}}}{\partial t} \qquad (9.6.)$$ [8]

9.5 The Gödel Spacetime

The Gödel spacetime is an exact solution of the Einstein field equations in which the stress-energy tensor contains two terms, the first representing the matter density of a homogeneous distribution of swirling dust particles, and the second associated with a nonzero cosmological constant.

The line element

$$ds^2 = \frac{1}{2\omega^2}\left((dt + e^x dz)^2 - dx^2 - dy^2 - \frac{1}{2}e^{2x}dz^2\right)$$
$$= \frac{1}{2\omega^2}\left(dt^2 + 2e^x dt dz - dx^2 - dy^2 + \frac{1}{2}e^{2x}dz^2\right)$$

The metric tensor

$$g_{ab} = \frac{1}{2\omega^2}\begin{Bmatrix} 1 & 0 & 0 & e^x \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ e^x & 0 & 0 & \frac{1}{2}e^{2x} \end{Bmatrix}$$

and its inverse

$$g^{ab} = 2\omega^2\begin{Bmatrix} -1 & 0 & 0 & 2e^{-x} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 2e^{-x} & 0 & 0 & -2e^{-2x} \end{Bmatrix}$$

The stress energy tensor

[7] Recall, if a charge is present the equation is Gauss law: $\nabla \cdot \bar{\mathbf{E}} = \frac{\rho}{\varepsilon_0}$

[8] Recall, if a charge is present the equation is Amperes law: $\nabla \times \bar{\mathbf{B}} = \mu_0 \bar{J} + \mu_0 \varepsilon_0 \frac{\partial \bar{\mathbf{E}}}{\partial t}$

$$T_{ab} = \frac{\rho}{2\omega^2}\begin{Bmatrix} 1 & 0 & 0 & e^x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e^x & 0 & 0 & e^{2x} \end{Bmatrix}$$

The Einstein equation for a metric with a cosmological constant and non-zero energy-momentum tensor

$$\begin{aligned} T_{ab} &= G_{ab} + g_{ab}\Lambda \\ \Rightarrow \quad g^{ab}T_{ab} &= g^{ab}G_{ab} + g^{ab}g_{ab}\Lambda \\ \Rightarrow^{9} \quad \rho &= g^{ab}\left(R_{ab} - \frac{1}{2}g_{ab}R\right) + 4\Lambda = g^{ab}R_{ab} - \frac{1}{2}g^{ab}g_{ab}R + 4\Lambda \\ &= R - \frac{1}{2}4R + 4\Lambda = -R + 4\Lambda \end{aligned}$$

To find R we work in the non-coordinate basis

9.5.1 The Basis one forms

$$\begin{aligned} \omega^{\hat{t}} &= \frac{1}{\sqrt{2}\omega}(dt + e^x dz) & dt &= \sqrt{2}\omega\omega^{\hat{t}} - 2\omega\omega^{\hat{z}} \\ \omega^{\hat{x}} &= \frac{1}{\sqrt{2}\omega}dx & dx &= \sqrt{2}\omega\omega^{\hat{x}} \\ \omega^{\hat{y}} &= \frac{1}{\sqrt{2}\omega}dy & dy &= \sqrt{2}\omega\omega^{\hat{y}} \\ \omega^{\hat{z}} &= \frac{1}{2\omega}e^x dz & dz &= 2\omega e^{-x}\omega^{\hat{z}} \end{aligned}$$

$$\eta^{\hat{\imath}\hat{\jmath}} = \begin{Bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{Bmatrix}$$

9.5.2 Cartan's First Structure equation

$$\begin{aligned} d\omega^{\hat{a}} &= -\Gamma^{\hat{a}}{}_{\hat{b}} \wedge \omega^{\hat{b}} \\ d\omega^{\hat{t}} &= d\frac{1}{\sqrt{2}\omega}(dt + e^x dz) = \frac{e^x}{\sqrt{2}\omega}dx \wedge dz = \frac{e^x}{\sqrt{2}\omega}\left(\sqrt{2}\omega\omega^{\hat{x}}\right) \wedge \left(2\omega e^{-x}\omega^{\hat{z}}\right) = 2\omega\omega^{\hat{x}} \wedge \omega^{\hat{z}} \\ &= -\omega\omega^{\hat{z}} \wedge \omega^{\hat{x}} + \omega\omega^{\hat{x}} \wedge \omega^{\hat{z}} \\ \Rightarrow \quad \Gamma^{\hat{t}}{}_{\hat{t}} &= \Gamma^{\hat{t}}{}_{\hat{y}} = 0 \\ \Gamma^{\hat{t}}{}_{\hat{x}} &= {}^{10}\Gamma^{\hat{x}}{}_{\hat{t}} = \omega\omega^{\hat{z}} \\ \Gamma^{\hat{t}}{}_{\hat{z}} &= {}^{11}\Gamma^{\hat{z}}{}_{\hat{t}} = -\omega\omega^{\hat{x}} \\ d\omega^{\hat{x}} &= d\omega^{\hat{y}} = 0 \\ d\omega^{\hat{z}} &= d\left(\frac{1}{2\omega}e^x dz\right) = \frac{1}{2\omega}e^x dx \wedge dz = \frac{1}{2\omega}e^x\left(\sqrt{2}\omega\omega^{\hat{x}}\right) \wedge \left(2\omega e^{-x}\omega^{\hat{z}}\right) \\ &= -\sqrt{2}\omega\omega^{\hat{z}} \wedge \omega^{\hat{x}} \qquad (9.8.) \\ \Rightarrow \quad \Gamma^{\hat{z}}{}_{\hat{t}} &= \Gamma^{\hat{z}}{}_{\hat{y}} = 0 \\ \Gamma^{\hat{z}}{}_{\hat{x}} &= {}^{12} - \Gamma^{\hat{x}}{}_{\hat{z}} = \sqrt{2}\omega\omega^{\hat{z}} \end{aligned}$$

