Content

Space-time		Line-element	Chap-ter
Anti de Sitter space-time	ds^2	$= -dt^2 + \cos^2(t)\, dr^2 + \cos^2(t)\sinh^2(r)\, d\theta^2 + \cos^2(t)\sinh^2(r)\sin^2\theta\, d\phi^2$	8
De Sitter space-time	ds^2	$= -dt^2 + a(t)^2\big(d\theta^2 + \sin^2\theta\,(d\phi^2 + \sin^2\phi\, d\psi^2)\big)$	8
Example: Four-dimensional space-time	ds^2	$= -dt^2 + L^2(t,r)dr^2 + B^2(t,r)d\phi^2 + M^2(t,r)dz^2$	8
Gravitationally collapse of an inho-mogeneous spherically symmetric dust cloud	ds^2	$= -dt^2 + e^{2b(t,r)}dr^2 + R^2(t,r)d\phi^2$	8

8 The Einstein Field Equations

8.1 [a]The vacuum Einstein equations

Prove that the Einstein field equations $G_{ab} = \kappa T_{ab}$ reduces to the vacuum Einstein equations $R_{ab} = 0$ if we set $T_{ab} = 0$.

The Einstein tensor

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R$$

If $G_{ab} = \kappa T_{ab} = 0$:

$$\Rightarrow \quad G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R = 0$$

$$\Rightarrow \quad R_{ab} = \frac{1}{2} g_{ab} R$$

Contracting with g^{ab}

$$\Rightarrow \quad g^{ab} R_{ab} = \frac{1}{2} g^{ab} g_{ab} R$$

using the definition

$$R = g^{ab} R_{ab}$$

and that in 4 dimensions $g^{ab} g_{ab} = 4$

$$\Rightarrow \quad R = \frac{1}{2} 4R = 2R$$

Now this can only be true if

$$R_{ab} = 0$$

8.2 The vacuum Einstein equations with a cosmological constant

Prove that the Einstein field equations $G_{ab} = \kappa T_{ab}$ reduces to $R_{ab} = g_{ab}\Lambda$ and $R = 4\Lambda^b$.

The Einstein equation in vacuum with a cosmological constant

$$0 = R_{ab} - \frac{1}{2} g_{ab} R + g_{ab}\Lambda$$

$$\Rightarrow \quad 0 = g^{ab} R_{ab} - \frac{1}{2} g^{ab} g_{ab} R + g_{ab} g^{ab}\Lambda = R - \frac{1}{2} 4R + 4\Lambda$$

$$\Rightarrow \quad R = 4\Lambda$$

Next we rewrite the Einstein equation

$$0 = R_{ab} - \frac{1}{2} g_{ab} R + g_{ab}\Lambda = R_{ab} - \frac{1}{2} g_{ab}(4\Lambda) + g_{ab}\Lambda = R_{ab} - g_{ab}\Lambda$$

$$\Rightarrow \quad R_{ab} = g_{ab}\Lambda$$ Q.E.D.

In the non-coordinate basis

$$R_{\hat{a}\hat{b}} = \eta_{\hat{a}\hat{b}}\Lambda$$

8.3 [c]General remarks on the Einstein equations with a cosmological constant

If we demand that the gravitational field equations are

(1) generally covariant

(2) be of second differential order in g_{ab}

(3) involve the energy-momentum T_{ab} linearly

it can be shown that the only equation which meets these requirements is

$$R_{ab} + \mu R g_{ab} + \Lambda g_{ab} = \kappa T_{ab}$$

where μ, Λ, and κ are constants.

The demand that T_{ab} satisfies the conservation equation

$$\nabla_b T^{ab} = 0$$

leads to

$$\mu = -\frac{1}{2}$$

Proof:

$$\nabla_b T^{ab} = 0$$

$$\Rightarrow \quad \nabla_b\left(R^{ab} + \mu R g^{ab} + \Lambda g^{ab}\right) = 0$$

$$\Rightarrow \quad \nabla_b R^{ab} + \mu \nabla_b\left(R g^{ab}\right) + \Lambda \nabla_b g^{ab} = 0$$ [1]

[1] $\nabla_b g^{ab} = 0$

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$$\Rightarrow \quad \nabla_b R^{ab} + \mu\left((\nabla_b R) g^{ab} + R(\nabla_b g^{ab})\right) = {}^{2}0$$
$$\Rightarrow \quad \nabla_b R^{ab} + \mu(\nabla_b R) g^{ab} = 0$$

Next we use the Bianchi identity:

$$\nabla_a R_{debc} + \nabla_b R_{deca} + \nabla_c R_{deab} = 0$$
$$\Rightarrow \quad g^{db}(\nabla_a R_{debc} + \nabla_b R_{deca} + \nabla_c R_{deab}) = 0$$
$$\Rightarrow \quad \nabla_a g^{db} R_{debc} + \nabla_b g^{db} R_{deca} + \nabla_c g^{db} R_{deab} = 0$$
$$\Rightarrow \quad \nabla_a R^b{}_{ebc} + \nabla_b R^b{}_{eca} + \nabla_c R^b{}_{eab} = 0$$
$$\Rightarrow \quad \nabla_a R_{ec} + \nabla_b R^b{}_{eca} - \nabla_c R_{ea} = 0$$
$$\Rightarrow \quad g^{ae}(\nabla_a R_{ec} + \nabla_b R^b{}_{eca} - \nabla_c R_{ea}) = 0$$
$$\Rightarrow \quad \nabla_a g^{ae} R_{ec} + \nabla_b g^{ae} R^b{}_{eca} - \nabla_c g^{ae} R_{ea} = 0$$
$$\Rightarrow^{3} \quad \nabla_a R^a{}_c + \nabla_b R^b{}_c - \nabla_c R = 0$$
$$\Rightarrow \quad 2\left(\nabla_a R^a{}_c - \frac{1}{2}\nabla_c R\right) = 0$$
$$\Rightarrow \quad 2g^{bc}\left(\nabla_a R^a{}_c - \frac{1}{2}\nabla_c R\right) = 0$$
$$\Rightarrow \quad 2\left(\nabla_a g^{bc} R^a{}_c - \frac{1}{2}(\nabla_c R) g^{bc}\right) = 0$$
$$\Rightarrow \quad 2\left(\nabla_a R^{ab} - \frac{1}{2}\nabla_a R g^{ab}\right) = 0$$

