15 Appendix A: Tensor Calculus

Content

15.1 The Einstein Summation Convention

15.1.1 [a]Definition

Any index repeated in a product is automatically summed on. A repeated index is named a dummy, other indices are named free indices.

$$A_i B^i = \sum_{i=0}^{n} A_i B^i \qquad i\ dummy, range\ n \quad n\ terms$$

$$\frac{\partial x^{\overline{\alpha}}}{\partial x^{\beta}} \xi^{\beta} = \sum_{\beta=0}^{n} \frac{\partial x^{\overline{\alpha}}}{\partial x^{\beta}} \xi^{\beta} \qquad \beta\ dummy, range\ n \quad \alpha\ free, range\ m \quad n\ terms \quad m\ equations$$

$$dx'^a = \sum_{b=0}^{n} \frac{\partial x'^a}{\partial x^b} dx^b = \frac{\partial x'^a}{\partial x^b} dx^b \qquad b\ dummy, range\ n \quad {}^{1}a\ free, range\ m \quad n\ terms \quad m\ equations$$

15.1.2 [b]Repeated Indices and dummies

$$Q = a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4 + a_5 b_5 + a_6 b_6 \qquad i\ dummy, range\ 6 \quad 6\ terms$$

$$Q = R^i{}_{jki} = R^1{}_{jk1} + R^2{}_{jk2} + R^3{}_{jk3} + R^4{}_{jk4} \qquad i\ dummy, range\ 4 \quad j,k\ free, range\ n \quad 4\ terms$$

[1] Notice: This describes a coordinate transformation so $n = m$

n^2 equations

$$Q = a_{i3}b_{i3} = a_{13}b_{13} + a_{23}b_{23} + a_{33}b_{33}$$

i dummy, range 3
3 terms

$$\begin{aligned} Q &= a_{ij}x_i x_j \\ &= a_{11}(x_1)^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{21}x_2x_1 \\ &+a_{22}(x_2)^2 + a_{23}x_2x_3 + a_{31}x_3x_1 + a_{32}x_3x_2 + a_{33}(x_{33})^2 \end{aligned}$$

i, j dummy, range 3
$3^2 = 9$ terms

$$a_{ii}x_k = a_{11}x_k + a_{22}x_k + \cdots a_{mm}x_k$$

i dummy, range m
k free, range n
m terms
k equations

$$a_{ij}x_j = a_{i1}x_1 + a_{i2}x_2 + \cdots a_{im}x_m$$

j dummy, range m
i free, range n
m terms
i equations

$$y_i = a_{ir}x_r$$

r dummy, range n
i free, range m
n terms
m equations

$$c^i_{jj} = c^i_{11} + c^i_{22} + c^i_{33} + c^i_{44}$$

j dummy, range 4
i free, range n
4 terms
n equations

$$\begin{aligned} d_1 &= c_{11}x_1 + c_{12}x_2 + c_{13}x_3 \\ d_2 &= c_{21}x_1 + c_{22}x_2 + c_{33}x_3 \\ d_3 &= c_{31}x_1 + c_{32}x_2 + c_{33}x_3 \\ \Rightarrow \quad d_i &= c_{ij}x_j \end{aligned}$$

j dummy, range 3
i free, range 3
3 terms
3 equations

$$b_j = a^1_j x_1 + a^2_j x_2 + a^3_j x_3 = a^i_j x_i$$

i dummy, range 3
j free, range n
3 terms
n equations

$$Q = c_i(x_i + y_i) = c_i z_i$$

i dummy, range n
n terms

$$Q = c_i x_i + c_j y_j$$

i dummy, range n
j dummy, range m
$n + m$ terms

$$Q = g^i_{jk}$$

i free, range n
j free, range m
k free, range q

$n \times m \times q$ terms

15.1.3 [c]Double and Triple Sums

$$\begin{aligned} y_1 &= c_{11}x_1 + c_{12}x_2 \\ y_2 &= c_{21}x_1 + c_{22}x_2 \\ \Rightarrow \quad y_i &= c_{ij}x_j \end{aligned}$$
j dummy, range 2; i free, range 2; 2 terms; 2 equations

$$z_{ij} = a_{ij}x_i y_j$$
i, j dummy, range n; n^2 terms

$$y_i = c_i^r a_{rs} x_s$$
r, s dummy, range n; i free, range m; n^2 terms; m equations

$$Q = a_{ij}b_{ji} = a_{11}b_{11} + a_{21}b_{12} + a_{31}b_{13} + a_{41}b_{14}$$
i dummy, range 4; j dummy, range 1; 4 terms

$$Q = a_{ji}b_{ji} = a_{11}b_{11} + a_{12}b_{12} + a_{13}b_{13}$$
i dummy, range 3; j dummy, range 1; 3 terms

$$Q = a^{ij}x_i x_j$$
i, j dummy, range n; n^2 terms

$$Q = a_{ij}b_{ji}$$
i, j dummy, range n; n^2 terms

$$Q = c_{rst}x^r y^s z^t$$
r, s, t dummy, range n; n^3 terms

$$\begin{aligned} Q &= (Z_{abc} + Z_{cab} + Z_{bca})X^a X^b X^c \\ &= Z_{abc}X^a X^b X^c + Z_{cab}X^a X^b X^c + Z_{bca}X^a X^b X^c \\ &= Z_{abc}X^a X^b X^c + Z_{abc}X^b X^c X^a + Z_{abc}X^c X^a X^b \\ &= 3Z_{abc}X^a X^b X^c \end{aligned}$$
a, b, c dummy, range n; n^3 terms