Summarizing the curvature one forms in a matrix:

[9] $g^{ab}T_{ab} = \frac{\rho}{2\omega^2}2\omega^2(-1 \cdot 1 + 2e^{-x}e^x + 2e^{-x}e^x - 2e^{-2x}e^{2x}) = \rho(-1 + 2 + 2 - 2) = \rho$

[10] $\Gamma^{\hat{t}}{}_{\hat{x}} = \eta^{\hat{t}\hat{t}}\Gamma_{\hat{t}\hat{x}} = -\eta^{\hat{t}\hat{t}}\Gamma_{\hat{x}\hat{t}} = -\eta^{\hat{t}\hat{t}}\eta_{\hat{x}\hat{x}}\Gamma^{\hat{x}}{}_{\hat{t}} = \Gamma^{\hat{x}}{}_{\hat{t}}$

[11] $\Gamma^{\hat{t}}{}_{\hat{z}} = \eta^{\hat{t}\hat{t}}\Gamma_{\hat{t}\hat{z}} = -\eta^{\hat{t}\hat{t}}\Gamma_{\hat{z}\hat{t}} = -\eta^{\hat{t}\hat{t}}\eta_{\hat{z}\hat{z}}\Gamma^{\hat{z}}{}_{\hat{t}} = \Gamma^{\hat{z}}{}_{\hat{t}}$

[12] $\Gamma^{\hat{z}}{}_{\hat{x}} = \eta^{\hat{z}\hat{z}}\Gamma_{\hat{z}\hat{x}} = -\eta^{\hat{z}\hat{z}}\Gamma_{\hat{x}\hat{z}} = -\eta^{\hat{z}\hat{z}}\eta_{\hat{x}\hat{x}}\Gamma^{\hat{x}}{}_{\hat{z}} = -\Gamma^{\hat{x}}{}_{\hat{z}}$

$$\Gamma^{\hat{a}}{}_{\hat{b}} = \begin{Bmatrix} 0 & \omega\omega^{\hat{z}} & 0 & -\omega\omega^{\hat{x}} \\ \omega\omega^{\hat{z}} & 0 & 0 & \sqrt{2}\omega\omega^{\hat{z}} \\ 0 & 0 & 0 & 0 \\ -\omega\omega^{\hat{x}} & -\sqrt{2}\omega\omega^{\hat{z}} & 0 & 0 \end{Bmatrix}$$

Where $\hat{a}$ refers to column and $\hat{b}$ to row.

9.5.3 The curvature two forms and the Riemann tensor

$$\Omega^{\hat{a}}{}_{\hat{b}} = d\Gamma^{\hat{a}}{}_{\hat{b}} + \Gamma^{\hat{a}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{b}} = \frac{1}{2} R^{\hat{a}}{}_{\hat{b}\hat{c}\hat{d}}\omega^{\hat{c}} \wedge \omega^{\hat{d}}$$
$$d\Gamma^{\hat{t}}{}_{\hat{x}} = d(\omega\omega^{\hat{z}}) = \omega d\omega^{\hat{z}} = {}^{13}\omega(-\sqrt{2}\omega\omega^{\hat{z}} \wedge \omega^{\hat{x}}) = -\sqrt{2}\omega^2\omega^{\hat{z}} \wedge \omega^{\hat{x}} = d\Gamma^{\hat{x}}{}_{\hat{t}}$$
$$d\Gamma^{\hat{x}}{}_{\hat{z}} = d(-\sqrt{2}\omega\omega^{\hat{z}}) = -\sqrt{2}\omega d\omega^{\hat{z}} = {}^{14} - \sqrt{2}\omega(-\sqrt{2}\omega\omega^{\hat{z}} \wedge \omega^{\hat{x}}) = 2\omega^2\omega^{\hat{z}} \wedge \omega^{\hat{x}} = -d\Gamma^{\hat{z}}{}_{\hat{x}}$$
$$\Omega^{\hat{t}}{}_{\hat{x}} = d\Gamma^{\hat{t}}{}_{\hat{x}} + \Gamma^{\hat{t}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{x}} = d\Gamma^{\hat{t}}{}_{\hat{x}} + \Gamma^{\hat{t}}{}_{\hat{z}} \wedge \Gamma^{\hat{z}}{}_{\hat{x}} = -\sqrt{2}\omega^2\omega^{\hat{z}} \wedge \omega^{\hat{x}} - \omega\omega^{\hat{x}} \wedge \sqrt{2}\omega\omega^{\hat{z}} = 0$$
$$\Omega^{\hat{x}}{}_{\hat{z}} = d\Gamma^{\hat{x}}{}_{\hat{z}} + \Gamma^{\hat{x}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{z}} = d\Gamma^{\hat{x}}{}_{\hat{z}} + \Gamma^{\hat{x}}{}_{\hat{t}} \wedge \Gamma^{\hat{t}}{}_{\hat{z}} = 2\omega^2\omega^{\hat{z}} \wedge \omega^{\hat{x}} + \omega\omega^{\hat{z}} \wedge (-\omega\omega^{\hat{x}}) = \omega^2\omega^{\hat{z}} \wedge \omega^{\hat{x}}$$

