Content

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14 Gravitational waves

14.1 Space-times

This document includes many different space-time examples. In order to keep track of them I have made this list- in alphabetical order.

Space-time		Line-element	Chap-ter
Aichelburg-Sexl Solution	ds^2	$= 4\mu \log(x^2 + y^2)\, du^2 + 2dudr - dx^2 - dy^2$	14
Brinkman space-time	ds^2	$= H(u,x,y)du^2 + 2dudv - dx^2 - dy^2$	14
Colliding gravitational waves	ds^2	$= \delta(u)(X^2 - Y^2)du^2 + 2dudr - dX^2 - dY^2$	14
Collision of a gravitational wave with a electromagnetic wave	ds^2	$= 2dudv - \cos^2 av\,(dx^2 + dy^2)$	14
Generalized Nariai space-time	ds^2	$= -A\Lambda v^2 du^2 + Bdudv - \frac{C}{\Omega^2}(dx^2 + dy^2)$	14
Impulsive gravitational wave	ds^2	$= 2dudv - [1 - v\Theta(v)]^2 dx^2 - [1 + v\Theta(v)]^2 dy^2$	14
Linearized metric	ds^2	$= (\eta_{ab} + \epsilon h_{ab})dx^a dx^b$	9, 12, 14
Nariai space-time	ds^2	$= -\Lambda v^2 du^2 + 2dudv - \frac{1}{\Omega^2}(dx^2 + dy^2)$	14
Penrose-Kahn metric	ds^2	$= 2dudv - (1-u)^2 dx^2 - (1+u)^2 dy^2$	14
Plane waves: $h_{ab} = h_{ab}(t-z)$	ds^2	$= (\eta_{ab} + \epsilon h_{ab})dx^a dx^b$	14
Rosen line element	ds^2	$= dUdV - a^2(U)dx^2 - b^2(U)dy^2$	14
Two interacting waves	ds^2	$= 2dudv - \cos^2 av\, dx^2 - \cosh^2 av\, dy^2$	14

14.2 [a]The delta – $\boldsymbol{\delta(u)}$ and heavy-side – $\boldsymbol{\Theta(u)}$ functions

14.2.1 Definitions

The delta-function

$$\delta(u) = \begin{cases} +\infty \; if \; u = 0 \\ 0 \; if \; u \neq 0 \end{cases};$$

$$\int_{-\infty}^{\infty} \delta(u) du = 1$$

he heavy-side-function

$$\Theta(u) = \frac{|x|+x}{2x} = \begin{cases} 0 \; if \; u \leq 0 \\ 1 \; if \; u > 0 \end{cases};$$

$$\frac{d\Theta(u)}{du} = \Theta'(u) = \delta(u);$$

$$\Theta(u) = \int_{-\infty}^{u} \delta(u) du$$

14.2.2 Examples-formulas

$$\begin{aligned}\int_{-\infty}^{\infty} f(u)\delta(u)du &= \int_{-\infty}^{\infty} f(u)\Theta'(u)du \\ &= [f(u)\Theta(u)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(u)\Theta(u)du \\ &= f(\infty) - \int_{0}^{\infty} f'(u)du \\ &= f(\infty) - \big(f(\infty) - f(0)\big) \\ &= f(0)\end{aligned}$$

if $f(u) = u$ we find

$$\int_{-\infty}^{\infty} f(u)\delta(u)du = \int_{-\infty}^{\infty} u\delta(u)du = \int_{-\infty}^{\infty} 0\cdot\delta(u)du = 0$$

We assume that $u\delta(u) = 0$.
Multiplying both sides with a test function $g(u)$ and integrating we get

$$\begin{aligned}u\delta(u) &= 0 \\ \int_{-\infty}^{\infty} ug(u)\delta(u)du &= \int_{-\infty}^{\infty} g(u)\cdot 0du \\ \Leftrightarrow \quad 0\cdot g(0) &= 0\end{aligned}$$

which is consistent with our initial assumption.
Next we calculate

$$\begin{aligned}\int_{-\infty}^{\infty} f(u)\delta'(u)du &= [f(u)\delta(u)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(u)\delta(u)du \\ &= [f(u)\delta(u)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(u)\Theta'(u)du \\ &= [f(u)\delta(u)]_{-\infty}^{\infty} - \left([f'(u)\Theta(u)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f''(u)\Theta(u)du\right) \\ &= 0 - \left(f'(\infty) - \int_{0}^{\infty} f''(u)du\right) \\ &= -\Big(f'(\infty) - \big(f'(\infty) - f'(0)\big)\Big) \\ &= -f'(0)\end{aligned}$$

if $f(u) = u$ we find

$$\int_{-\infty}^{\infty} f(u)\delta'(u)du = \int_{-\infty}^{\infty} u\delta'(u)du = -1$$

Next we assume that $-\delta(u) = u\delta'(u)$.
Multiplying both sides with a test function $f(u)$ and integrating we get

$$\begin{aligned}\int_{-\infty}^{\infty} -f(u)\delta(u)du &= \int_{-\infty}^{\infty} f(u)u\delta'(u)du \\ \Leftrightarrow \quad -f(0) &= -(f(u)u)'(u=0) = -\big(f'(u)\cdot u + f(u)\big)(u=0) = -f(0)\end{aligned}$$

which is consistent with our initial assumption and we can therefore conclude that

$$-\delta(u) = u\delta'(u)$$

Collecting the results

$$\begin{aligned}\Theta'(u) &= \delta(u) \\ \int_{-\infty}^{\infty} f(u)\delta(u)du &= f(0)\end{aligned}$$

$$\begin{aligned}
\int_{-\infty}^{\infty} f(u)\delta'(u)du &= {}^{1} -f'(0) \\
u\delta(u) &= u\Theta'(u) = 0 \\
u\delta'(u) &= -\delta(u)
\end{aligned}$$

14.3 [b]Linearized Metric

14.3.1 The metric tensor and it's inverse

The metric tensor

$$g_{ab} = {}^{2}\eta_{ab} + \epsilon h_{ab}$$

We assume the inverse metric tensor can be written as

$$g^{ab} = \eta^{ab} + k\epsilon h^{ab}$$

We calculate, ignoring terms of order ϵ^2

$$\begin{aligned}
\delta_a^c &= g_{ab}g^{bc} \\
&= (\eta_{ab} + \epsilon h_{ab})(\eta^{bc} + k\epsilon h^{bc}) \\
&= \eta_{ab}\eta^{bc} + \epsilon(h_{ab}\eta^{bc} + k\eta_{ab}h^{bc}) \\
&= \delta_a^c + \epsilon(h_{ab}\eta^{bc} + k\eta_{ab}h^{bc}) + k\epsilon^2 h_{ab}h^{bc} \\
\Rightarrow \quad h_{ab}\eta^{bc} &= -k\eta_{ab}h^{bc} = -k\eta_{ab}\eta^{bd}\eta^{ce}h_{de} = -k\delta_a^d\eta^{ce}h_{de} = -k\eta^{ce}h_{ae} = -k\eta^{cb}h_{ab} \\
\Rightarrow \quad k &= -1 \\
\Rightarrow \quad g^{ab} &= \eta^{ab} - \epsilon h^{ab}
\end{aligned}$$

14.3.2 Christoffel symbols.

Ignoring terms of order ϵ^2.

$$\begin{aligned}
\Gamma^a{}_{bc} &= g^{ad}\Gamma_{bcd} = \frac{1}{2}g^{ad}\left(\frac{\partial g_{bd}}{\partial x^c} + \frac{\partial g_{cd}}{\partial x^b} - \frac{\partial g_{bc}}{\partial x^d}\right) \\
&= \frac{1}{2}(\eta^{ad} - \epsilon h^{ad})\epsilon\left(\frac{\partial h_{bd}}{\partial x^c} + \frac{\partial h_{cd}}{\partial x^b} - \frac{\partial h_{bc}}{\partial x^d}\right) \\
&= \frac{\epsilon}{2}\eta^{ad}\left(\frac{\partial h_{bd}}{\partial x^c} + \frac{\partial h_{cd}}{\partial x^b} - \frac{\partial h_{bc}}{\partial x^d}\right)
\end{aligned}$$