Now if we compare with

$$\nabla_b R^{ab} + \mu(\nabla_b R) g^{ab} = 0$$

we see that

$$\mu = -\frac{1}{2}$$

8.4 [d]Using the contracted Bianchi identities, prove that: $\nabla_b G^{ab} = 0$

Expressions needed:

Bianchi identity:

$$0 = \nabla_a R_{debc} + \nabla_b R_{deca} + \nabla_c R_{deab} \tag{8.1.}$$
$$0 = \nabla_c g_{ab}$$

We want to prove

$$0 = \nabla_c g^{ab}$$

If

$$0 = \nabla_c g^{ab}$$
$$\Leftrightarrow \quad 0 = g_{da} g_{eb} \nabla_c g^{ab}$$
$$\Leftrightarrow \quad 0 = \nabla_c g_{da} g_{eb} g^{ab}$$
$$\Leftrightarrow \quad 0 = \nabla_c g_{de}$$
$$\Leftrightarrow \quad 0 = \nabla_c g^{ab} \tag{8.2.}$$

Riemann tensor:

$$R_{abcd} = -R_{abdc} \tag{8.3.}$$

Ricci tensor:

$$R^c{}_{acb} = R_{ab} \tag{8.4.}$$

Ricci scalar:

$$R = g^{ab} R_{ab} = R^a{}_a \tag{8.5.}$$
$$g^{ae} R^b{}_{eca} = g^{ae} R_e{}^b{}_{ac} = R^{ab}{}_{ac} = R^b{}_c \tag{8.6.}$$

The Einstein tensor:

[2] $\nabla_b g^{ab} = 0$

[3] $g^{ae} R^b{}_{eca} = g^{ae} R_e{}^b{}_{ac} = R^{ab}{}_{ac} = R^b{}_c$

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R \tag{8.7.}$$

Kronecker delta

$$g^a{}_c = g^{ab} g_{bc} = \delta^a_c \tag{8.8.}$$

$$\nabla_a G^a{}_c = \nabla_a \left(R^a{}_c - \frac{1}{2} g^a{}_c R \right) = \nabla_a R^a{}_c - \frac{1}{2} \nabla_a g^a{}_c R = {}^4\nabla_a R^a{}_c - \frac{1}{2} g^a{}_c \nabla_a R$$

$$= \nabla_a R^a{}_c - \frac{1}{2} \delta^a_c \nabla_a R = \nabla_a R^a{}_c - \frac{1}{2} \nabla_c R \tag{8.9.}$$

The proof:

$$0 = {}^5 g^{db} (\nabla_a R_{debc} + \nabla_b R_{deca} + \nabla_c R_{deab})$$

$$\Leftrightarrow \quad 0 = {}^6 \nabla_a g^{db} R_{debc} + \nabla_b g^{db} R_{deca} + \nabla_c g^{db} R_{deab}$$

$$\Leftrightarrow \quad 0 = \nabla_a R^b{}_{ebc} + \nabla_b R^b{}_{eca} + \nabla_c R^b{}_{eab}$$

$$\Leftrightarrow \quad 0 = {}^7 \nabla_a R_{ec} + \nabla_b R^b{}_{eca} - \nabla_c R^b{}_{eba}$$

$$\Leftrightarrow \quad 0 = {}^8 \nabla_a R_{ec} + \nabla_b R^b{}_{eca} - \nabla_c R_{ea}$$

$$0 = {}^9 g^{ae} \left(\nabla_a R_{ec} + \nabla_b R^b{}_{eca} - \nabla_c R_{ea} \right)$$

$$\Leftrightarrow \quad 0 = \nabla_a g^{ae} R_{ec} + \nabla_b g^{ae} R^b{}_{eca} - \nabla_c g^{ae} R_{ea}$$

$$\Leftrightarrow \quad 0 = \nabla_a R^a{}_c + \nabla_b g^{ae} R^b{}_{eca} - \nabla_c R^a{}_a$$

$$\Leftrightarrow \quad 0 = {}^{10} \nabla_a R^a{}_c + \nabla_b g^{ae} R^b{}_{eca} - \nabla_c R$$

$$\Leftrightarrow \quad 0 = {}^{11} \nabla_a R^a{}_c + \nabla_b R^b{}_c - \nabla_c R$$

$$\Leftrightarrow \quad 0 = 2 \left[\nabla_a R^a{}_c - \frac{1}{2} \nabla_c R \right]$$

$$\Leftrightarrow \quad 0 = {}^{12} \nabla_a G^a{}_c$$

$$0 = {}^{13} g^{bc} \nabla_a G^a{}_c = \nabla_a g^{bc} G^a{}_c = \nabla_a G^{ab}$$

Q.E.D.

This is a very important result because it leads to the conservation laws of the right hand side of the Einstein equation, which we will look into later.

$$\nabla_a T^{ab} = 0$$

8.5 e2+1 dimensions: Gravitational collapse of an inhomogeneous spherically symmetric dust cloud.

8.5.1 The components of the curvature tensor for the metric in 2+1 dimensions using Cartan's structure equations

The line element:

$$ds^2 = -dt^2 + e^{2b(t,r)} dr^2 + R^2(t,r) d\phi^2$$

The Basis one forms

$$\omega^{\hat{t}} = dt$$

$$\omega^{\hat{r}} = e^{b(t,r)} dr \qquad dr = e^{-b(t,r)} \omega^{\hat{r}} \qquad \eta^{ij} = \begin{Bmatrix} -1 & & \\ & 1 & \\ & & 1 \end{Bmatrix}$$

$$\omega^{\hat{\phi}} = R(t,r) d\phi \qquad d\phi = \frac{1}{R(t,r)} \omega^{\hat{\phi}}$$