Summarized in a matrix

$$\Omega^{\hat{a}}{}_{\hat{b}} = \begin{Bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega^2\omega^{\hat{x}} \wedge \omega^{\hat{z}} \\ 0 & 0 & 0 & 0 \\ 0 & -\omega^2\omega^{\hat{x}} \wedge \omega^{\hat{z}} & 0 & 0 \end{Bmatrix}$$

Where $\hat{a}$ refers to column and $\hat{b}$ to row

Now we can see that the only nonzero element of the Riemann tensor in the non-coordinate basis is

$$R^{\hat{x}}{}_{\hat{z}\hat{x}\hat{z}} = -\omega^2$$

9.5.4 The Ricci Tensor and Ricci Scalar

$$R_{\hat{a}\hat{b}} = R^{\hat{c}}{}_{\hat{a}\hat{c}\hat{b}}$$

The non-zero elements

$$R_{\hat{x}\hat{x}} = R^{\hat{c}}{}_{\hat{x}\hat{c}\hat{x}} = R^{\hat{z}}{}_{\hat{x}\hat{z}\hat{x}} = R^{\hat{x}}{}_{\hat{z}\hat{x}\hat{z}} = -\omega^2$$
$$R_{\hat{z}\hat{z}} = R^{\hat{c}}{}_{\hat{z}\hat{c}\hat{z}} = R^{\hat{x}}{}_{\hat{z}\hat{x}\hat{z}} = -\omega^2$$

Summarized in a matrix:

$$R_{\hat{a}\hat{b}} = \begin{Bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\omega^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\omega^2 \end{Bmatrix}$$

Where $\hat{a}$ refers to column and $\hat{b}$ to row

<u>The Ricci scalar:</u>

$$R = \eta^{\hat{a}\hat{b}}R_{\hat{a}\hat{b}} = \eta^{\hat{t}\hat{t}}R_{\hat{t}\hat{t}} + \eta^{\hat{x}\hat{x}}R_{\hat{x}\hat{x}} + \eta^{\hat{y}\hat{y}}R_{\hat{y}\hat{y}} + \eta^{\hat{z}\hat{z}}R_{\hat{z}\hat{z}} = -R_{\hat{x}\hat{x}} - R_{\hat{z}\hat{z}} = 2\omega^2$$

Now we can find Λ and ρ.

$$T_{\hat{y}\hat{y}} = R_{\hat{y}\hat{y}} - \frac{1}{2}\eta_{\hat{y}\hat{y}}R + \eta_{\hat{y}\hat{y}}\Lambda$$
$$\Rightarrow \quad 0 = -\frac{1}{2}(-1)(2\omega^2) - \Lambda$$
$$\Rightarrow \quad \Lambda = \omega^2$$
$$\rho = -R + 4\Lambda = 2\omega^2$$

The term ρ represents the matter density of a homogeneous distribution of swirling dust particles, and the second term $\Lambda = \omega^2$ is associated with a nonzero cosmological constant.
Now we can also see the connection to the energy tensor.

[13] Eq. (9.8.)
[14] Eq. (9.8.)

$$T_{\hat{a}\hat{b}} = R_{\hat{a}\hat{b}} - \frac{1}{2}\eta_{\hat{a}\hat{b}}R + \eta_{\hat{a}\hat{b}}\Lambda = R_{\hat{a}\hat{b}} - \eta_{\hat{a}\hat{b}}\left(\frac{1}{2}R - \Lambda\right) = R_{\hat{a}\hat{b}} = \begin{Bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\omega^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\omega^2 \end{Bmatrix}$$

$$T_{ab} = R_{ab} = \begin{Bmatrix} 1 & 0 & 0 & e^x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e^x & 0 & 0 & e^{2x} \end{Bmatrix}$$

9.5.5 Other formulations of the Gödel spacetime

9.5.5.1 [k]Hawking-Ellis

The line element

$$ds^2 = -dt^2 + dx^2 - \frac{1}{2}\exp\left(2\sqrt{2}\omega x\right)dy^2 + dz^2 - 2\exp\left(\sqrt{2}\omega x\right)dtdy$$

If we compare this with the former line element

$$ds'^2 = \frac{1}{2\omega^2}\left(dt'^2 + 2e^{x'}dt'dz' - dx'^2 - dy'^2 + \frac{1}{2}e^{2x'}dz'^2\right)$$

We find

$$t' = \sqrt{2}\omega t$$
$$x' = \sqrt{2}\omega x$$
$$y' = \sqrt{2}\omega z$$
$$z' = \sqrt{2}\omega y$$

And can conclude that

$$R = -2\omega^2$$
$$\Lambda = -\omega^2$$

due to the change in time signature.