14.3.3 The Riemann tensor.

Ignoring terms of order ϵ^2

$$\begin{aligned}
R^a{}_{bcd} &= \partial_c\Gamma^a{}_{bd} - \partial_d\Gamma^a{}_{bc} + \Gamma^e{}_{bd}\Gamma^a{}_{ec} - \Gamma^e{}_{bc}\Gamma^a{}_{ed} \\
&= \partial_c\left(\frac{\epsilon}{2}\eta^{ae}\left(\frac{\partial h_{be}}{\partial x^d} + \frac{\partial h_{de}}{\partial x^b} - \frac{\partial h_{bd}}{\partial x^e}\right)\right) - \partial_d\left(\frac{\epsilon}{2}\eta^{af}\left(\frac{\partial h_{bf}}{\partial x^c} + \frac{\partial h_{cf}}{\partial x^b} - \frac{\partial h_{bc}}{\partial x^f}\right)\right) \\
&= \frac{\epsilon}{2}\left(\eta^{ae}\left(\frac{\partial h_{be}}{\partial x^c\partial x^d} + \frac{\partial h_{de}}{\partial x^b\partial x^c} - \frac{\partial h_{bd}}{\partial x^c\partial x^e}\right) - \eta^{af}\left(\frac{\partial h_{bf}}{\partial x^c\partial x^d} + \frac{\partial h_{cf}}{\partial x^b\partial x^d} - \frac{\partial h_{bc}}{\partial x^d\partial x^f}\right)\right) \\
&= \frac{\epsilon}{2}\eta^{ae}\left(\frac{\partial h_{be}}{\partial x^c\partial x^d} + \frac{\partial h_{de}}{\partial x^b\partial x^c} - \frac{\partial h_{bd}}{\partial x^c\partial x^e} - \frac{\partial h_{be}}{\partial x^c\partial x^d} - \frac{\partial h_{ce}}{\partial x^b\partial x^d} + \frac{\partial h_{bc}}{\partial x^d\partial x^e}\right) \\
&= \frac{\epsilon}{2}\eta^{ae}\left(\frac{\partial h_{de}}{\partial x^b\partial x^c} - \frac{\partial h_{bd}}{\partial x^c\partial x^e} - \frac{\partial h_{ce}}{\partial x^b\partial x^d} + \frac{\partial h_{bc}}{\partial x^d\partial x^e}\right) \\
\Rightarrow \quad R_{abcd} &= \frac{\epsilon}{2}\left(\frac{\partial h_{da}}{\partial x^b\partial x^c} - \frac{\partial h_{bd}}{\partial x^c\partial x^a} - \frac{\partial h_{ca}}{\partial x^b\partial x^d} + \frac{\partial h_{bc}}{\partial x^d\partial x^a}\right)
\end{aligned}$$

14.3.4 The Ricci tensor.

$$\begin{aligned}
R_{ab} &= R^c{}_{acb} \\
&= \partial_c\Gamma^c{}_{ab} - \partial_b\Gamma^c{}_{ac}
\end{aligned}$$

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[1] The general formula is $\int_{-\infty}^{\infty} f(u)\delta^{(n)}(u)du = (-1)^n f^{(n)}(0)$

[2] ϵ is a small dimensionless parameter of order $\frac{v}{c}$ so $\epsilon \ll 1$

$$= \partial_c\left(\frac{\epsilon}{2}\eta^{cd}\left(\frac{\partial h_{ad}}{\partial x^b}+\frac{\partial h_{bd}}{\partial x^a}-\frac{\partial h_{ab}}{\partial x^d}\right)\right)-\partial_b\left(\frac{\epsilon}{2}\eta^{cd}\left(\frac{\partial h_{ad}}{\partial x^c}+\frac{\partial h_{cd}}{\partial x^a}-\frac{\partial h_{ac}}{\partial x^d}\right)\right)$$
$$= \frac{\epsilon}{2}\eta^{cd}\left(\frac{\partial h_{ad}}{\partial x^b\partial x^c}+\frac{\partial h_{bd}}{\partial x^a\partial x^c}-\frac{\partial h_{ab}}{\partial x^c\partial x^d}-\frac{\partial h_{ad}}{\partial x^b\partial x^c}-\frac{\partial h_{cd}}{\partial x^a\partial x^b}+\frac{\partial h_{ac}}{\partial x^b\partial x^d}\right)$$
$$= \frac{\epsilon}{2}\eta^{cd}\left(\frac{\partial h_{bd}}{\partial x^a\partial x^c}-\frac{\partial h_{ab}}{\partial x^c\partial x^d}-\frac{\partial h_{cd}}{\partial x^a\partial x^b}+\frac{\partial h_{ac}}{\partial x^b\partial x^d}\right)$$
$$=^{3\,4\,5\,6}\frac{\epsilon}{2}\left(\frac{\partial h^c{}_b}{\partial x^a\partial x^c}-Wh_{ab}-\frac{\partial h}{\partial x^a\partial x^b}+\frac{\partial h^c{}_a}{\partial x^b\partial x^c}\right)$$

14.3.5 The Ricci scalar.

Ignoring terms of order ϵ^2.

$$R = g^{ab}R_{ab}$$
$$= \eta^{ab}\frac{\epsilon}{2}\left(\frac{\partial h^c{}_b}{\partial x^a\partial x^c}-Wh_{ab}-\frac{\partial h}{\partial x^a\partial x^b}+\frac{\partial h^c{}_a}{\partial x^b\partial x^c}\right)$$
$$= \frac{\epsilon}{2}\left(\frac{\partial h^{ac}}{\partial x^a\partial x^c}-Wh-Wh+\frac{\partial h^{bc}}{\partial x^b\partial x^c}\right)$$
$$= \epsilon\left(\frac{\partial h^{ab}}{\partial x^a\partial x^b}-Wh\right)$$

14.3.6 The Einstein tensor.

Ignoring terms of order ϵ^2.

$$G_{ab} = R_{ab}-\frac{1}{2}g_{ab}R$$
$$= R_{ab}-\frac{1}{2}\eta_{ab}R$$
$$= \frac{\epsilon}{2}\left(\frac{\partial h^c{}_b}{\partial x^a\partial x^c}-Wh_{ab}-\frac{\partial h}{\partial x^a\partial x^b}+\frac{\partial h^c{}_a}{\partial x^b\partial x^c}\right)-\frac{\epsilon}{2}\eta_{ab}\left(\frac{\partial h^{ab}}{\partial x^a\partial x^b}-Wh\right)$$
$$= \frac{\epsilon}{2}\left(\frac{\partial h^c{}_b}{\partial x^a\partial x^c}-Wh_{ab}-\frac{\partial h}{\partial x^a\partial x^b}+\frac{\partial h^c{}_a}{\partial x^b\partial x^c}-\eta_{ab}\frac{\partial h^{cd}}{\partial x^c\partial x^c}+\eta_{ab}Wh\right)$$

14.3.7 [c]Gauge transformation - The Einstein Gauge

If we require that $R^a{}_{bcd}$, R_{ab} and R are unchanged under a gauge-transformation of first order in ε, we can show that this is fulfilled by the coordinate transformations

$$x^a \to x^{a'} = x^a + \varepsilon\phi^a$$
$$h_{ab} \to h'_{ab} = h_{ab} - \phi_{a,b} - \phi_{b,a}$$
$$\psi^a{}_{b,a} \to \psi'^a{}_{b,a} = \psi^a{}_{b,a} - \Box\phi_b$$

where ϕ^a is a function of position and $\left|\phi^a{}_{,b}\right| \ll 1$. We have

$$R^a{}_{bcd} = \frac{1}{2}\varepsilon\eta^{ae}\left(-\frac{\partial^2 h_{bd}}{\partial x^c\partial x^f}+\frac{\partial^2 h_{df}}{\partial x^c\partial x^b}+\frac{\partial^2 h_{bc}}{\partial x^d\partial x^f}-\frac{\partial^2 h_{cf}}{\partial x^d\partial x^b}\right)$$
$$R_{ab} = \frac{1}{2}\varepsilon\left(\frac{\partial^2 h^c{}_a}{\partial x^b\partial x^c}+\frac{\partial^2 h^c{}_b}{\partial x^a\partial x^c}-Wh_{ab}-\frac{\partial^2 h}{\partial x^a\partial x^b}\right)$$
$$R = \varepsilon\left(\frac{\partial^2 h^{cd}}{\partial x^c\partial x^d}-Wh\right)$$

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[3] Assuming $h_{ab} = h_{ba}$

[4] Defining $W = \eta^{cd}\partial_c\partial_d$

[5] Defining $h = \eta^{cd}h_{cd}$

[6] $\eta^{cd}\frac{\partial h_{ac}}{\partial x^b\partial x^d} = \frac{\partial h^d{}_a}{\partial x^b\partial x^d} = \frac{\partial h^c{}_a}{\partial x^b\partial x^c}$

$$\psi_{ab} = h_{ab} - \frac{1}{2}\eta_{ab}h$$

The Einstein gauge transformation is a coordinate transformation that leaves $R^a{}_{bcd}, R_{ab}$ and R unchanged. The coordinate transformation that will do this is

$$x^a \rightarrow x^{a'} = x^a + \varepsilon\phi^a$$

In order to show this you only have to convince yourself that the line element is unchanged. Checking