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[4] use eq.(8.8.)
[5] Multiply eq.(8.1.) by g^{db}:
[6] use eq.(8.2.)(8.2.)
[7] use eq. (8.3.) and eq. (8.4.)
[8] use eq. (8.4.)
[9] Multiply by g^{ae}
[10] use eq. (8.5.)
[11] use eq. (8.6.)
[12] use eq. (8.9.)
[13] Multiply by g^{bc}

Cartan's First Structure equation and the calculation of the curvature two-forms

$$
\begin{aligned}
d\omega^{\hat a} &= -\Gamma^{\hat a}{}_{\hat b} \wedge \omega^{\hat b} \\
\Gamma^{\hat a}{}_{\hat b} &= \Gamma^{\hat a}{}_{\hat b \hat c}\omega^{\hat c} \\
d\omega^{\hat t} &= 0 \\
d\omega^{\hat r} &= d\left(e^{b(t,r)}dr\right) = \dot b e^{b(t,r)} dt \wedge dr = \dot b \omega^{\hat t} \wedge \omega^{\hat r} \\
d\omega^{\hat \Phi} &= d(R(t,r)d\phi) = \dot R dt \wedge d\phi + R' dr \wedge d\phi = \frac{\dot R}{R}\omega^{\hat t} \wedge \omega^{\hat \Phi} + \frac{R'}{R} e^{-b(t,r)}\omega^{\hat r} \wedge \omega^{\hat \Phi}
\end{aligned}
$$

Summarizing the curvature one forms in a matrix:

$$
\Gamma^{\hat a}{}_{\hat b} = \begin{Bmatrix} 0 & \dot b \omega^{\hat r} & \frac{\dot R}{R}\omega^{\hat \Phi} \\ \dot b \omega^{\hat r} & 0 & \frac{R'}{R} e^{-b(t,r)}\omega^{\hat \Phi} \\ \frac{\dot R}{R}\omega^{\hat \Phi} & -\frac{R'}{R} e^{-b(t,r)}\omega^{\hat \Phi} & 0 \end{Bmatrix}
$$

Where $\hat a$ refers to column and $\hat b$ to row.

8.5.1.1 The curvature two forms

$$
\begin{aligned}
\Omega^{\hat a}{}_{\hat b} &= d\Gamma^{\hat a}{}_{\hat b} + \Gamma^{\hat a}{}_{\hat c} \wedge \Gamma^{\hat c}{}_{\hat b} = \frac{1}{2} R^{\hat a}{}_{\hat b \hat c \hat d}\omega^{\hat c} \wedge \omega^{\hat d} \\
d\Gamma^{\hat r}{}_{\hat t} &= d\left(\dot b \omega^{\hat r}\right) = d\left(\dot b e^{b(t,r)} dr\right) = \left[\ddot b e^{b(t,r)} + \left(\dot b\right)^2 e^{b(t,r)}\right] dt \wedge dr \\
&= -\left[\ddot b + \left(\dot b\right)^2\right]\omega^{\hat r} \wedge \omega^{\hat t} \\
\Gamma^{\hat r}{}_{\hat c} \wedge \Gamma^{\hat c}{}_{\hat t} &= \Gamma^{\hat r}{}_{\hat t} \wedge \Gamma^{\hat t}{}_{\hat t} + \Gamma^{\hat r}{}_{\hat r} \wedge \Gamma^{\hat r}{}_{\hat t} + \Gamma^{\hat r}{}_{\hat \Phi} \wedge \Gamma^{\hat \Phi}{}_{\hat t} = 0 \\
\Rightarrow \quad \Omega^{\hat r}{}_{\hat t} &= -\left[\ddot b + \left(\dot b\right)^2\right]\omega^{\hat r} \wedge \omega^{\hat t} \\
d\Gamma^{\hat \Phi}{}_{\hat t} &= d\left(\frac{\dot R}{R}\omega^{\hat \Phi}\right) = d\left(\dot R(t,r) d\phi\right) = \ddot R dt \wedge d\phi + \left(\dot R\right)' dr \wedge d\phi \\
&= -\frac{\ddot R}{R}\omega^{\hat \Phi} \wedge \omega^{\hat t} - \frac{\left(\dot R\right)'}{R} e^{-b(t,r)}\omega^{\hat \Phi} \wedge \omega^{\hat r} \\
\Gamma^{\hat \Phi}{}_{\hat c} \wedge \Gamma^{\hat c}{}_{\hat t} &= \Gamma^{\hat \Phi}{}_{\hat t} \wedge \Gamma^{\hat t}{}_{\hat t} + \Gamma^{\hat \Phi}{}_{\hat r} \wedge \Gamma^{\hat r}{}_{\hat t} + \Gamma^{\hat \Phi}{}_{\hat \Phi} \wedge \Gamma^{\hat \Phi}{}_{\hat t} = \Gamma^{\hat \Phi}{}_{\hat r} \wedge \Gamma^{\hat r}{}_{\hat t} = \frac{R'}{R} e^{-b(t,r)}\omega^{\hat \Phi} \wedge \dot b \omega^{\hat r} \\
\Rightarrow \quad \Omega^{\hat \Phi}{}_{\hat t} &= -\frac{\ddot R}{R}\omega^{\hat \Phi} \wedge \omega^{\hat t} - \left(\frac{\left(\dot R\right)'}{R} - \frac{R'\dot b}{R}\right) e^{-b(t,r)}\omega^{\hat \Phi} \wedge \omega^{\hat r} \\
d\Gamma^{\hat \Phi}{}_{\hat r} &= d\left(\frac{R'}{R} e^{-b(t,r)}\omega^{\hat \Phi}\right) = d\left(R' e^{-b(t,r)} d\phi\right) \\
&= \left[\left(\dot R\right)' e^{-b(t,r)} - R'\dot b e^{-b(t,r)}\right] dt \wedge d\phi + \left[R'' e^{-b(t,r)} + R' b' e^{-b(t,r)}\right] dr \wedge d\phi \\
&= -\left[\left(\dot R\right)' - R'\dot b\right]\frac{e^{-b(t,r)}}{R}\,\omega^{\hat \Phi} \wedge \omega^{\hat t} - \left[R'' + R'b'\right]\frac{e^{-2b(t,r)}}{R}\omega^{\hat \Phi} \wedge \omega^{\hat r} \\
\Gamma^{\hat \Phi}{}_{\hat c} \wedge \Gamma^{\hat c}{}_{\hat r} &= \Gamma^{\hat \Phi}{}_{\hat t} \wedge \Gamma^{\hat t}{}_{\hat r} + \Gamma^{\hat \Phi}{}_{\hat r} \wedge \Gamma^{\hat r}{}_{\hat r} + \Gamma^{\hat \Phi}{}_{\hat \Phi} \wedge \Gamma^{\hat \Phi}{}_{\hat r} = \Gamma^{\hat \Phi}{}_{\hat t} \wedge \Gamma^{\hat t}{}_{\hat r} = \frac{\dot R \dot b}{R}\omega^{\hat \Phi} \wedge \omega^{\hat r} \\
\Rightarrow \quad \Omega^{\hat \Phi}{}_{\hat r} &= -\left[\left(\dot R\right)' - R'\dot b\right]\frac{e^{-b(t,r)}}{R}\,\omega^{\hat \Phi} \wedge \omega^{\hat t} - \left(\left[R'' + R'b'\right]\frac{e^{-2b(t,r)}}{R} - \frac{\dot R \dot b}{R}\right)\omega^{\hat \Phi} \wedge \omega^{\hat r}
\end{aligned}
$$