9.5.5.2 [l]McMahon

The line element

$$ds^2 = -dt^2 + dx^2 - \frac{1}{2}\exp(2\omega x)dy^2 + dz^2 - 2\exp(\omega x)dtdy$$

If we compare this with the former line element

$$ds'^2 = \frac{1}{2\omega^2}\left(dt'^2 + 2e^{x'}dt'dz' - dx'^2 - dy'^2 + \frac{1}{2}e^{2x'}dz'^2\right)$$

We find

$$t' = \omega t$$
$$x' = \omega x$$
$$y' = \omega z$$
$$z' = \omega y$$

And can conclude that

$$R = \omega^2$$
$$\Lambda = \frac{\omega^2}{2}$$

[15]Because the two line-elements are conformally related by a factor $\frac{1}{2}$.

[15] $ds'^2 = \frac{1}{2}ds^2$

9.6 mThe Einstein Cylinder

9.6.1 The line element

The Einstein cylinder has the line element

$$ds^2 = -dt^2 + (a_0)^2\left(d\theta^2 + \sin^2\theta\,(d\phi^2 + \sin^2\phi\,d\psi^2)\right)$$

Make the coordinate transformation

$$r = \sin\theta$$

$$\Rightarrow \quad dr = d(\sin\theta) = \cos\theta\,d\theta$$

$$\Rightarrow \quad ds^2 = -dt^2 + (a_0)^2\left(\frac{dr^2}{\cos^2\theta} + r^2(d\phi^2 + \sin^2\phi\,d\psi^2)\right)$$

$$= -dt^2 + (a_0)^2\left(\frac{dr^2}{1-\sin^2\theta} + r^2(d\phi^2 + \sin^2\phi\,d\psi^2)\right)$$

$$= -dt^2 + (a_0)^2\left(\frac{dr^2}{1-r^2} + r^2(d\phi^2 + \sin^2\phi\,d\psi^2)\right)$$

9.6.2 The Ricci tensor

To find the Ricci tensor we can compare with calculations made for the Robertson Walker space-time and find a proper transformation of the coordinates.

The Einstein cylinder

$$ds^2 = -dt^2 + \frac{(a_0)^2}{1-r^2}dr^2 + (a_0 r)^2 d\phi^2 + (a_0 r)^2\sin^2\phi\,d\psi^2$$

The Robertson Walker line element

$$ds^2 = -dt'^2 + \frac{a^2(t')}{1-kr'^2}dr'^2 + a^2(t')r'^2 d\theta'^2 + a^2(t')r'^2\sin^2\theta'\,d\phi'^2$$

Comparing the line elements

$$dt = dt'$$

$$\frac{a_0}{\sqrt{1-r^2}}dr = \frac{a(t')}{\sqrt{1-kr'^2}}dr'$$

$$a_0 r d\phi = a(t')r'd\theta'$$

$$a_0 r\sin\phi\,d\psi = a(t')r'\sin\theta'\,d\phi'$$

A proper transformation would be

$$t' = t$$

$$a(t') = a_0$$

$$k = 1$$

$$r' = r$$

$$\theta' = \phi$$

$$\phi' = \psi$$

The Ricci tensor in the non-coordinate basis is[16]

$$R_{\hat{t}\hat{t}} = -3\frac{\ddot{a}}{a} = 0$$

$$R_{\hat{r}\hat{r}} = \frac{\ddot{a}}{a} + 2\frac{(\dot{a}^2 + k)}{a^2} = \frac{2}{a_0^2}$$

$$R_{\hat{\phi}\hat{\phi}} = \frac{\ddot{a}}{a} + 2\frac{(\dot{a}^2 + k)}{a^2} = \frac{2}{a_0^2}$$

$$R_{\hat{\psi}\hat{\psi}} = \frac{\ddot{a}}{a} + 2\frac{(\dot{a}^2 + k)}{a^2} = \frac{2}{a_0^2}$$

[16] The chapter is named: The Robertson Walker metric

The Ricci scalar

$$R = \eta^{\hat{a}\hat{b}}R_{\hat{a}\hat{b}} = \eta^{\hat{t}\hat{t}}R_{\hat{t}\hat{t}} + \eta^{\hat{r}\hat{r}}R_{\hat{r}\hat{r}} + \eta^{\hat{\phi}\hat{\phi}}R_{\hat{\phi}\hat{\phi}} + \eta^{\hat{\psi}\hat{\psi}}R_{\hat{\psi}\hat{\psi}} = 3 \cdot \frac{2}{a_0^2}$$