15.1.4 [d]Substitutions

$$\begin{aligned} y_i &= a_{ij}x_j \\ \Rightarrow \quad Q &= g_{ij}y_i y_j = g_{ij}(a_{ik}x_k)(a_{jl}x_l) = g_{ij}a_{ik}a_{jl}x_k x_l = b_{kl}x_k x_l \end{aligned}$$
i, j, k, l dummy, range n; $b_{kl} = g_{ij}a_{ik}a_{jl}$; n^2 terms

$$\begin{aligned} y_i &= T^{jj}{}_i \\ \Rightarrow \quad x_j &= b^i{}_j T^{kk}{}_i \end{aligned}$$
i dummy, range n; k dummy; j free, range m; n terms; m equations

$$y_i = b_{ij}x_j$$
j, k dummy, range n

$$\Rightarrow \quad z_i = a_{ij}y_j = a_{ij}b_{jk}x_k = c_{ik}x_k$$

i free, range m; $c_{ik} = a_{ij}b_{jk}$; n terms; m equations

$$y_i = a_{ij}x_j$$
$$\Rightarrow \quad Q = b_{ij}y_i x_j = b_{ij}(a_{ir}x_r)x_j = a_{ir}b_{ij}x_r x_j$$

i, j, r dummy, range n; n^3 terms

$$y_i = b_{ij}x_j$$
$$\begin{aligned} \Rightarrow \quad Q &= a_{ijk}y_i y_j y_k \\ &= a_{ijk}(b_{il}x_l)(b_{jm}x_m)(b_{kn}x_n) \\ &= a_{ijk}b_{il}b_{jm}b_{kn}x_l x_m x_n \\ &= c_{lmn}x_l x_m x_n \end{aligned}$$

$i, j\ k, l, m, n$ dummy, range q; $c_{lmn} = a_{ijk}b_{il}b_{jm}b_{kn}$; q^3 terms

$$a_{ij} = i - j$$
$$Q = a_{ij}x_i x_j = -a_{ji}x_i x_j$$
$$\begin{aligned} \Rightarrow \quad Q &= \sum a_{nn}(x_n)^2 + \sum_{n \neq m} a_{nm}x_n x_m \\ &= \sum (n-n)(x_n)^2 + \sum_{n \neq m}(n-m)x_n x_m \\ &= {}^{2}0 \end{aligned}$$

i, j dummy, range n; n^2 terms

15.2 [e]Kronecker Delta

15.2.1 [f]Definition

$$\delta_b^a = \begin{cases} 1 \ if \ a = b \\ 0 \ if \ a \neq b \end{cases}$$

15.2.2 Repeated indices, sums and substitutions

$$\delta_j^i x_i = x_j$$

i dummy; j free, range n; n equations

$$\delta_a^b X^a = X^b$$

a dummy; b free, range n; n equations

$$\delta_a^b X_b = X_a$$

b dummy; a free, range n; n equations

$$\delta_i^j x_i x_j = x_i(\delta_i^j x_j) = x_i x_i = (x_i)^2$$

i, j dummy, range n; n terms

$$\delta_j^r a_{ir}x_i = (\delta_j^r a_{ir})x_i = a_{ij}x_i$$

i dummy, range n; j free, range m; n terms

[2] The off-diagonal elements cancel off in pairs

$m\ equations$

$$\delta_i^i = n$$
i dummy, range n; n terms

$$\delta_i^j \delta_j^i = \delta_i^i = n$$
i, j dummy, range n; n terms

$$\delta_i^j \delta_j^k c_{ik} = \delta_i^k c_{ik} = c_{ii}$$
i dummy, range m; k dummy; j dummy, range n; m terms

$$\begin{aligned} y_i &= b_{ir} x_r \\ a_{ir} b_{rj} &= \delta_i^j \\ \Rightarrow \quad T^i &= g_r^i a_{rs} y_s = g_r^i a_{rs} (b_{sj} x_j) = g_r^i \delta_r^j x_j = g_r^i x_r \end{aligned}$$
r, s, j dummy range n; i, free range m; n terms; i equations

$$\begin{aligned} y_i &= c_{ij} x_j \\ b^{ij} c_{ik} &= \delta_k^j \\ \Rightarrow \quad Q &= b^{ij} y_i y_j = b^{ij} (c_{ik} x_k)(c_{jl} x_l) = \delta_k^j c_{jl} x_k x_l = c_{kl} x_k x_l \end{aligned}$$
$i, j, k\ l$ dummy, range n; n^2 terms

$$\begin{aligned} y^i &= a^{ij} x_j \\ b_{ir} a^{rj} &= \delta_i^j \\ \Rightarrow \quad x_i &= \delta_i^j x_j = (b_{ir} a^{rj}) x_j = b_{ir} y^r \end{aligned}$$
j, r dummy, range n; i free, range m; n terms; m equations

$$\delta_a^b \delta_b^c \delta_c^d = \delta_a^b \delta_b^d = \delta_a^d$$
b, c dummy; a, d free