$$\begin{aligned}
ds^2 &= g_{ab}dx^a dx^b = (\eta_{ab} + \varepsilon h_{ab})dx^a dx^b \\
ds'^2 &= g_{a'b'}dx^{a'}dx^{b'}1 \\
&= (\eta'_{ab} + \varepsilon h'_{ab})dx^{a'}dx^{b'} \\
&= {}^7(\eta_{ab} + \varepsilon h'_{ab})d(x^a + \varepsilon\phi^a)d(x^b + \varepsilon\phi^b) \\
&= (\eta_{ab} + \varepsilon h'_{ab})\left(\frac{\partial x^a}{\partial x^c}dx^c + \varepsilon\frac{\partial\phi^a}{\partial x^c}dx^c\right)\left(\frac{\partial x^b}{\partial x^d}dx^d + \varepsilon\frac{\partial\phi^b}{\partial x^d}dx^d\right) \\
&= (\eta_{ab} + \varepsilon h'_{ab})(\delta^a_c dx^c + \varepsilon\phi^a{}_{,c}dx^c)(\delta^b_d dx^d + \varepsilon\phi^b{}_{,d}dx^d) \\
&= (\eta_{ab} + \varepsilon h'_{ab})(\delta^a_c\delta^b_d dx^c dx^d + \varepsilon\phi^a{}_{,c}\delta^b_d dx^c dx^d + \varepsilon\phi^b{}_{,d}\delta^a_c dx^c dx^d) + \varepsilon^2 \ldots \\
&= (\eta_{ab} + \varepsilon h'_{ab})(dx^a dx^b + \varepsilon\phi^a{}_{,c}dx^c dx^b + \varepsilon\phi^b{}_{,d}dx^a dx^d) \\
&= (\eta_{ab} + \varepsilon h'_{ab})(dx^a dx^b + \varepsilon\eta^{ae}\phi_{e,c}dx^c dx^b + \varepsilon\eta^{bf}\phi_{f,d}\,dx^a dx^d) \\
&= (\eta_{ab} + \varepsilon h'_{ab})dx^a dx^b + \eta_{ab}(\varepsilon\eta^{ae}\phi_{e,c}dx^c dx^b + \varepsilon\eta^{bf}\phi_{f,d}\,dx^a dx^d) + \varepsilon^2 \ldots \\
&= (\eta_{ab} + \varepsilon h'_{ab})dx^a dx^b + (\varepsilon\delta^e_b\phi_{e,c}dx^c dx^b + \varepsilon\delta^f_a\phi_{f,d}\,dx^a dx^d) \\
&= (\eta_{ab} + \varepsilon h'_{ab})dx^a dx^b + (\varepsilon\phi_{b,c}dx^c dx^b + \varepsilon\phi_{a,d}\,dx^a dx^d)
\end{aligned}$$

Renaming the dummy variables

$$\begin{aligned}
&= (\eta_{ab} + \varepsilon h'_{ab})dx^a dx^b + (\varepsilon\phi_{b,a}dx^a dx^b + \varepsilon\phi_{a,b}\,dx^a dx^b) \\
&= (\eta_{ab} + \varepsilon h'_{ab} + \varepsilon\phi_{b,a} + \varepsilon\phi_{a,b})dx^a dx^b \\
&= (\eta_{ab} + \varepsilon h_{ab})dx^a dx^b
\end{aligned}$$

If

$$h'_{ab} = h_{ab} - \phi_{b,a} - \phi_{a,b} \qquad \text{(I)}$$

Next we are going to investigate the transformation of the derivative of the trace reverse

$$\psi^a{}_{b,a} \rightarrow \psi'^a{}_{b,a} = \psi^a{}_{b,a} - \Box\phi_b \qquad \text{(II)}$$

$$\begin{aligned}
\psi'^a{}_{b,a} &= \eta^{ac}\psi'_{cb,a} \\
&= \eta^{ac}\left(h'_{cb,a} - \frac{1}{2}\eta_{cb}h'_{,a}\right) \\
&= \eta^{ac}\left(h'_{cb,a} - \frac{1}{2}\eta_{cb}h'^d{}_{d,a}\right) \\
&= \eta^{ac}\left(h'_{cb,a} - \frac{1}{2}\eta_{cb}\eta^{ed}h'_{ed,a}\right) \\
&= \eta^{ac}(h_{cb,a} - \phi_{b,ca} - \phi_{c,ba}) - \frac{1}{2}\eta^{ac}\eta_{cb}\eta^{ed}(h_{ed,a} - \phi_{d,ea} - \phi_{e,da}) \\
&= \eta^{ac}\left(h_{cb,a} - \frac{1}{2}\eta_{cb}\eta^{ed}h_{ed,a}\right) - \eta^{ac}\left(\phi_{b,ca} + \phi_{c,ba} - \frac{1}{2}\eta_{cb}\eta^{ed}(\phi_{d,ea} + \phi_{e,da})\right) \\
&= \eta^{ac}\left(h_{cb,a} - \frac{1}{2}\eta_{cb}h^d{}_{d,a}\right) - \eta^{ac}\phi_{b,ca} - \eta^{ac}\left(\phi_{c,ba} - \frac{1}{2}\eta_{cb}\eta^{ed}(\phi_{d,ea} + \phi_{e,da})\right) \\
&= \eta^{ac}\left(h_{cb,a} - \frac{1}{2}\eta_{cb}h_{,a}\right) - \Box\phi_b - \left(\eta^{ac}\phi_{c,ba} - \frac{1}{2}\eta^{ac}\eta_{cb}\eta^{ed}(\phi_{d,ea} + \phi_{e,da})\right) \\
&= \eta^{ac}\psi_{cb,a} - \Box\phi_b - \left(\eta^{ac}\phi_{c,ba} - \frac{1}{2}\delta^a_b\eta^{ed}(\phi_{d,ea} + \phi_{e,da})\right)
\end{aligned}$$

[7] $\eta'_{ab} = \eta_{ab}$

$$= \psi^a{}_{b,a} - \Box\phi_b - \left(\phi^a{}_{,ba} - \frac{1}{2}\delta^a_b\left(\phi^e{}_{,ea} + \phi^d{}_{,da}\right)\right)$$
$$= {}^8\psi^a{}_{b,a} - \Box\phi_b - \left(\phi^a{}_{,ba} - \frac{1}{2}\left(\phi^e{}_{,eb} + \phi^d{}_{,db}\right)\right)$$
$$= \psi^a{}_{b,a} - \Box\phi_b - \left(\phi^a{}_{,ba} - \frac{1}{2}\left(\phi^a{}_{,ab} + \phi^a{}_{,ab}\right)\right)$$
$$\psi'^a{}_{b,a} = \psi^a{}_{b,a} - \Box\phi_b \qquad \text{(II)}$$

The choice of $\psi'^a{}_{b,a} = 0$ leads to[9]

$$\psi'^a{}_{b,a} = h'^a{}_{b,a} - \frac{1}{2}h'_{,b} = \eta^{ac}\psi'_{cb,a} = \eta^{ac}\left(h'_{cb,a} - \frac{1}{2}\eta_{cb}h'_{,a}\right) = \eta^{ac}h'_{cb,a} - \frac{1}{2}\delta^a_b h'_{,a} = 0$$

14.4 Plane waves

14.4.1 dThe Riemann tensor of a plane wave

Here we want to show that the Riemann tensor only depends on h_{xx}, h_{xy}, h_{yx} and h_{yy}. For symmetry reasons it is only necessary to show that the Riemann tensor does not depend on h_{tt} and h_{tx}. The Riemann tensor

$$R^a{}_{bcd} = \frac{1}{2}\varepsilon\eta^{af}\left(\frac{\partial^2 h_{df}}{\partial x^c \partial x^b} - \frac{\partial^2 h_{bd}}{\partial x^c \partial x^f} + \frac{\partial^2 h_{bc}}{\partial x^d \partial x^f} - \frac{\partial^2 h_{cf}}{\partial x^d \partial x^b}\right)$$

For plane waves we have

$$h_{ab} = h_{ab}(t-z)$$
$$\Rightarrow \quad \frac{\partial h_{ab}}{\partial x} = \frac{\partial h_{ab}}{\partial y} = 0$$

We also need

$$\frac{\partial h_{ab}}{\partial z} = \frac{\partial(t-z)}{\partial z}\frac{\partial h_{ab}}{\partial(t-z)} = -\frac{\partial h_{ab}}{\partial(t-z)}$$
$$\frac{\partial h_{ab}}{\partial t} = \frac{\partial(t-z)}{\partial t}\frac{\partial h_{ab}}{\partial(t-z)} = \frac{\partial h_{ab}}{\partial(t-z)}$$
$$\Rightarrow \quad \frac{\partial h_{ab}}{\partial z} = -\frac{\partial h_{ab}}{\partial t}$$
$$\frac{\partial^2 h_{ab}}{\partial z^2} = \frac{\partial^2 h_{ab}}{\partial t^2}$$

and

$$h_{ab} = \begin{pmatrix} h_{tt} & h_{tx} & h_{ty} & h_{tz} \\ h_{xt} & h_{xx} & h_{xy} & h_{xz} \\ h_{yt} & h_{yx} & h_{yy} & h_{yz} \\ h_{zt} & h_{zx} & h_{zy} & h_{zz} \end{pmatrix}$$
$$= \begin{pmatrix} h_{tt} & h_{tx} & h_{ty} & -\frac{1}{2}(h_{tt}+h_{zz}) \\ h_{tx} & h_{xx} & h_{xy} & -h_{tx} \\ h_{ty} & h_{xy} & -h_{xx} & -h_{ty} \\ -\frac{1}{2}(h_{tt}+h_{zz}) & -h_{tx} & -h_{ty} & h_{zz} \end{pmatrix}$$

The Minkowski

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[8] Renaming the dummy variables