Summarized in a matrix:

$$\Omega^{\hat{a}}{}_{\hat{b}} = \begin{Bmatrix} 0 & -\left[\ddot{b}+\left(\dot{b}\right)^2\right]\omega^{\hat{t}}\wedge\omega^{\hat{r}} & \begin{bmatrix} -\frac{\ddot{R}}{R}\omega^{\hat{\Phi}}\wedge\omega^{\hat{t}} \\ -\left(\frac{\left(\dot{R}\right)'}{R}-\frac{R'\dot{b}}{R}\right)e^{-b(t,r)}\omega^{\hat{\Phi}}\wedge\omega^{\hat{r}} \end{bmatrix} \\ S & 0 & \begin{bmatrix} -\left[\left(\dot{R}\right)'-R'\dot{b}\right]\frac{e^{-b(t,r)}}{R}\omega^{\hat{\Phi}}\wedge\omega^{\hat{t}} \\ -\left([R''+R'b']\frac{e^{-2b(t,r)}}{R}-\frac{\dot{R}\dot{b}}{R}\right)\omega^{\hat{\Phi}}\wedge\omega^{\hat{r}} \end{bmatrix} \\ S & AS & 0 \end{Bmatrix}$$

Where $\hat{a}$ refers to column and $\hat{b}$ to row.

8.5.1.2 The Riemann Tensor

Now we can find the independent elements of the Riemann tensor in the non-coordinate basis:

$$\mathrm{R}^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}}(A) = -\left[\ddot{b}+\left(\dot{b}\right)^2\right] \quad R^{\hat{\Phi}}{}_{\hat{t}\hat{\Phi}\hat{t}}(B) = -\frac{\ddot{R}}{R}$$

$$R^{\hat{\Phi}}{}_{\hat{t}\hat{\Phi}\hat{r}}(C) = -\left[\left(\dot{R}\right)'+R'\dot{b}\right]\frac{e^{-b(t,r)}}{R}$$

$$R^{\hat{\Phi}}{}_{\hat{r}\hat{\Phi}\hat{r}}(D) = -\left([R''+R'b']\frac{e^{-2b(t,r)}}{R}-\frac{\dot{R}\dot{b}}{R}\right)$$

Where A, B, C and D will be used later to make the calculations easier

8.5.2 Find the components of the Riemann tensor for the metric in 2+1 dimensions – alternative solution

The line element:

$$ds^2 = -dt^2 + e^{2b(t,r)}dr^2 + R^2(t,r)d\phi^2$$

Now we can compare with the Tolman-Bondi – de Sitter line element, where the primes should not be mistaken for the derivative d/dr.

$$ds^2 = dt'^2 - e^{-2\psi(t',r')}dr'^2 - R^2(t',r')d\theta'^2 - R^2(t',r')\sin^2\theta'\,d\phi'^2$$

And chose:

$$\begin{aligned} dt' &= dt \\ e^{-\psi(t',r')}dr' &= e^{b(t,r)}dr \\ R(t',r')d\theta' &= 0 \\ R(t',r')\sin\theta'\,d\phi' &= R(t,r)d\phi \end{aligned}$$

Comparing the two metrics we see: $d\phi' = d\phi, \theta' = \frac{\pi}{2}, R(t',r') = R(t,r), dt' = dt$

Next we can use the former calculations of the Tolman-Bondi – de Sitter metric to find the Riemann and Einstein tensor for the 2+1 metric.

But first we need to find

$$\begin{aligned} \dot{\psi} &= \frac{d\psi(t',r')}{dt'} = e^{-\psi(t',r')}\frac{d}{dt'}\left(e^{\psi(t',r')}\right) = e^{b(t,r)}\frac{dr}{dr'}\frac{d}{dt}\left(e^{-b(t,r)}\frac{dr'}{dr}\right) \\ &= -\frac{db(t,r)}{dt} = -\dot{b}(t,r) \\ \ddot{\psi} &= \frac{d^2\psi(t',r')}{dt'^2} = \frac{d}{dt}\left(-\dot{b}\right) = -\ddot{b}(t,r) \\ \psi' &= \frac{d\psi(t',r')}{dr'} = e^{-\psi(t',r')}\frac{d}{dr'}\left(e^{\psi(t',r')}\right) \\ &= e^{-\psi(t',r')}\frac{dr}{dr'}\frac{d}{dr}\left(e^{-b(t,r)}\frac{dr'}{dr}\right) = -e^{-\psi(t',r')}e^{-b(t,r)}b'(t,r) \end{aligned}$$

$$\dot{R} = \frac{dR(t',r')}{dt'} = \frac{dR(t,r)}{dt} = \dot{R}(t,r)$$
$$\ddot{R} = \frac{d^2R(t',r')}{dt'^2} = \frac{d^2R(t,r)}{dt^2} = \ddot{R}(t,r)$$
$$R' = \frac{dR(t',r')}{dr'} = \frac{dr}{dr'}\frac{dR(t,r)}{dr} = e^{-\psi(t',r')}e^{-b(t,r)}R'(t,r)$$
$$\dot{R}' = \frac{d^2R(t',r')}{dt'dr'} = \frac{d}{dr'}\left(\frac{dR(t',r')}{dt'}\right) = \frac{dr}{dr'}\frac{d}{dr}\left(\dot{R}(t,r)\right) = e^{-\psi(t',r')}e^{-b(t,r)}\dot{R}'(t,r)$$