To find the Ricci tensor in the coordinate basis we need the basis one-forms

$$\omega^{\hat{t}} = dt$$

$$\omega^{\hat{r}} = \frac{a_0}{\sqrt{1-r^2}}dr \qquad dr = \frac{\sqrt{1-r^2}}{a_0}\omega^{\hat{r}}$$

$$\omega^{\hat{\phi}} = a_0 r\, d\phi \qquad d\phi = \frac{1}{a_0 r}\omega^{\hat{\phi}} \qquad \eta^{ij} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$\omega^{\hat{\psi}} = a_0 r \sin\phi\, d\psi \qquad d\psi = \frac{1}{a_0 r \sin\phi}\omega^{\hat{\psi}}$$

The transformation

$$R_{ab} = \Lambda^{\hat{c}}{}_a \Lambda^{\hat{d}}{}_b R_{\hat{c}\hat{d}}$$

$$\Rightarrow \quad R_{tt} = 0$$

$$R_{rr} = \Lambda^{\hat{c}}{}_r \Lambda^{\hat{d}}{}_r R_{\hat{c}\hat{d}} = \left(\Lambda^{\hat{r}}{}_r\right)^2 R_{\hat{r}\hat{r}} = \frac{a_0^2}{1-r^2}\frac{2}{a_0^2} = \frac{2}{1-r^2}$$

$$R_{\phi\phi} = \Lambda^{\hat{c}}{}_\phi \Lambda^{\hat{d}}{}_\phi R_{\hat{c}\hat{d}} = \left(\Lambda^{\hat{\phi}}{}_\phi\right)^2 R_{\hat{\phi}\hat{\phi}} = (a_0 r)^2 \frac{2}{a_0^2} = 2r^2$$

$$R_{\psi\psi} = \Lambda^{\hat{c}}{}_\psi \Lambda^{\hat{d}}{}_\psi R_{\hat{c}\hat{d}} = \left(\Lambda^{\hat{\psi}}{}_\psi\right)^2 R_{\hat{\psi}\hat{\psi}} = (a_0 r \sin\phi)^2 \frac{2}{a_0^2} = 2r^2 \sin^2\phi$$

Summarized in a matrix

$$R_{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \dfrac{2}{1-r^2} & 0 & 0 \\ 0 & 0 & 2r^2 & 0 \\ 0 & 0 & 0 & 2r^2\sin^2\phi \end{pmatrix}$$

Where a refers to column and b to row

9.6.3 The Einstein Equations

To find the Einstein tensor we once more copy the result from the Robertson Walker metric

$$G_{\hat{t}\hat{t}} = 3\frac{\dot{a}^2 + k}{a^2} = \frac{3}{a_0^2}$$

$$G_{\hat{r}\hat{r}} = -\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2}\right) = -\frac{1}{a_0^2}$$

$$G_{\hat{\phi}\hat{\phi}} = -\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2}\right) = -\frac{1}{a_0^2}$$

$$G_{\hat{\psi}\hat{\psi}} = -\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2}\right) = -\frac{1}{a_0^2}$$

Summarized in a matrix

$$G_{\hat{a}\hat{b}} = \begin{Bmatrix} \frac{3}{a_0^2} & 0 & 0 & 0 \\ 0 & -\frac{1}{a_0^2} & 0 & 0 \\ 0 & 0 & -\frac{1}{a_0^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{a_0^2} \end{Bmatrix}$$

Where a refers to column and b to row

9.6.4 The Stress Energy Tensor

In case of a perfect fluid, the stress energy tensor is

$$\begin{aligned} T_{\hat{a}\hat{b}} &= \mu u_{\hat{a}} u_{\hat{b}} + p(\eta_{\hat{a}\hat{b}} + u_{\hat{a}} u_{\hat{b}}) = \mu \eta_{\hat{a}\hat{a}} u^{\hat{a}} u_{\hat{b}} + p\left(\eta_{\hat{a}\hat{b}} + \eta_{\hat{a}\hat{a}} u^{\hat{a}} u_{\hat{b}}\right) \\ \Rightarrow \quad T_{\hat{t}\hat{t}} &= {}^{17}\mu \eta_{\hat{t}\hat{t}} u^{\hat{t}} u_{\hat{t}} + p\left(\eta_{\hat{t}\hat{t}} + \eta_{\hat{t}\hat{t}} u^{\hat{t}} u_{\hat{t}}\right) = \mu(-1)(-1) + p\big((-1) + (-1)(-1)\big) = \mu \\ T_{\hat{r}\hat{r}} &= \mu \eta_{\hat{r}\hat{r}} u^{\hat{r}} u_{\hat{r}} + p\left(\eta_{\hat{r}\hat{r}} + \eta_{\hat{r}\hat{r}} u^{\hat{r}} u_{\hat{r}}\right) = p \\ T_{\hat{\Phi}\hat{\Phi}} &= \mu \eta_{\hat{\Phi}\hat{\Phi}} u^{\hat{\Phi}} u_{\hat{\Phi}} + p\left(\eta_{\hat{\Phi}\hat{\Phi}} + \eta_{\hat{\Phi}\hat{\Phi}} u^{\hat{\Phi}} u_{\hat{\Phi}}\right) = p \\ T_{\hat{\Psi}\hat{\Psi}} &= \mu \eta_{\hat{\Psi}\hat{\Psi}} u^{\hat{\Psi}} u_{\hat{\Psi}} + p\left(\eta_{\hat{\Psi}\hat{\Psi}} + \eta_{\hat{\Psi}\hat{\Psi}} u^{\hat{\Psi}} u_{\hat{\Psi}}\right) = p \end{aligned}$$