[9] However I don't know how to show that the Riemann-tensor keeps the same form if we make this choice

$$\eta^{ab} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

<u>The dependence on h_{tt}</u>
<u>$d = f = t\ (\Rightarrow a = t)$:</u>

$$R^t{}_{bct} = \frac{1}{2}\varepsilon\eta^{tt}\left(\frac{\partial^2 h_{tt}}{\partial x^c \partial x^b} - \frac{\partial^2 h_{bt}}{\partial x^c \partial t} + \frac{\partial^2 h_{bc}}{\partial^2 t} - \frac{\partial^2 h_{ct}}{\partial t \partial x^b}\right)$$

<u>$b = t$:</u> $$R^t{}_{tct} = \frac{1}{2}\varepsilon\left(\frac{\partial^2 h_{tt}}{\partial x^c \partial t} - \frac{\partial^2 h_{tt}}{\partial x^c \partial t} + \frac{\partial^2 h_{tc}}{\partial^2 t} - \frac{\partial^2 h_{ct}}{\partial^2 t}\right) = 0$$

<u>$b = x$:</u> $$R^t{}_{xct} = \frac{1}{2}\varepsilon\left(\frac{\partial^2 h_{tt}}{\partial x^c \partial x} - \frac{\partial^2 h_{xt}}{\partial x^c \partial t} + \frac{\partial^2 h_{xc}}{\partial^2 t} - \frac{\partial^2 h_{ct}}{\partial t \partial x}\right) = \frac{1}{2}\varepsilon\left(-\frac{\partial^2 h_{xt}}{\partial x^c \partial t} + \frac{\partial^2 h_{xc}}{\partial^2 t}\right)$$

<u>$c = t$:</u> $$R^t{}_{xtt} = \frac{1}{2}\varepsilon\left(-\frac{\partial^2 h_{xt}}{\partial^2 t} + \frac{\partial^2 h_{xt}}{\partial^2 t}\right) = 0$$

<u>$c = x$:</u> $$R^t{}_{xxt} = \frac{1}{2}\varepsilon\left(-\frac{\partial^2 h_{xt}}{\partial x \partial t} + \frac{\partial^2 h_{xx}}{\partial^2 t}\right) = \frac{1}{2}\varepsilon\frac{\partial^2 h_{xx}}{\partial^2 t}$$

<u>$c = y$:</u> $$R^t{}_{xyt} = \frac{1}{2}\varepsilon\left(-\frac{\partial^2 h_{xt}}{\partial y \partial t} + \frac{\partial^2 h_{xy}}{\partial^2 t}\right) = \frac{1}{2}\varepsilon\frac{\partial^2 h_{xy}}{\partial^2 t}$$

<u>$c = z$:</u> $$R^t{}_{xzt} = \frac{1}{2}\varepsilon\left(-\frac{\partial^2 h_{xt}}{\partial z \partial t} + \frac{\partial^2 h_{xz}}{\partial^2 t}\right) = \frac{1}{2}\varepsilon\left(\frac{\partial^2 h_{xt}}{\partial^2 t} - \frac{\partial^2 h_{tx}}{\partial^2 t}\right) = 0$$

<u>$b = y$:</u> $$R^t{}_{yct} = \frac{1}{2}\varepsilon\left(\frac{\partial^2 h_{tt}}{\partial x^c \partial y} - \frac{\partial^2 h_{yt}}{\partial x^c \partial t} + \frac{\partial^2 h_{yc}}{\partial^2 t} - \frac{\partial^2 h_{ct}}{\partial t \partial y}\right) = \frac{1}{2}\varepsilon\left(-\frac{\partial^2 h_{yt}}{\partial x^c \partial t} + \frac{\partial^2 h_{yc}}{\partial^2 t}\right)$$

<u>$c = t$:</u> $$R^t{}_{ytt} = \frac{1}{2}\varepsilon\left(-\frac{\partial^2 h_{yt}}{\partial^2 t} + \frac{\partial^2 h_{yt}}{\partial^2 t}\right) = 0$$

<u>$c = x$:</u> $$R^t{}_{yxt} = \frac{1}{2}\varepsilon\left(-\frac{\partial^2 h_{yt}}{\partial x \partial t} + \frac{\partial^2 h_{yx}}{\partial^2 t}\right) = \frac{1}{2}\varepsilon\frac{\partial^2 h_{yx}}{\partial^2 t}$$

<u>$c = y$:</u> $$R^t{}_{yyt} = \frac{1}{2}\varepsilon\left(-\frac{\partial^2 h_{yt}}{\partial y \partial t} + \frac{\partial^2 h_{yy}}{\partial^2 t}\right) = \frac{1}{2}\varepsilon\frac{\partial^2 h_{yy}}{\partial^2 t}$$

<u>$c = z$:</u> $$R^t{}_{yzt} = \frac{1}{2}\varepsilon\left(-\frac{\partial^2 h_{yt}}{\partial z \partial t} + \frac{\partial^2 h_{yz}}{\partial^2 t}\right) = \frac{1}{2}\varepsilon\left(\frac{\partial^2 h_{yt}}{\partial^2 t} - \frac{\partial^2 h_{yt}}{\partial^2 t}\right) = 0$$

<u>$b = z$:</u> $$R^t{}_{zct} = \frac{1}{2}\varepsilon\left(\frac{\partial^2 h_{tt}}{\partial x^c \partial z} - \frac{\partial^2 h_{zt}}{\partial x^c \partial t} + \frac{\partial^2 h_{zc}}{\partial^2 t} - \frac{\partial^2 h_{ct}}{\partial t \partial z}\right)$$

<u>$c = t$:</u> $$R^t{}_{ztt} = \frac{1}{2}\varepsilon\left(\frac{\partial^2 h_{tt}}{\partial t \partial z} - \frac{\partial^2 h_{zt}}{\partial^2 t} + \frac{\partial^2 h_{zt}}{\partial^2 t} - \frac{\partial^2 h_{tt}}{\partial t \partial z}\right) = 0$$

<u>$c = x$:</u> $$R^t{}_{zxt} = \frac{1}{2}\varepsilon\left(\frac{\partial^2 h_{tt}}{\partial x \partial z} - \frac{\partial^2 h_{zt}}{\partial x \partial t} + \frac{\partial^2 h_{zx}}{\partial^2 t} - \frac{\partial^2 h_{xt}}{\partial t \partial z}\right) = \frac{1}{2}\varepsilon\left(-\frac{\partial^2 h_{tx}}{\partial^2 t} + \frac{\partial^2 h_{xt}}{\partial^2 t}\right) = 0$$

<u>$c = y$:</u> $$R^t{}_{zyt} = \frac{1}{2}\varepsilon\left(\frac{\partial^2 h_{tt}}{\partial y \partial z} - \frac{\partial^2 h_{zt}}{\partial y \partial t} + \frac{\partial^2 h_{zy}}{\partial^2 t} - \frac{\partial^2 h_{yt}}{\partial t \partial z}\right) = \frac{1}{2}\varepsilon\left(-\frac{\partial^2 h_{ty}}{\partial^2 t} + \frac{\partial^2 h_{yt}}{\partial^2 t}\right) = 0$$

<u>$c = z$:</u> $$R^t{}_{zzt} = \frac{1}{2}\varepsilon\left(\frac{\partial^2 h_{tt}}{\partial^2 z} - \frac{\partial^2 h_{zt}}{\partial z \partial t} + \frac{\partial^2 h_{zz}}{\partial^2 t} - \frac{\partial^2 h_{zt}}{\partial t \partial z}\right)$$

$$= \frac{1}{2}\varepsilon\left(\frac{\partial^2 h_{tt}}{\partial^2 t} + \frac{\partial^2\left(-\frac{1}{2}(h_{tt} + h_{zz})\right)}{\partial^2 t} + \frac{\partial^2 h_{zz}}{\partial^2 t} + \frac{\partial^2\left(-\frac{1}{2}(h_{tt} + h_{zz})\right)}{\partial^2 t}\right)$$

$$= 0$$

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$b = d = t$:

$$R^a{}_{tct} = \frac{1}{2}\varepsilon\eta^{af}\left(\frac{\partial^2 h_{tf}}{\partial x^c \partial t} - \frac{\partial^2 h_{tt}}{\partial x^c \partial x^f} + \frac{\partial^2 h_{tc}}{\partial t \partial x^f} - \frac{\partial^2 h_{cf}}{\partial^2 t}\right)$$

$a = x (\Rightarrow f = x)$:

$$R^x{}_{tct} = \frac{1}{2}\varepsilon\eta^{xx}\left(\frac{\partial^2 h_{tx}}{\partial x^c \partial t} - \frac{\partial^2 h_{tt}}{\partial x^c \partial x} + \frac{\partial^2 h_{tc}}{\partial t \partial x} - \frac{\partial^2 h_{cx}}{\partial^2 t}\right) = -\frac{1}{2}\varepsilon\left(\frac{\partial^2 h_{tx}}{\partial x^c \partial t} - \frac{\partial^2 h_{cx}}{\partial^2 t}\right)$$