8.5.2.1 The Riemann tensor

Tolman –Bondi – de Sitter 2+1

$$\mathrm{R}^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}} = \left[\ddot{\psi} - (\dot{\psi})^2\right] \quad \Rightarrow \quad \mathrm{R}^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}}(A) = -\left[\ddot{b} + (\dot{b})^2\right]$$
$$R^{\hat{\theta}}{}_{\hat{t}\hat{\theta}\hat{t}} = -\frac{\ddot{R}}{R}$$
$$R^{\hat{\theta}}{}_{\hat{t}\hat{\theta}\hat{r}} = -\left[(\dot{R})' + R'\dot{\psi}\right]\frac{e^{\psi(t,r)}}{R}$$
$$R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} = -\left[(R'' + R'\psi')\frac{e^{2\psi(t,r)}}{R} + \frac{\dot{R}\dot{\psi}}{R}\right]$$
$$R^{\hat{\phi}}{}_{\hat{t}\hat{\phi}\hat{t}} = -\frac{\ddot{R}}{R} \quad \Rightarrow \quad R^{\hat{\phi}}{}_{\hat{t}\hat{\phi}\hat{t}}(B) = -\frac{\ddot{R}}{R}$$
$$R^{\hat{\phi}}{}_{\hat{t}\hat{\phi}\hat{r}} = -\left[(\dot{R})' + R'\dot{\psi}\right]\frac{e^{\psi(t,r)}}{R} \quad \Rightarrow \quad R^{\hat{\phi}}{}_{\hat{t}\hat{\phi}\hat{r}}(C) = -\left[(\dot{R})' - R'\dot{b}\right]\frac{e^{-b(t,r)}}{R}$$
$$R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{r}} = -\left[(R'' + R'\psi')\frac{e^{2\psi(t,r)}}{R} + \frac{\dot{R}\dot{\psi}}{R}\right] \quad \Rightarrow \quad R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{r}}(D) = -\left[(R'' - R'b')\frac{e^{-2b(t,r)}}{R} - \frac{\dot{R}\dot{b}}{R}\right]$$
$$R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{\theta}} = \left[\frac{1}{R^2} + \frac{(\dot{R})^2}{R^2} - \frac{(R')^2}{R^2}e^{2\psi(t,r)}\right]$$

Where A, B, C and D will be used later to make the calculations easier

8.5.3 Find the components of the Einstein tensor in the coordinate basis for the metric in 2+1 dimensions.

8.5.3.1 The Ricci tensor

$$R_{\hat{a}\hat{b}} = R^{\hat{c}}{}_{\hat{a}\hat{c}\hat{b}}$$
$$R_{\hat{t}\hat{t}} = R^{\hat{c}}{}_{\hat{t}\hat{c}\hat{t}} = R^{\hat{t}}{}_{\hat{t}\hat{t}\hat{t}} + R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}} + R^{\hat{\phi}}{}_{\hat{t}\hat{\phi}\hat{t}} = R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}} + R^{\hat{\phi}}{}_{\hat{t}\hat{\phi}\hat{t}} = -\left[\ddot{b} + (\dot{b})^2\right] - \frac{\ddot{R}}{R} = A + B$$
$$R_{\hat{r}\hat{t}} = R^{\hat{c}}{}_{\hat{r}\hat{c}\hat{t}} = R^{\hat{t}}{}_{\hat{r}\hat{t}\hat{t}} + R^{\hat{r}}{}_{\hat{r}\hat{r}\hat{t}} + R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{t}} = R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{t}} = -\left[(\dot{R})' - R'\dot{b}\right]\frac{e^{-b(t,r)}}{R} = C$$
$$R_{\hat{\phi}\hat{t}} = R^{\hat{c}}{}_{\hat{\phi}\hat{c}\hat{t}} = R^{\hat{t}}{}_{\hat{\phi}\hat{t}\hat{t}} + R^{\hat{r}}{}_{\hat{\phi}\hat{r}\hat{t}} + R^{\hat{\theta}}{}_{\hat{\phi}\hat{\theta}\hat{t}} + R^{\hat{\phi}}{}_{\hat{\phi}\hat{\phi}\hat{t}} = 0$$
$$R_{\hat{r}\hat{r}} = R^{\hat{c}}{}_{\hat{r}\hat{c}\hat{r}} = R^{\hat{t}}{}_{\hat{r}\hat{t}\hat{r}} + R^{\hat{r}}{}_{\hat{r}\hat{r}\hat{r}} + R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{r}} = -R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}} + R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{r}}$$
$$= \left[\ddot{b} + (\dot{b})^2\right] - \left[(R'' - R'b')\frac{e^{-2b(t,r)}}{R} - \frac{\dot{R}\dot{b}}{R}\right] = -A + D$$
$$R_{\hat{\phi}\hat{\phi}} = R^{\hat{c}}{}_{\hat{\phi}\hat{c}\hat{\phi}} = R^{\hat{t}}{}_{\hat{\phi}\hat{t}\hat{\phi}} + R^{\hat{r}}{}_{\hat{\phi}\hat{r}\hat{\phi}} + R^{\hat{\phi}}{}_{\hat{\phi}\hat{\phi}\hat{\phi}} = -R^{\hat{\phi}}{}_{\hat{t}\hat{\phi}\hat{t}} + R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{r}}$$
$$= \frac{\ddot{R}}{R} - \left[(R'' - R'b')\frac{e^{-2b(t,r)}}{R} - \frac{\dot{R}\dot{b}}{R}\right] = -B + D$$