Summarized in a matrix

$$T_{\hat{a}\hat{b}} = \begin{Bmatrix} \mu & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{Bmatrix}$$

Where a refers to column and b to row

$$\begin{aligned} \Rightarrow \quad p &= -\frac{1}{a_0^2} < 0 \\ \mu &= \frac{3}{a_0^2} \end{aligned}$$

9.6.5 The Einstein tensor with a cosmological constant

A negative pressure is problematic in classical physics and we include a cosmological constant. The Einstein equation with a cosmological constant

$$G_{\hat{a}\hat{b}} = T_{\hat{a}\hat{b}} - \eta_{\hat{a}\hat{b}}\Lambda$$

$$\Rightarrow \quad \begin{Bmatrix} \frac{3}{a_0^2} & 0 & 0 & 0 \\ 0 & -\frac{1}{a_0^2} & 0 & 0 \\ 0 & 0 & -\frac{1}{a_0^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{a_0^2} \end{Bmatrix} = \begin{Bmatrix} \mu & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{Bmatrix} - \Lambda \begin{Bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{Bmatrix}$$

$$\begin{aligned} \Rightarrow \quad \mu &= \frac{3}{a_0^2} - \Lambda \\ p &= -\frac{1}{a_0^2} + \Lambda \end{aligned}$$

[17] $u_{\hat{a}} = (1,0,0,0), u^{\hat{a}} = \eta^{\hat{a}\hat{b}} u_{\hat{b}} = (-1,0,0,0)$

And we can conclude that the Einstein cylinder is a solution of the Einstein equations with a positive cosmological constant with a perfect fluid source with positive energy and pressure if

$$\frac{3}{a_0^2} > \Lambda > \frac{1}{a_0^2} > 0$$

9.7 [n]The Newtonian Approximation – The right hand side!

The newtionian approximation is characterized by a weak gravitational field and bodies of low masses and velocities.
The Einstein space-time in a weak and slowly varying gravitational field can be described by the linearized metric which differs only slightly from the Minkowsky space-time.

$$\begin{aligned} ds^2 &= g_{ab}dx^a dx^b \\ g_{ab} &= {}^{18}\eta_{ab} + \epsilon h_{ab} \\ \eta_{ab} &= \begin{Bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{Bmatrix} \end{aligned}$$

We look at a particle in a weak field with velocity $v^i = \frac{dx^i}{dx^0}, |v| \ll c$.

$$\begin{aligned} \Rightarrow \quad ds^2 &= g_{oo}(dx^0)^2 + \sum_i^{1,2,3} \left[(g_{0i} + g_{i0})dx^i dx^0 + g_{ii}(dx^i)^2\right] \\ \Rightarrow \quad \left(\frac{ds}{dx^0}\right)^2 &= g_{oo} + \sum_i^{1,2,3} \left[(g_{0i} + g_{i0})\left(\frac{dx^i}{dx^0}\right) + g_{ii}\left(\frac{dx^i}{dx^0}\right)^2\right] \\ &= g_{oo} + \sum_i^{1,2,3} \left[(g_{0i} + g_{i0})(v^i) + g_{ii}(v^i)^2\right] \\ &= (1 + \epsilon h_{00}) + \sum_i^{1,2,3} \left[(\epsilon h_{0i} + \epsilon h_{i0})v^i + (-1 + \epsilon h_{ii})(v^i)^2\right] \rightarrow 1 + \epsilon h_{00} \end{aligned} \tag{9.9.}$$

i.e. in the approximation where $v \ll c$, the time component h_{00} is dominant with respect to the space-components (h_{0i}, h_{i0}, h_{ii}).