$c = x$: $$R^x{}_{txt} = -\frac{1}{2}\varepsilon\left(\frac{\partial^2 h_{tx}}{\partial x \partial t} - \frac{\partial^2 h_{xx}}{\partial^2 t}\right) = \frac{1}{2}\varepsilon\frac{\partial^2 h_{xx}}{\partial^2 t}$$

$c = y$: $$R^x{}_{tyt} = -\frac{1}{2}\varepsilon\left(\frac{\partial^2 h_{tx}}{\partial y \partial t} - \frac{\partial^2 h_{yx}}{\partial^2 t}\right) = \frac{1}{2}\varepsilon\frac{\partial^2 h_{yx}}{\partial^2 t}$$

$c = z$: $$R^x{}_{tzt} = -\frac{1}{2}\varepsilon\left(\frac{\partial^2 h_{tx}}{\partial z \partial t} - \frac{\partial^2 h_{zx}}{\partial^2 t}\right) = -\frac{1}{2}\varepsilon\left(-\frac{\partial^2 h_{tx}}{\partial^2 t} + \frac{\partial^2 h_{tx}}{\partial^2 t}\right) = 0$$

$a = y (\Rightarrow f = y)$:

$$R^y{}_{tct} = \frac{1}{2}\varepsilon\eta^{yy}\left(\frac{\partial^2 h_{ty}}{\partial x^c \partial t} - \frac{\partial^2 h_{tt}}{\partial x^c \partial y} + \frac{\partial^2 h_{tc}}{\partial t \partial y} - \frac{\partial^2 h_{cy}}{\partial^2 t}\right) = -\frac{1}{2}\varepsilon\left(\frac{\partial^2 h_{ty}}{\partial x^c \partial t} - \frac{\partial^2 h_{cy}}{\partial^2 t}\right)$$

$c = x$: $$R^y{}_{txt} = -\frac{1}{2}\varepsilon\left(\frac{\partial^2 h_{ty}}{\partial x \partial t} - \frac{\partial^2 h_{xy}}{\partial^2 t}\right) = \frac{1}{2}\varepsilon\frac{\partial^2 h_{xy}}{\partial^2 t}$$

$c = y$: $$R^y{}_{tyt} = -\frac{1}{2}\varepsilon\left(\frac{\partial^2 h_{ty}}{\partial y \partial t} - \frac{\partial^2 h_{yy}}{\partial^2 t}\right) = \frac{1}{2}\varepsilon\frac{\partial^2 h_{yy}}{\partial^2 t}$$

$c = z$: $$R^y{}_{tzt} = -\frac{1}{2}\varepsilon\left(\frac{\partial^2 h_{ty}}{\partial z \partial t} - \frac{\partial^2 h_{zy}}{\partial^2 t}\right) = -\frac{1}{2}\varepsilon\left(-\frac{\partial^2 h_{ty}}{\partial^2 t} + \frac{\partial^2 h_{ty}}{\partial^2 t}\right) = 0$$

$a = z (\Rightarrow f = z)$:

$$R^z{}_{tct} = \frac{1}{2}\varepsilon\eta^{zz}\left(\frac{\partial^2 h_{tz}}{\partial x^c \partial t} - \frac{\partial^2 h_{tt}}{\partial x^c \partial z} + \frac{\partial^2 h_{tc}}{\partial t \partial z} - \frac{\partial^2 h_{cz}}{\partial^2 t}\right)$$

$$= -\frac{1}{2}\varepsilon\left(\frac{\partial^2 h_{tz}}{\partial x^c \partial t} - \frac{\partial^2 h_{tt}}{\partial x^c \partial z} + \frac{\partial^2 h_{tc}}{\partial t \partial z} - \frac{\partial^2 h_{cz}}{\partial^2 t}\right)$$

$c = x$: $$R^z{}_{txt} = -\frac{1}{2}\varepsilon\left(\frac{\partial^2 h_{tz}}{\partial x \partial t} - \frac{\partial^2 h_{tt}}{\partial x \partial z} + \frac{\partial^2 h_{tx}}{\partial t \partial z} - \frac{\partial^2 h_{xz}}{\partial^2 t}\right) = 0$$

$c = y$: $$R^z{}_{tyt} = -\frac{1}{2}\varepsilon\left(\frac{\partial^2 h_{tz}}{\partial y \partial t} - \frac{\partial^2 h_{tt}}{\partial y \partial z} + \frac{\partial^2 h_{ty}}{\partial t \partial z} - \frac{\partial^2 h_{yz}}{\partial^2 t}\right) = 0$$

$c = z$: $$R^z{}_{tzt} = -\frac{1}{2}\varepsilon\left(\frac{\partial^2 h_{tz}}{\partial z \partial t} - \frac{\partial^2 h_{tt}}{\partial^2 z} + \frac{\partial^2 h_{tz}}{\partial t \partial z} - \frac{\partial^2 h_{zz}}{\partial^2 t}\right) = 0$$

The dependence on h_{tx}

$d = t, f = x (\Rightarrow a = x)$:

$$R^x{}_{bct} = \frac{1}{2}\varepsilon\eta^{xx}\left(\frac{\partial^2 h_{tx}}{\partial x^c \partial x^b} - \frac{\partial^2 h_{bt}}{\partial x^c \partial x} + \frac{\partial^2 h_{bc}}{\partial t \partial x} - \frac{\partial^2 h_{cx}}{\partial t \partial x^b}\right) = -\frac{1}{2}\varepsilon\left(\frac{\partial^2 h_{tx}}{\partial x^c \partial x^b} - \frac{\partial^2 h_{cx}}{\partial t \partial x^b}\right)$$

$b = x$: $R^x{}_{xct} = 0$

$b = y$: $R^x{}_{yct} = 0$

$b = z$: $$R^x{}_{zct} = -\frac{1}{2}\varepsilon\left(\frac{\partial^2 h_{tx}}{\partial x^c \partial z} - \frac{\partial^2 h_{cx}}{\partial t \partial z}\right)$$

$b = t, d = x$:

$$R^a{}_{tcx} = \frac{1}{2}\varepsilon\eta^{af}\left(\frac{\partial^2 h_{xf}}{\partial x^c \partial t} - \frac{\partial^2 h_{tx}}{\partial x^c \partial x^f} + \frac{\partial^2 h_{tc}}{\partial x \partial x^f} - \frac{\partial^2 h_{cf}}{\partial x \partial t}\right) = \frac{1}{2}\varepsilon\eta^{af}\left(\frac{\partial^2 h_{xf}}{\partial x^c \partial t} - \frac{\partial^2 h_{tx}}{\partial x^c \partial x^f}\right)$$

$a = t (\Rightarrow f = t)$:

$$R^t{}_{tcx} = \frac{1}{2}\varepsilon\eta^{tt}\left(\frac{\partial^2 h_{xt}}{\partial x^c \partial t} - \frac{\partial^2 h_{tx}}{\partial x^c \partial t}\right) = 0$$

$\underline{a = x(\Rightarrow f = x):}$

$$R^x{}_{tcx} = \frac{1}{2}\varepsilon\eta^{xx}\left(\frac{\partial^2 h_{xx}}{\partial x^c \partial t} - \frac{\partial^2 h_{tx}}{\partial x^c \partial x}\right) = -\frac{1}{2}\varepsilon\frac{\partial^2 h_{xx}}{\partial x^c \partial t}$$

$\underline{a = y(\Rightarrow f = y):}$

$$R^y{}_{tcx} = \frac{1}{2}\varepsilon\eta^{yy}\left(\frac{\partial^2 h_{xy}}{\partial x^c \partial t} - \frac{\partial^2 h_{tx}}{\partial x^c \partial y}\right) = -\frac{1}{2}\varepsilon\frac{\partial^2 h_{xy}}{\partial x^c \partial t}$$

$\underline{a = z(\Rightarrow f = z):}$

$$R^z{}_{tcx} = \frac{1}{2}\varepsilon\eta^{zz}\left(\frac{\partial^2 h_{xz}}{\partial x^c \partial t} - \frac{\partial^2 h_{tx}}{\partial x^c \partial z}\right) = \frac{1}{2}\varepsilon\eta^{zz}\left(-\frac{\partial^2 h_{tx}}{\partial x^c \partial t} + \frac{\partial^2 h_{tx}}{\partial x^c \partial t}\right) = 0$$

$\underline{b = t, c = x:}$

$$R^a{}_{txd} = \frac{1}{2}\varepsilon\eta^{af}\left(\frac{\partial^2 h_{df}}{\partial x \partial t} - \frac{\partial^2 h_{td}}{\partial x \partial x^f} + \frac{\partial^2 h_{tx}}{\partial x^d \partial x^f} - \frac{\partial^2 h_{xf}}{\partial x^d \partial t}\right) = \frac{1}{2}\varepsilon\eta^{af}\left(\frac{\partial^2 h_{tx}}{\partial x^d \partial x^f} - \frac{\partial^2 h_{xf}}{\partial x^d \partial t}\right)$$

$\underline{a = t(\Rightarrow f = t):}$

$$R^t{}_{txd} = \frac{1}{2}\varepsilon\eta^{tt}\left(\frac{\partial^2 h_{tx}}{\partial x^d \partial t} - \frac{\partial^2 h_{xt}}{\partial x^d \partial t}\right) = 0$$