Summarized in a matrix:

$$R_{\hat{a}\hat{b}} = \begin{Bmatrix} -\left[\ddot{b}+\left(\dot{b}\right)^2\right]-\frac{\ddot{R}}{R} & -\left[\left(\dot{R}\right)'-R'\dot{b}\right]\frac{e^{-b}}{R} & 0 \\ S & \left[\ddot{b}+\left(\dot{b}\right)^2\right]-\left[(R''-R'b')\frac{e^{-2b}}{R}-\frac{\dot{R}\dot{b}}{R}\right] & 0 \\ 0 & 0 & \frac{\ddot{R}}{R}-\left[(R''-R'b')\frac{e^{-2b}}{R}-\frac{\dot{R}\dot{b}}{R}\right] \end{Bmatrix}$$

Where $\hat{a}$ refers to column and $\hat{b}$ to row

8.5.3.2 The Ricci scalar

$$R = \eta^{\hat{a}\hat{b}}R_{\hat{a}\hat{b}}$$

$$R = \eta^{\hat{t}\hat{t}}R_{\hat{t}\hat{t}} + \eta^{\hat{r}\hat{r}}R_{\hat{r}\hat{r}} + \eta^{\hat{\phi}\hat{\phi}}R_{\hat{\phi}\hat{\phi}} = -R_{\hat{t}\hat{t}} + R_{\hat{r}\hat{r}} + R_{\hat{\phi}\hat{\phi}} = -(A+B)+(-A+D)+(-B+D)$$

$$= -2A-2B+2D = -2R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}} - 2R^{\hat{\phi}}{}_{\hat{t}\hat{\phi}\hat{t}} + 2R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{r}}$$

$$= 2\left[\ddot{b}+\left(\dot{b}\right)^2\right] + 2\frac{\ddot{R}}{R} - 2\left[(R''-R'b')\frac{e^{-2b(t,r)}}{R}-\frac{\dot{R}\dot{b}}{R}\right]$$

8.5.3.3 The Einstein tensor

$$G_{\hat{a}\hat{b}} = R_{\hat{a}\hat{b}} - \frac{1}{2}\eta_{\hat{a}\hat{b}}R$$

$$G_{\hat{t}\hat{t}} = R_{\hat{t}\hat{t}} - \frac{1}{2}\eta_{\hat{t}\hat{t}}R = R_{\hat{t}\hat{t}} + \frac{1}{2}R = A+B+\frac{1}{2}(-2A-2B+2D) = D = R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{r}}$$

$$= -\left[(R''-R'b')\frac{e^{-2b}}{R}-\frac{\dot{R}\dot{b}}{R}\right]$$

$$G_{\hat{r}\hat{t}} = R_{\hat{r}\hat{t}} - \frac{1}{2}\eta_{\hat{r}\hat{t}}R = R_{\hat{r}\hat{t}} = R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{t}} = -\left[\left(\dot{R}\right)'-R'\dot{b}\right]\frac{e^{-b}}{R}$$

$$G_{\hat{\theta}\hat{t}} = R_{\hat{\theta}\hat{t}} - \frac{1}{2}\eta_{\hat{\theta}\hat{t}}R = 0$$

$$G_{\hat{r}\hat{r}} = R_{\hat{r}\hat{r}} - \frac{1}{2}\eta_{\hat{r}\hat{r}}R = R_{\hat{r}\hat{r}} - \frac{1}{2}R = -A+D-\frac{1}{2}(-2A-2B+2D) = B = R^{\hat{\phi}}{}_{\hat{t}\hat{\phi}\hat{t}} = -\frac{\ddot{R}}{R}$$

$$G_{\hat{\phi}\hat{\phi}} = R_{\hat{\phi}\hat{\phi}} - \frac{1}{2}\eta_{\hat{\phi}\hat{\phi}}R = R_{\hat{\phi}\hat{\phi}} - \frac{1}{2}R = -B+D-\frac{1}{2}(-2A-2B+2D) = A = R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}}$$

$$= -\left[\ddot{b}+\left(\dot{b}\right)^2\right]$$

Summarized in a matrix:

$$G_{\hat{a}\hat{b}} = \begin{Bmatrix} -\left[(R''-R'b')\frac{e^{-2b}}{R}-\frac{\dot{R}\dot{b}}{R}\right] & -\left[\left(\dot{R}\right)'-R'\dot{b}\right]\frac{e^{-b}}{R} & 0 \\ S & -\frac{\ddot{R}}{R} & 0 \\ 0 & 0 & -\left[\ddot{b}+\left(\dot{b}\right)^2\right] \end{Bmatrix}$$

Where $\hat{a}$ refers to column and $\hat{b}$ to row

8.5.3.4 The Einstein tensor in the coordinate basis

The transformation

$$G_{ab} = \Lambda^{\hat{c}}{}_a\Lambda^{\hat{d}}{}_bG_{\hat{c}\hat{d}}$$

$$G_{tt} = \Lambda^{\hat{c}}{}_t\Lambda^{\hat{d}}{}_tG_{\hat{c}\hat{d}} = \left(\Lambda^{\hat{t}}{}_t\right)^2G_{\hat{t}\hat{t}} = -\left[(R''-R'b')\frac{e^{-2b}}{R}-\frac{\dot{R}\dot{b}}{R}\right]$$

$$G_{rt} = \Lambda^{\hat{c}}{}_r \Lambda^{\hat{d}}{}_t G_{\hat{c}\hat{d}} = \Lambda^{\hat{r}}{}_r \Lambda^{\hat{t}}{}_t G_{\hat{r}\hat{t}} = -\frac{\left[(\dot{R})' - R'\dot{b}\right]}{R}$$
$$G_{rr} = \Lambda^{\hat{c}}{}_r \Lambda^{\hat{d}}{}_r G_{\hat{c}\hat{d}} = \left(\Lambda^{\hat{r}}{}_r\right)^2 G_{\hat{r}\hat{r}} = -\frac{\ddot{R}}{R} e^{2b}$$
$$G_{\phi\phi} = \Lambda^{\hat{c}}{}_\phi \Lambda^{\hat{d}}{}_\phi G_{\hat{c}\hat{d}} = \left(\Lambda^{\hat{\phi}}{}_\phi\right)^2 G_{\hat{\phi}\hat{\phi}} = -R^2\left[\ddot{b} + (\dot{b})^2\right]$$