9.7.1 [19]The Ricci tensor

In this linearized theory we have the Christoffel symbols

$$\begin{aligned} \Gamma^a{}_{bc} &= \frac{1}{2}\epsilon\eta^{ad}\left(\frac{\partial h_{bd}}{\partial x^c} + \frac{\partial h_{cd}}{\partial x^b} - \frac{\partial h_{bc}}{\partial x^d}\right) \\ R_{ab} &= \partial_c\Gamma^c{}_{ab} - \partial_b\Gamma^c{}_{ac} \\ \Rightarrow \quad R_{00} &= \partial_c\Gamma^c{}_{00} - \partial_0\Gamma^c{}_{0c} \end{aligned} \tag{9.10.}$$

Assuming[20] the time derivatives $\partial_0 h_{ab}$ are small compared to the space derivatives $\partial_i h_{ab}$

$$\begin{aligned} \Rightarrow \quad R_{00} &= \partial_c\Gamma^c{}_{00} \\ &= \partial_0\Gamma^0{}_{00} + \sum_i^{1,2,3} \partial_i\Gamma^i{}_{00} \end{aligned}$$

---

[18] ϵ is a small dimensionless parameter of order $\frac{v}{c}$ so $\epsilon \ll 1$

[19] For a detailed calculation of the Christoffel symbols, the Riemann and Ricci tensors, the Ricci scalar and the Einstein equation (The left hand side!) see a later chapter named "Linearized metric"

[20] In the Newtonian approximation we work in a regime where frequencies are low and wavelengths long. This is sometimes stated as the 'slow-motion approximation' and has the effect that all derivatives with respect to time $(\frac{\partial}{\partial x^0})$ are negligible.

$$= \sum_i^{1,2,3} \partial_i \left(\frac{1}{2} \epsilon \eta^{id} \left(\frac{\partial h_{0d}}{\partial x^0} + \frac{\partial h_{0d}}{\partial x^0} - \frac{\partial h_{00}}{\partial x^d} \right) \right)$$

$$= -\frac{1}{2} \epsilon \sum_i^{1,2,3} \eta^{ii} \frac{\partial^2 h_{00}}{\partial x^i \partial x^i}$$

$$= {}^{21} \frac{1}{2} \epsilon \nabla^2 h_{00} \tag{9.11.}$$

9.7.2 The Stress-energy Tensor

Earlier[22] we found that

$$R_{ab} = \kappa \left(T_{ab} - \frac{1}{2} g^{ab} T_{ab} g_{ab} \right) = \kappa \rho_{ab}$$

Where

$$\rho_{ab} = T_{ab} - \frac{1}{2} g^{ab} T_{ab} g_{ab} = T_{ab} - \frac{1}{2} T g_{ab}$$

The stress-energy tensor is[23]

$$T_{ab} = \mu u_a u_b$$

In the simplest case, pure matter no pressure[24], $u_a = \left(\frac{dx^0}{ds}, \frac{dx^i}{ds} \right) = {}^{25}\left(\frac{1}{\sqrt{1-\epsilon h_{00}}}, 0,0,0 \right)$ and the only non-zero element in the stress energy tensor is

$$T_{00} = \mu (u_0)^2 = \mu \left(\frac{1}{\sqrt{1-\epsilon h_{00}}} \right)^2 = \frac{\mu}{1-\epsilon h_{00}} \approx \mu (1 + \epsilon h_{00}) \to \mu$$

and

$$T = g^{ab} T_{ab} = g^{00} T_{00} = g^{00} \mu$$

$$\Rightarrow \quad \rho_{00} = T_{00} - \frac{1}{2} T g_{00} = {}^{26} \mu - \frac{1}{2} \mu g^{00} g_{00} = \frac{1}{2} \mu$$

Now, because

$$R_{00} = {}^{27} \frac{1}{2} \epsilon \nabla^2 h_{00} = \kappa \rho_{00} = \frac{1}{2} \kappa \mu$$

We can see the equivalence with the Poisson's equation for gravity (the variation in the gravitational field, ϕ ,caused by a mass distribution, ρ)

$$\nabla^2 \phi = {}^{28} 4\pi G_N \rho$$

9.7.3 The Equation of Motion

We can also find the equation of motion in this approximation. In GR a test particle follows geodesics described x^i

$$\frac{d^2 x^i}{ds^2} = {}^{29} - \Gamma^i{}_{bc} \frac{dx^b}{ds} \frac{dx^c}{ds} = -\Gamma^i{}_{00} \left(\frac{dx^0}{ds} \right)^2 - \Gamma^i{}_{bc} \frac{dx^b}{ds} \frac{dx^c}{ds} \tag{9.12.}$$

---

[21] $\nabla^2 = \sum_i^{1,2,3} \frac{\partial^2}{\partial x^i \partial x^i}$

[22] Eq. (9.1.): $R_{ab} = \kappa T_{ab} - \frac{\kappa T^a_a}{(n-1)} g_{ab} = \kappa \left(T_{ab} - \frac{1}{2} g^{ab} T_{ab} g_{ab} \right) = \kappa \rho_{ab} \; if \; n+1 = 4$

[23] Eq. (9.2.): $T^{ab} = \rho u^a u^b$

[24] Eq. (9.3.)

[25] $\frac{dx^0}{ds} = \frac{1}{\sqrt{-1+\epsilon h_{00}}}$ eq. (9.9.)

[26] $g^{00} g_{00} \approx 1$ because the off-diagonal elements are $\ll 1$.

[27] Eq. (9.11.)