$\underline{a = x(\Rightarrow f = x):}$

$$R^x{}_{txd} = \frac{1}{2}\varepsilon\eta^{xx}\left(\frac{\partial^2 h_{tx}}{\partial x^d \partial x} - \frac{\partial^2 h_{xx}}{\partial x^d \partial t}\right) = \frac{1}{2}\varepsilon\frac{\partial^2 h_{xx}}{\partial x^d \partial t}$$

$\underline{a = y(\Rightarrow f = y):}$

$$R^y{}_{txd} = \frac{1}{2}\varepsilon\eta^{yy}\left(\frac{\partial^2 h_{tx}}{\partial x^d \partial y} - \frac{\partial^2 h_{xy}}{\partial x^d \partial t}\right) = \frac{1}{2}\varepsilon\frac{\partial^2 h_{xy}}{\partial x^d \partial t}$$

$\underline{a = z(\Rightarrow f = z):}$

$$R^z{}_{txd} = \frac{1}{2}\varepsilon\eta^{zz}\left(\frac{\partial^2 h_{tx}}{\partial x^d \partial z} - \frac{\partial^2 h_{xz}}{\partial x^d \partial t}\right) = -\frac{1}{2}\varepsilon\left(-\frac{\partial^2 h_{tx}}{\partial x^d \partial t} + \frac{\partial^2 h_{tx}}{\partial x^d \partial t}\right) = 0$$

$\underline{c = t, f = x(\Rightarrow a = x):}$

$$R^x{}_{btd} = \frac{1}{2}\varepsilon\eta^{xx}\left(\frac{\partial^2 h_{dx}}{\partial t \partial x^b} - \frac{\partial^2 h_{bd}}{\partial t \partial x} + \frac{\partial^2 h_{bt}}{\partial x^d \partial x} - \frac{\partial^2 h_{tx}}{\partial x^d \partial x^b}\right) = -\frac{1}{2}\varepsilon\left(\frac{\partial^2 h_{dx}}{\partial t \partial x^b} - \frac{\partial^2 h_{tx}}{\partial x^d \partial x^b}\right)$$

The nonzero calculated elements of the Riemann tensor, from which we can conclude that the Riemann tensor only depends on h_{xx}, h_{xy}, h_{yx} and h_{yy}:

$$R^t{}_{xxt} = \frac{1}{2}\varepsilon\frac{\partial^2 h_{xx}}{\partial^2 t} \qquad R^x{}_{txt} = \frac{1}{2}\varepsilon\frac{\partial^2 h_{xx}}{\partial^2 t} \qquad R^y{}_{txt} = \frac{1}{2}\varepsilon\frac{\partial^2 h_{xy}}{\partial^2 t}$$

$$R^t{}_{xyt} = \frac{1}{2}\varepsilon\frac{\partial^2 h_{xy}}{\partial^2 t} \qquad R^x{}_{txz} = -\frac{1}{2}\varepsilon\frac{\partial^2 h_{xx}}{\partial^2 t} \qquad R^y{}_{txz} = -\frac{1}{2}\varepsilon\frac{\partial^2 h_{xy}}{\partial^2 t}$$

$$R^t{}_{yxt} = \frac{1}{2}\varepsilon\frac{\partial^2 h_{yx}}{\partial^2 t} \qquad R^x{}_{tyt} = \frac{1}{2}\varepsilon\frac{\partial^2 h_{yx}}{\partial^2 t} \qquad R^y{}_{tyt} = \frac{1}{2}\varepsilon\frac{\partial^2 h_{yy}}{\partial^2 t}$$

$$R^t{}_{yyt} = \frac{1}{2}\varepsilon\frac{\partial^2 h_{yy}}{\partial^2 t} \qquad R^x{}_{tzx} = \frac{1}{2}\varepsilon\frac{\partial^2 h_{xx}}{\partial^2 t} \qquad R^y{}_{tzx} = \frac{1}{2}\varepsilon\frac{\partial^2 h_{xy}}{\partial^2 t}$$

$$R^x{}_{zxt} = -\frac{1}{2}\varepsilon\frac{\partial^2 h_{xx}}{\partial^2 t}$$

$$R^x{}_{zyt} = -\frac{1}{2}\varepsilon\frac{\partial^2 h_{yx}}{\partial^2 t}$$

14.4.2 [e]The line element of a plane wave in the Einstein gauge

The perturbation

$$h_{ab} = \begin{pmatrix} h_{tt} & h_{tx} & h_{ty} & -\frac{1}{2}(h_{tt}+h_{zz}) \\ h_{tx} & h_{xx} & h_{xy} & -h_{tx} \\ h_{ty} & h_{xy} & -h_{xx} & -h_{ty} \\ -\frac{1}{2}(h_{tt}+h_{zz}) & -h_{tx} & -h_{ty} & h_{zz} \end{pmatrix}$$

the perturbation in the Einstein gauge[f]

$$h'_{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h'_{xx} & h'_{xy} & 0 \\ 0 & h'_{xy} & -h'_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with the transformation

$$h'_{ab} = h_{ab} - \phi_{b,a} - \phi_{a,b}$$

where we assume the plane wave condition $\phi_a = \phi_a(t-z)$.
h_{xx} and h_{xy} are unchanged by the transformation:

$$h'_{xx} = h_{xx} - \phi_{x,x} - \phi_{x,x} = h_{xx} - 2\frac{\partial \phi_x}{\partial x} = h_{xx}$$

$$h'_{xy} = h_{xy} - \phi_{y,x} - \phi_{x,y} = h_{xy} - \frac{\partial \phi_y}{\partial x} - \frac{\partial \phi_x}{\partial y} = h_{xy}$$

Choosing the remaining elements $h'_{ab} = 0$ leaves h_{xx} and h_{xy} unchanged

$$h'_{tt} = h_{tt} - \phi_{t,t} - \phi_{t,t} = h_{tt} - 2\frac{\partial \phi_t}{\partial t} = 0$$

$$\Leftrightarrow \quad h_{tt} = 2\frac{\partial \phi_t}{\partial t}$$

$$h'_{tx} = h_{tx} - \phi_{x,t} - \phi_{t,x} = h_{tx} - \frac{\partial \phi_x}{\partial t} - \frac{\partial \phi_t}{\partial x} = h_{tx} - \frac{\partial \phi_x}{\partial t} = 0$$

$$\Leftrightarrow \quad h_{tx} = \frac{\partial \phi_x}{\partial t}$$

$$h'_{ty} = h_{ty} - \phi_{y,t} - \phi_{t,y} = h_{ty} - \frac{\partial \phi_y}{\partial t} - \frac{\partial \phi_y}{\partial x} = h_{ty} - \frac{\partial \phi_y}{\partial t} = 0$$

$$\Leftrightarrow \quad h_{ty} = \frac{\partial \phi_y}{\partial t}$$

$$h'_{tz} = h_{tz} - \phi_{z,t} - \phi_{t,z} = h_{tz} - \frac{\partial \phi_z}{\partial t} - \frac{\partial \phi_t}{\partial z} = 0$$

$$h'_{zt} = h_{zt} - \phi_{t,z} - \phi_{z,t} = h_{zt} - \frac{\partial \phi_t}{\partial z} - \frac{\partial \phi_z}{\partial t} = 0$$

$$\Leftrightarrow \quad h_{tz} = h_{zt} = \frac{\partial \phi_z}{\partial t} + \frac{\partial \phi_t}{\partial z}$$

$$h'_{zz} = h_{zz} - \phi_{z,z} - \phi_{z,z} = h_{zz} - 2\frac{\partial \phi_z}{\partial z} = 0$$

$$\Leftrightarrow \quad h_{zz} = 2\frac{\partial \phi_z}{\partial z}$$

14.4.3 [g]The line element of a plane wave

With

$$h_{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_{xx} & h_{xy} & 0 \\ 0 & h_{xy} & -h_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

we find the line element

$$ds^2 = g_{ab} dx^a dx^b$$

$$= (\eta_{ab} + \varepsilon h_{ab})dx^a dx^b$$
$$= {}^{10}dt^2 - (1 - \varepsilon h_{xx})dx^2 + \varepsilon h_{xy}dxdy + \varepsilon h_{yx}dydx - (1 + \varepsilon h_{xx})dy^2 - dz^2$$
$$= dt^2 - (1 - \varepsilon h_{xx})dx^2 - (1 + \varepsilon h_{xx})dy^2 - dz^2 + 2\varepsilon h_{xy}dxdy$$

$\underline{h_{xy} = 0:}$ $$ds^2 = dt^2 - (1 - \varepsilon h_{xx})dx^2 - (1 + \varepsilon h_{xx})dy^2 - dz^2$$

$\underline{h_{xx} = 0:}$ $$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 + 2\varepsilon h_{xy}dxdy$$