Summarized in a matrix

$$G_{ab} = \begin{Bmatrix} -\left[(R'' - R'b')\frac{e^{-2b}}{R} - \frac{\dot{R}\dot{b}}{R}\right] & -\frac{\left[(\dot{R})' - R'\dot{b}\right]}{R} & 0 \\ S & -\frac{\ddot{R}}{R}e^{2b} & 0 \\ 0 & 0 & -R^2\left[\ddot{b} + (\dot{b})^2\right] \end{Bmatrix}$$

Where a refers to column and b to row

8.5.4 The Einstein equations of the metric in 2+1 dimensions.

Given the Einstein equation (if $c = G = 1$):

$$G_{\hat{a}\hat{b}} + \Lambda\eta_{\hat{a}\hat{b}} = \kappa T_{\hat{a}\hat{b}}$$

with $\Lambda = -\lambda^2$ you get

$$G_{\hat{a}\hat{b}} - \lambda^2\eta_{\hat{a}\hat{b}} = \kappa T_{\hat{a}\hat{b}}$$

and the stress-energy tensor:

$$T_{\hat{a}\hat{b}} = \kappa \begin{Bmatrix} \rho & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{Bmatrix}$$

You can find the Einstein – equations

$$\begin{Bmatrix} -\left[(R'' - R'b')\frac{e^{-2b}}{R} - \frac{\dot{R}\dot{b}}{R}\right] & -\left[(\dot{R})' - R'\dot{b}\right]\frac{e^{-b}}{R} & 0 \\ S & -\frac{\ddot{R}}{R} & 0 \\ 0 & 0 & -\left[\ddot{b} + (\dot{b})^2\right] \end{Bmatrix} - \lambda^2 \begin{Bmatrix} -1 & & \\ & 1 & \\ & & 1 \end{Bmatrix} = \kappa \begin{Bmatrix} \rho & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{Bmatrix}$$

$$G_{\hat{t}\hat{t}}: \quad -\left[(R'' - R'b')\frac{e^{-2b}}{R} - \frac{\dot{R}\dot{b}}{R}\right] + \lambda^2 = \kappa\rho \qquad (8.10.)$$
$$G_{\hat{r}\hat{t}}: \quad -\left[(\dot{R})' - R'\dot{b}\right]\frac{e^{-b}}{R} = 0$$
$$\Leftrightarrow \quad (\dot{R})' - R'\dot{b} = 0 \qquad (8.11.)$$
$$G_{\hat{r}\hat{r}}: \quad -\frac{\ddot{R}}{R} - \lambda^2 = 0$$
$$\Leftrightarrow \quad \ddot{R} + \lambda^2 R = 0 \qquad (8.12.)$$
$$G_{\hat{\phi}\hat{\phi}}: \quad -\left[\ddot{b} + (\dot{b})^2\right] - \lambda^2 = 0 \qquad (8.13.)$$

8.6 [f]The de Sitter Spacetime

The de Sitter spacetime is an example of the Robertson Walker [14]space-time in vacuum with positive curvature and a cosmological constant. The solution to the Einstein equations are the Friedman equations[15]. The line element

$$ds^2 = -dt^2 + \frac{1}{k^2}\cosh^2(kt)\left(d\theta^2 + \sin^2\theta\,(d\phi^2 + \sin^2\phi\,d\psi^2)\right)$$

8.6.1 The Friedmann equations with a cosmological constant

$$0 = \frac{3}{a^2}(1+\dot{a}^2) - \Lambda$$

$$0 = 2\frac{\ddot{a}}{a} + \frac{1}{a^2}(1+\dot{a}^2) - \Lambda$$

We chose $k^2 = \frac{\Lambda}{3}$

$$0 = \frac{1}{a^2}(1+\dot{a}^2) - k^2$$

$$\Rightarrow \quad -1 = \dot{a}^2 - k^2a^2$$

$$0 = 2\frac{\ddot{a}}{a} + \frac{1}{a^2}(1+\dot{a}^2) - 3k^2 = 2\frac{\ddot{a}}{a} + k^2 - 3k^2 = 2\frac{\ddot{a}}{a} - 2k^2$$

$$\Rightarrow \quad 0 = \ddot{a} - k^2a \qquad (8.14.)$$

8.6.2 Solving the Friedmann equations

These equations we can solve. The general solution to eq.(8.14.) is

$$a = Ae^{kt} + Be^{-kt}$$

$$\Rightarrow \quad a^2 = \left(Ae^{kt} + Be^{-kt}\right)^2 = A^2e^{2kt} + B^2e^{-2kt} + 2AB$$

$$\Rightarrow \quad \dot{a} = Ake^{kt} - Bke^{-kt}$$

$$\Rightarrow \quad \dot{a}^2 = \left(Ake^{kt} - Bke^{-kt}\right)^2 = k^2(A^2e^{2kt} + B^2e^{-2kt} - 2AB)$$

$$\ddot{a} = Ak^2e^{kt} + Bk^2e^{-kt} = k^2a$$

$$\Rightarrow \quad 0 = \ddot{a} - k^2a = k^2a - k^2a = 0$$

$$\Rightarrow \quad -1 = \dot{a}^2 - k^2a^2 = k^2\left(A^2e^{2kt} + B^2e^{-2kt} - 2AB\right) - k^2\left(A^2e^{2kt} + B^2e^{-2kt} + 2AB\right)$$

$$= -4k^2AB$$

$$\Rightarrow \quad \frac{1}{k^2} = 4AB$$

We choose $A = B$

$$\Rightarrow \quad a^2 = \left(Ae^{kt} + Be^{-kt}\right)^2 = A^2\left(e^{kt} + e^{-kt}\right)^2 = \frac{1}{4k^2}2^2\cosh^2(kt) = \frac{1}{k^2}\cosh^2(kt)$$

And find the de Sitter line element

$$ds^2 = -dt^2 + \frac{1}{k^2}\cosh^2(kt)\left(d\theta^2 + \sin^2\theta\,(d\phi^2 + \sin^2\phi\,d\psi^2)\right)$$