[28] If we identify $\frac{1}{2} \epsilon h_{00} = \phi$ and $4\pi G_N \rho = \frac{1}{2} \kappa \mu$

[29] The general equation is: $\frac{d^2 x^a}{ds^2} + \Gamma^a{}_{bc} \frac{dx^b}{ds} \frac{dx^c}{ds} = 0$

Rewriting left hand side of eq.(9.12.)

$$\frac{d^2x^i}{ds^2} = \left(\frac{dx^0}{ds}\right)^2 \frac{d^2x^i}{(dx^0)^2}$$

Rewriting right hand side of eq. (9.12.)

$$\Gamma^i{}_{bc}\frac{dx^b}{ds}\frac{dx^c}{ds} = \Gamma^i{}_{bc}\left(\frac{dx^0}{ds}\right)^2\frac{dx^b}{dx^0}\frac{dx^c}{dx^0} = \Gamma^i{}_{bc}\left(\frac{dx^0}{ds}\right)^2 v^b v^c \to {}^{30}0 \qquad b,c = 1,2,3$$

$$\Rightarrow \quad \left(\frac{dx^0}{ds}\right)^2 \frac{d^2x^i}{(dx^0)^2} = -\Gamma^i{}_{00}\left(\frac{dx^0}{ds}\right)^2$$

$$\Rightarrow \quad \frac{d^2x^i}{(dx^0)^2} = -\Gamma^i{}_{00}$$

We use eq. (9.10.) to find

$$\Gamma^i{}_{00} = \frac{1}{2}\epsilon\eta^{id}\left(\frac{\partial h_{0d}}{\partial x^0} + \frac{\partial h_{0d}}{\partial x^0} - \frac{\partial h_{00}}{\partial x^d}\right)$$

$$= {}^{31} -\frac{1}{2}\epsilon\eta^{ii}\frac{\partial h_{00}}{\partial x^i}$$

$$= \frac{1}{2}\epsilon\frac{\partial h_{00}}{\partial x^i} \qquad i = 1,2,3$$

$$\Rightarrow \quad \frac{d^2x^i}{(dx^0)^2} = -\Gamma^i{}_{00} = -\frac{1}{2}\epsilon\frac{\partial h_{00}}{\partial x^i} \qquad (9.13.)$$

Because $x^0 = t$ we can rewrite eq. (9.13.)

$$\frac{d^2x^i}{dt^2} = -\frac{1}{2}\epsilon\frac{\partial h_{00}}{\partial x^i}$$

Again we see the equivalence to Newton's law, where the left side represent the force (acceleration) and the right side the gradient of the potential ($\phi = \frac{1}{2}\epsilon h_{00}$). Recall

$$\bar{F} = -\nabla\phi$$

With

$$\phi = -\frac{GMm}{r}$$

Leads to Newton's famous equation

$$F = -\frac{GMm}{r^2}$$

Now, this was a classical Newtonian interpretation. [32]Later we will use eq(9.11.) (9.13.) to calculate a relativistic time dilation.

Bibliografi

Carroll, S. M. (2004). *An Introduction to General Relativity, Spacetime and Geometry.* San Fransisco, CA: Addison Wesley.

Choquet-Bruhat, Y. (2015). *Introduction to General Relativity, Black Holes and Cosmology.* Oxford: Oxford University Press.

d'Inverno, R. (1992). *Introducing Einstein's Relativity.* Oxford: Clarendon Press.

Ellis, S. H. (1973). *The Large Scale Structure of Space-time.* Cambridge: Cambridge University Press.

McMahon, D. (2006). *Relativity Demystified.* McGraw-Hill Companies, Inc.

[30] In the low velocity approximation

[31] In the Newtonian approximation we work in a regime where frequencies are low and wavelengths long. This is sometimes stated as the 'slow-motion approximation' and has the effect that all derivatives with respect to time ($\frac{\partial}{\partial x^0}$) are negligible.

[32] (Chapter 12)

Penrose, R. (2004). *The Road to Reality.* New York: Vintage Books.
Weinberg, S. (2008). *Cosmology.* Oxford University Press.

[a] (Penrose, 2004, s. 445)
[b] (Choquet-Bruhat, 2015, s. 70), (Penrose, 2004, s. 464)
[c] (McMahon, 2006, p. 160), (Carroll, 2004, pp. 33-35)
[d] (Choquet-Bruhat, 2015, s. 71)
[e] (McMahon, 2006, p. 158)
[f] (McMahon, 2006, p. 159)
[g] (McMahon, 2006, p. 164)
[h] (Choquet-Bruhat, 2015, s. 71)
[i] (Choquet-Bruhat, 2015, s. 71)
[j] http://en.wikipedia.org/wiki/G%C3%B6del_metric
[k] (Ellis, 1973, p. 165)
[l] (McMahon, 2006, p. 326)
[m] (Choquet-Bruhat, 2015, s. 95)
[n] (Choquet-Bruhat, 2015, s. 75), (Weinberg, 2008, s. 516), (McMahon, 2006, p. 136), (d'Inverno, 1992, p. 165), (Ellis, 1973, p. 168)