Considering the following transformation

$$dx' = \frac{dx - dy}{\sqrt{2}}$$
$$dy' = \frac{dx + dy}{\sqrt{2}}$$
$$\Rightarrow \quad dx = \frac{1}{\sqrt{2}}(dx' + dy')$$
$$dy = -\frac{1}{\sqrt{2}}(dx' - dy')$$
$$\Rightarrow \quad dx^2 = \frac{1}{2}dx'^2 + \frac{1}{2}dy'^2 + dx'dy'$$
$$dy^2 = \frac{1}{2}dx'^2 + \frac{1}{2}dy'^2 - dx'dy'$$
$$dxdy = -\frac{1}{2}(dx'^2 - dy'^2)$$
$$dx^2 + dy^2 = dx'^2 + dy'^2$$

we can rewrite the line element

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 + 2\varepsilon h_{xy}dxdy$$
$$= dt^2 - dx'^2 - dy'^2 - dz^2 - \varepsilon h_{xy}(dx'^2 - dy'^2)$$
$$= dt^2 - \left(1 + \varepsilon h_{xy}\right)dx'^2 - \left(1 - \varepsilon h_{xy}\right)dy'^2 - dz^2$$

14.5 [h]The Rosen line element

<u>The line element:</u>

$$ds^2 = dUdV - a^2(U)dx^2 - b^2(U)dy^2$$

<u>The metric tensor:</u>

$$g_{ab} = \begin{pmatrix} & \frac{1}{2} & & \\ \frac{1}{2} & & & \\ & & -a^2(U) & \\ & & & -b^2(U) \end{pmatrix}$$

And the inverse

[10] $= (\eta_{tt} + \varepsilon h_{tt})dt^2 + (\eta_{tx} + \varepsilon h_{tx})dtdx + \left(\eta_{ty} + \varepsilon h_{ty}\right)dtdy + (\eta_{tz} + \varepsilon h_{tz})dtdz + (\eta_{xt} + \varepsilon h_{xt})dxdt + (\eta_{xx} + \varepsilon h_{xx})dx^2 + \left(\eta_{xy} + \varepsilon h_{xy}\right)dxdy + (\eta_{xz} + \varepsilon h_{xz})dxdz + \left(\eta_{yt} + \varepsilon h_{yt}\right)dydt + \left(\eta_{yx} + \varepsilon h_{yx}\right)dydx + \left(\eta_{yy} + \varepsilon h_{yy}\right)dy^2 + \left(\eta_{yz} + \varepsilon h_{yz}\right)dydz + (\eta_{zt} + \varepsilon h_{zt})dzdt + (\eta_{zx} + \varepsilon h_{zx})dzdx + \left(\eta_{zy} + \varepsilon h_{zy}\right)dzdy + (\eta_{zz} + \varepsilon h_{zz})dy^2 =$

$$g^{ab} = \begin{Bmatrix} 2 & & & \\ & 2 & & \\ & & -\frac{1}{a^2(U)} & \\ & & & -\frac{1}{b^2(U)} \end{Bmatrix}$$

14.5.1 [i]The non-zero Christoffel symbols

$$\Gamma_{abc} = \frac{1}{2}(\partial_a g_{bc} + \partial_b g_{ca} - \partial_c g_{ab}) \qquad \Gamma^a{}_{bc} = g^{ad}\Gamma_{bcd}$$

$$\Gamma_{Uxx} = \Gamma_{xUx} = {}^{11} -\frac{1}{2}\left(\partial_U\left(a^2(U)\right)\right) = -a\dot{a} \quad \Rightarrow \quad \Gamma^x{}_{xU} = \Gamma^x{}_{Ux} = {}^{12} g^{xx}\Gamma_{xUx} = \frac{\dot{a}}{a}$$

$$\Gamma_{xxU} = {}^{13}\frac{1}{2}\left(\partial_U\left(a^2(U)\right)\right) = a\dot{a} \quad \Rightarrow \quad \Gamma^V{}_{xx} = g^{Vd}\Gamma_{xxd} = g^{VU}\Gamma_{xxU} = 2a\dot{a}$$

$$\Gamma_{Uyy} = \Gamma_{yUy} = {}^{14} -\frac{1}{2}\left(\partial_U\left(b^2(U)\right)\right) = -b\dot{b} \quad \Rightarrow \quad \Gamma^y{}_{yU} = \Gamma^y{}_{Uy} = {}^{15} g^{yy}\Gamma_{yUy} = \frac{\dot{b}}{b}$$

$$\Gamma_{yyU} = {}^{16} -\frac{1}{2}(\partial_U g_{yy}) = \frac{1}{2}\left(\partial_U\left(b^2(U)\right)\right) = b\dot{b} \quad \Rightarrow \quad \Gamma^V{}_{yy} = g^{Vd}\Gamma_{yyd} = g^{VU}\Gamma_{yyu} = 2b\dot{b}$$

14.5.2 The basis one forms

Finding the basis one forms is not so obvious, we write:

$$\begin{aligned} ds^2 &= dUdV - a^2(U)dx^2 - b^2(U)dy^2 \\ &= \left(\omega^{\hat{0}}\right)^2 - \left(\omega^{\hat{1}}\right)^2 - \left(\omega^{\hat{2}}\right)^2 - \left(\omega^{\hat{3}}\right)^2 \\ &= \left(\omega^{\hat{0}} + \omega^{\hat{1}}\right)\left(\omega^{\hat{0}} - \omega^{\hat{1}}\right) - \left(\omega^{\hat{2}}\right)^2 - \left(\omega^{\hat{3}}\right)^2 \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} dU &= \omega^{\hat{0}} + \omega^{\hat{1}} \\ dV &= \omega^{\hat{0}} - \omega^{\hat{1}} \end{aligned}$$

$$\omega^{\hat{0}} = \frac{1}{2}(dU + dV) \qquad dU = \omega^{\hat{0}} + \omega^{\hat{1}}$$

$$\omega^{\hat{1}} = \frac{1}{2}(dU - dV) \qquad dV = \omega^{\hat{0}} - \omega^{\hat{1}}$$

$$\omega^{\hat{2}} = a(U)dx \qquad dx = \frac{1}{a(U)}\omega^{\hat{2}}$$

$$\omega^{\hat{3}} = b(U)dy \qquad dy = \frac{1}{b(U)}\omega^{\hat{3}}$$

$$\eta^{\hat{\imath}\hat{\jmath}} = \begin{Bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{Bmatrix}$$

14.5.3 Cartan's First Structure equation and the calculation of the curvature one-forms

$$d\omega^{\hat{a}} = -\Gamma^{\hat{a}}{}_{\hat{b}} \wedge \omega^{\hat{b}}$$

$$d\omega^{\hat{0}} = d\left(\frac{1}{2}(dU + dV)\right) = 0$$

---

[11] $= \frac{1}{2}(\partial_U g_{xx} + \partial_x g_{Ux} - \partial_x g_{Ux}) = \frac{1}{2}(\partial_U g_{xx}) =$

[12] $= g^{xd}\Gamma_{xUd} =$

[13] $= \frac{1}{2}(\partial_x g_{xU} + \partial_x g_{xU} - \partial_U g_{xx}) = -\frac{1}{2}(\partial_U g_{xx}) =$

[14] $= \frac{1}{2}\left(\partial_U g_{yy} + \partial_y g_{Uy} - \partial_y g_{Uy}\right) = \frac{1}{2}\left(\partial_U g_{yy}\right) =$

[15] $= g^{yd}\Gamma_{yUd} =$

[16] $= \frac{1}{2}\left(\partial_y g_{yU} + \partial_y g_{Uy} - \partial_U g_{yy}\right) =$

$$d\omega^{\hat{1}} = d\left(\frac{1}{2}(dU - dV)\right) = 0$$
$$d\omega^{\hat{2}} = d(a(U)dx) = \frac{da}{dU}dU \wedge dx = \frac{da}{dU}\left(\omega^{\hat{0}} + \omega^{\hat{1}}\right) \wedge \frac{1}{a(U)}\omega^{\hat{2}} = -\frac{1}{a}\frac{da}{dU}\omega^{\hat{2}} \wedge \left(\omega^{\hat{0}} + \omega^{\hat{1}}\right)$$
$$d\omega^{\hat{3}} = d(b(U)dy) = \frac{db}{dU}dU \wedge dy = \frac{db}{dU}\left(\omega^{\hat{0}} + \omega^{\hat{1}}\right) \wedge \frac{1}{b(U)}\omega^{\hat{3}} = -\frac{1}{b}\frac{db}{dU}\omega^{\hat{3}} \wedge \left(\omega^{\hat{0}} + \omega^{\hat{1}}\right)$$

The curvature one-forms summarized in a matrix

$$\Gamma^{\hat{a}}{}_{\hat{b}} = \left\{\begin{matrix} 0 & 0 & \frac{1}{a}\frac{da}{dU}\omega^{\hat{2}}(A) & \frac{1}{b}\frac{db}{dU}\omega^{\hat{3}}(B) \\ 0 & 0 & \frac{1}{a}\frac{da}{dU}\omega^{\hat{2}}(A) & \frac{1}{b}\frac{db}{dU}\omega^{\hat{3}}(B) \\ \frac{1}{a}\frac{da}{dU}\omega^{\hat{2}}(A) & -\frac{1}{a}\frac{da}{dU}\omega^{\hat{2}}(-A) & 0 & 0 \\ \frac{1}{b}\frac{db}{dU}\omega^{\hat{3}}(B) & -\frac{1}{b}\frac{db}{dU}\omega^{\hat{3}}(-B) & 0 & 0 \end{matrix}\right\}$$