8.7 [g]The Anti-de Sitter Spacetime

The line element

$$ds^2 = -dt^2 + \cos^2(t)\,dr^2 + \cos^2(t)\sinh^2(r)\,d\theta^2 + \cos^2(t)\sinh^2(r)\sin^2\theta\,d\phi^2$$

To show that this spacetime is a solution to the Einstein vacuum equation with a cosmological constant $\Lambda = -3$ we can compare to the Tolman-Bondi de Sitter spacetime[16]

[14] $ds^2 = -dt^2 + a(t)^2\left(d\theta^2 + \sin^2\theta\,(d\phi^2 + \sin^2\phi\,d\psi^2)\right)$

[15] See the chapter named: *The Einstein tensor and Friedmann-equations for the Robertson Walker metric*

[16] See the chapter named: *The Einstein Tensor of the Tolman-Bondi de Sitter Metric*

$$ds^2 = {}^{17}dt^2 - e^{-2\psi(t,r)}dr^2 - R^2(t,r)d\theta^2 - R^2(t,r)\sin^2\theta\, d\phi^2$$

Comparing the two metrics

$$e^{-2\psi(t,r)} = \cos^2(t)$$
$$R^2(t,r) = \cos^2(t)\sinh^2(r)$$
$$\Rightarrow \quad R(t,r) = \pm\cos(t)\sinh(r)$$

We need

$$\frac{d\left(e^{-2\psi(t,r)}\right)}{dt} = \frac{d(\cos^2(t))}{dt} = -2\cos(t)\sin(t) = -2\dot{\psi}e^{-2\psi(t,r)}$$
$$\Rightarrow \quad \dot{\psi} = \frac{\cos(t)\sin(t)}{e^{-2\psi(t,r)}} = \frac{\cos(t)\sin(t)}{\cos^2(t)} = \tan(t)$$
$$\ddot{\psi} = \frac{d\tan(t)}{dt} = \frac{1}{\cos^2(t)}$$
$$\psi' = 0$$
$$\dot{R} = \frac{d(\pm\cos(t)\sinh(r))}{dt} = \mp\sin(t)\sinh(r)$$
$$\ddot{R} = \mp\cos(t)\sinh(r)$$
$$4\frac{\ddot{R}}{R} = 4\frac{\mp\cos(t)\sinh(r)}{\pm\cos(t)\sinh(r)} = -4$$
$$R' = \pm\cos(t)\cosh(r)$$
$$R'' = \pm\cos(t)\sinh(r) = R(t,r)$$
$$\Rightarrow \quad 2\left[\ddot{\psi} - \left(\dot{\psi}\right)^2\right] = 2\left[\frac{1}{\cos^2(t)} - \tan^2(t)\right] = 2\frac{1}{\cos^2(t)}[1-\sin^2(t)] = 2$$
$$(R'' + R'\psi')\frac{e^{2\psi(t,r)}}{R} = e^{2\psi(t,r)} = \frac{1}{\cos^2(t)}$$
$$\frac{\dot{R}\dot{\psi}}{R} = \frac{\mp\sin(t)\sinh(r)\tan(t)}{\pm\cos(t)\sinh(r)} = -\tan^2(t)$$
$$\frac{\left(\dot{R}\right)^2}{R^2} = \frac{(\mp\sin(t)\sinh(r))^2}{\cos^2(t)\sinh^2(r)} = \tan^2(t)$$
$$\frac{(R')^2}{R^2}e^{2\psi(t,r)} = \frac{(\pm\cos(t)\cosh(r))^2}{\cos^2(t)\sinh^2(r)\cos^2(t)} = \frac{1}{\cos^2(t)\tanh^2(r)}$$

8.7.1 The Ricci scalar

$$R = 2\left[\ddot{\psi} - \left(\dot{\psi}\right)^2\right] - 4\frac{\ddot{R}}{R} + 4\left[(R''+R'\psi')\frac{e^{2\psi(t,r)}}{R} + \frac{\dot{R}\dot{\psi}}{R}\right] - 2\left[\frac{1}{R^2} + \frac{\left(\dot{R}\right)^2}{R^2} - \frac{(R')^2}{R^2}e^{2\psi(t,r)}\right]$$
$$= 2 - (-4) + 4\left[\frac{1}{\cos^2(t)} - \tan^2(t)\right] - 2\left[\frac{1}{\cos^2(t)\sinh^2(r)} + \tan^2(t) - \frac{1}{\cos^2(t)\tanh^2(r)}\right]$$
$$= 10 - \frac{2}{\cos^2(t)}\left[\sin^2(t) + \frac{1}{\sinh^2(r)}(1-\cosh^2(r))\right] = 10 - \frac{2}{\cos^2(t)}[\sin^2(t) - 1] = 12$$

Hence the Ricci scalar we are looking for is $R = -12$

The vacuum Einstein equation for a metric with a cosmological constant is

$$0 = R_{ab} - \frac{1}{2}g_{ab}R + g_{ab}\Lambda$$
$$\Rightarrow \quad 0 = {}^{18}g^{ab}R_{ab} - \frac{1}{2}g^{ab}g_{ab}R + g^{ab}g_{ab}\Lambda = R - \frac{1}{2}4R + 4\Lambda = -R + 4\Lambda$$
$$\Rightarrow \quad \Lambda = \frac{R}{4} = -\frac{12}{4} = -3$$

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[17] Notice the two spacetimes do not have the same signature and therefore thee Ricci scalar changes sign. Chapter 2.

[18] Notice: This is equivalent with the former paragraph 8.2

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[a] (McMahon, 2006, p. 138)

[b] An excellent qualitative explanation of the cosmological constant, you can find in (Greene, 2004, s. 273-279)

[c] (d'Inverno, 1992, p. 172)

[d] (McMahon, 2006, p. 152), (d'Inverno, 1992, p. 87), (Kay, 1988, s. 120, 125)

[e] (McMahon, 2006, pp. 139-150)

[f] (Choquet-Bruhat, 2015, s. 96), (Ellis, 1973, p. 124)

[g] (Choquet-Bruhat, 2015, s. 97), (Ellis, 1973, p. 124)