15.2.3 Partial differentiation

$$\begin{aligned} y_k &= \frac{\partial}{\partial x_k}(a_{ij} x_i x_j) \\ &= a_{ij} x_j \frac{\partial x_i}{\partial x_k} + a_{ij} x_i \frac{\partial x_j}{\partial x_k} \\ &= a_{ij} x_j \delta_{ik} + a_{ij} x_i \delta_{jk} \\ &= a_{kj} x_j + a_{ik} x_i \\ &= a_{ki} x_i + a_{ik} x_i \\ &= (a_{ki} + a_{ik}) x_i \end{aligned}$$
i, j dummy, range n; k free, range m; n terms; m equations

$$\begin{aligned} b_{lk} &= \frac{\partial^2}{\partial x_k \partial x_l}(a_{ij} x_i x_j) \\ &= \frac{\partial}{\partial x_k}\left[\frac{\partial}{\partial x_l}(a_{ij} x_i x_j)\right] \\ &= \frac{\partial}{\partial x_k}[(a_{li} + a_{il}) x_i] \\ &= (a_{li} + a_{il}) \frac{\partial x_i}{\partial x_k} \end{aligned}$$
i, j dummy; k, l free, range n; n^2 equations

$$= (a_{li} + a_{il})\delta_{ik}$$
$$= a_{lk} + a_{kl}$$

$$b_k = \frac{\partial}{\partial x_k}(a_{11}x_1 + a_{12}x_2 + a_{13}x_3)$$
$$= \frac{\partial}{\partial x_k}(a_{1i}x_i)$$
$$= a_{1i}\frac{\partial x_i}{\partial x_k}$$
$$= a_{1i}\delta_{ik}$$
$$= a_{1k}$$

$i\ dummy, range\ 3$
$k\ free, range\ 3$
$3\ equations$

$$b_{ik} = \frac{\partial}{\partial x_k}(a_{ij}x_j) = a_{ij}\frac{\partial x_j}{\partial x_k} = a_{ij}\delta_{jk} = a_{ik}$$

$j\ dummy, range\ l$
$i\ free, range\ m$
$k\ free, range\ n$
$l\ terms$
$m * n\ equations$

$$y_k = \frac{\partial}{\partial x_k}\left[a_{ij}x_i(x_j)^2\right]$$
$$= a_{ij}(x_j)^2\frac{\partial x_i}{\partial x_k} + 2a_{ij}x_ix_j\frac{\partial x_j}{\partial x_k}$$
$$= a_{ij}(x_j)^2\delta_{ik} + 2a_{ij}x_ix_j\delta_{jk}$$
$$= a_{kj}(x_j)^2 + 2a_{ik}x_ix_k$$
$$= a_{kj}(x_j)^2 + 2a_{jk}x_jx_k$$
$$= a_{kj}\left[(x_j)^2 + 2x_jx_k\right]$$

$i, j\ dummy, range\ m$
$k\ free, range\ n, no\ summation$
$m\ terms$
$n\ equations$

$$y_l = \frac{\partial}{\partial x_l}(a_{ijk}x_ix_jx_k)$$
$$= a_{ijk}x_jx_k\frac{\partial x_i}{\partial x_l} + a_{ijk}x_ix_k\frac{\partial x_j}{\partial x_l} + a_{ijk}x_ix_j\frac{\partial x_k}{\partial x_l}$$
$$= a_{ijk}x_jx_k\delta_l^i + a_{ijk}x_ix_k\delta_l^j + a_{ijk}x_ix_j\delta_l^k$$
$$= a_{ljk}x_jx_k + a_{ilk}x_ix_k + a_{ijl}x_ix_j$$
$$= a_{lik}x_ix_k + a_{ilk}x_ix_k + a_{ikl}x_ix_k$$
$$= (a_{lik} + a_{ilk} + a_{ikl})x_ix_k$$

$i, j, k\ dummy, range\ m$
$l\ free, range\ n$
$m^2\ terms$
$n\ equations$

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[a] (Charles Misner, 2017, s. 9), (McMahon, 2006, s. 28), (d'Inverno, 1992, p. 59)

[b] (Kay, 1988, s. 1)
[c] (d'Inverno, 1992, p. 67), (Kay, 1988, s. 2)
[d] (Kay, 1988, s. 2)
[e] (Kay, 1988, s. 3)
[f] (d'Inverno, 1992, p. 59)