Where $\hat{a}$ refers to column and $\hat{b}$ to row and A and B will be used later, to make the calculations easier

14.5.4 The curvature two forms

$$\Omega^{\hat{a}}{}_{\hat{b}} = d\Gamma^{\hat{a}}{}_{\hat{b}} + \Gamma^{\hat{a}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{b}} = \frac{1}{2}R^{\hat{a}}{}_{\hat{b}\hat{c}\hat{d}}\omega^{\hat{c}} \wedge \omega^{\hat{d}}$$

First we will calculate

$$\begin{aligned} dA &= d\left(\frac{1}{a}\frac{da}{dU}\omega^{\hat{2}}\right) \\ &= d\left(\frac{da}{dU}dx\right) \\ &= \frac{d^2a}{dU^2}dU \wedge dx \\ &= \frac{d^2a}{dU^2}\left(\omega^{\hat{0}} + \omega^{\hat{1}}\right) \wedge \frac{1}{a}\omega^{\hat{2}} \\ &= \frac{1}{a}\frac{d^2a}{dU^2}\left(\omega^{\hat{0}} \wedge \omega^{\hat{2}} + \omega^{\hat{1}} \wedge \omega^{\hat{2}}\right) \\ dB &= d\left(\frac{1}{b}\frac{db}{dU}\omega^{\hat{3}}\right) \\ &= d\left(\frac{db}{dU}dy\right) \\ &= \frac{d^2b}{dU^2}dU \wedge dy \\ &= \frac{d^2b}{dU^2}\left(\omega^{\hat{0}} + \omega^{\hat{1}}\right) \wedge \frac{1}{b}\omega^{\hat{3}} \\ &= \frac{1}{b}\frac{d^2b}{dU^2}\left(\omega^{\hat{0}} \wedge \omega^{\hat{3}} + \omega^{\hat{1}} \wedge \omega^{\hat{3}}\right) \end{aligned}$$

Now we are ready to calculate the curvature two-forms

$$\begin{aligned} \Omega^{\hat{0}}{}_{\hat{0}} &= d\Gamma^{\hat{0}}{}_{\hat{0}} + \Gamma^{\hat{0}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{0}} = \Gamma^{\hat{0}}{}_{\hat{0}} \wedge \Gamma^{\hat{0}}{}_{\hat{0}} + \Gamma^{\hat{0}}{}_{\hat{1}} \wedge \Gamma^{\hat{1}}{}_{\hat{0}} + \Gamma^{\hat{0}}{}_{\hat{2}} \wedge \Gamma^{\hat{2}}{}_{\hat{0}} + \Gamma^{\hat{0}}{}_{\hat{3}} \wedge \Gamma^{\hat{3}}{}_{\hat{0}} = 0 \\ \Omega^{\hat{1}}{}_{\hat{0}} &= d\Gamma^{\hat{1}}{}_{\hat{0}} + \Gamma^{\hat{1}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{0}} = \Gamma^{\hat{1}}{}_{\hat{0}} \wedge \Gamma^{\hat{0}}{}_{\hat{0}} + \Gamma^{\hat{1}}{}_{\hat{1}} \wedge \Gamma^{\hat{1}}{}_{\hat{0}} + \Gamma^{\hat{1}}{}_{\hat{2}} \wedge \Gamma^{\hat{2}}{}_{\hat{0}} + \Gamma^{\hat{1}}{}_{\hat{3}} \wedge \Gamma^{\hat{3}}{}_{\hat{0}} = 0 \\ \Omega^{\hat{2}}{}_{\hat{0}} &= d\Gamma^{\hat{2}}{}_{\hat{0}} + \Gamma^{\hat{2}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{0}} \\ &= d\Gamma^{\hat{2}}{}_{\hat{0}} + \Gamma^{\hat{2}}{}_{\hat{0}} \wedge \Gamma^{\hat{0}}{}_{\hat{0}} + \Gamma^{\hat{2}}{}_{\hat{1}} \wedge \Gamma^{\hat{1}}{}_{\hat{0}} + \Gamma^{\hat{2}}{}_{\hat{2}} \wedge \Gamma^{\hat{2}}{}_{\hat{0}} + \Gamma^{\hat{2}}{}_{\hat{3}} \wedge \Gamma^{\hat{3}}{}_{\hat{0}} \\ &= \frac{1}{a}\frac{d^2a}{dU^2}\left(\omega^{\hat{0}} \wedge \omega^{\hat{2}} + \omega^{\hat{1}} \wedge \omega^{\hat{2}}\right) \end{aligned}$$

$$
\begin{aligned}
\Omega^{\hat{3}}{}_{\hat{0}} &= d\Gamma^{\hat{3}}{}_{\hat{0}} + \Gamma^{\hat{3}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{0}} \\
&= d\Gamma^{\hat{3}}{}_{\hat{0}} + \Gamma^{\hat{3}}{}_{\hat{0}} \wedge \Gamma^{\hat{0}}{}_{\hat{0}} + \Gamma^{\hat{3}}{}_{\hat{1}} \wedge \Gamma^{\hat{1}}{}_{\hat{0}} + \Gamma^{\hat{3}}{}_{\hat{2}} \wedge \Gamma^{\hat{2}}{}_{\hat{0}} + \Gamma^{\hat{3}}{}_{\hat{3}} \wedge \Gamma^{\hat{3}}{}_{\hat{0}} \\
&= \frac{1}{b}\frac{d^2b}{dU^2}\left(\omega^{\hat{0}} \wedge \omega^{\hat{3}} + \omega^{\hat{1}} \wedge \omega^{\hat{3}}\right) \\
\Omega^{\hat{1}}{}_{\hat{1}} &= d\Gamma^{\hat{1}}{}_{\hat{1}} + \Gamma^{\hat{1}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{1}} = \Gamma^{\hat{1}}{}_{\hat{0}} \wedge \Gamma^{\hat{0}}{}_{\hat{1}} + \Gamma^{\hat{1}}{}_{\hat{1}} \wedge \Gamma^{\hat{1}}{}_{\hat{1}} + \Gamma^{\hat{1}}{}_{\hat{2}} \wedge \Gamma^{\hat{2}}{}_{\hat{1}} + \Gamma^{\hat{1}}{}_{\hat{3}} \wedge \Gamma^{\hat{3}}{}_{\hat{1}} = 0 \\
\Omega^{\hat{2}}{}_{\hat{1}} &= d\Gamma^{\hat{2}}{}_{\hat{1}} + \Gamma^{\hat{2}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{1}} \\
&= d\Gamma^{\hat{2}}{}_{\hat{1}} + \Gamma^{\hat{2}}{}_{\hat{0}} \wedge \Gamma^{\hat{0}}{}_{\hat{1}} + \Gamma^{\hat{2}}{}_{\hat{1}} \wedge \Gamma^{\hat{1}}{}_{\hat{1}} + \Gamma^{\hat{2}}{}_{\hat{2}} \wedge \Gamma^{\hat{2}}{}_{\hat{1}} + \Gamma^{\hat{2}}{}_{\hat{3}} \wedge \Gamma^{\hat{3}}{}_{\hat{1}} \\
&= \frac{1}{a}\frac{d^2a}{dU^2}\left(\omega^{\hat{0}} \wedge \omega^{\hat{2}} + \omega^{\hat{1}} \wedge \omega^{\hat{2}}\right) \\
\Omega^{\hat{3}}{}_{\hat{1}} &= d\Gamma^{\hat{3}}{}_{\hat{1}} + \Gamma^{\hat{3}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{1}} \\
&= d\Gamma^{\hat{3}}{}_{\hat{1}} + \Gamma^{\hat{3}}{}_{\hat{0}} \wedge \Gamma^{\hat{0}}{}_{\hat{1}} + \Gamma^{\hat{3}}{}_{\hat{1}} \wedge \Gamma^{\hat{1}}{}_{\hat{1}} + \Gamma^{\hat{3}}{}_{\hat{2}} \wedge \Gamma^{\hat{2}}{}_{\hat{1}} + \Gamma^{\hat{3}}{}_{\hat{3}} \wedge \Gamma^{\hat{3}}{}_{\hat{1}} \\
&= \frac{1}{b}\frac{d^2b}{dU^2}\left(\omega^{\hat{0}} \wedge \omega^{\hat{3}} + \omega^{\hat{1}} \wedge \omega^{\hat{3}}\right) \\
\Omega^{\hat{2}}{}_{\hat{2}} &= d\Gamma^{\hat{2}}{}_{\hat{2}} + \Gamma^{\hat{2}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{2}} = \Gamma^{\hat{2}}{}_{\hat{0}} \wedge \Gamma^{\hat{0}}{}_{\hat{2}} + \Gamma^{\hat{2}}{}_{\hat{1}} \wedge \Gamma^{\hat{1}}{}_{\hat{2}} + \Gamma^{\hat{2}}{}_{\hat{2}} \wedge \Gamma^{\hat{2}}{}_{\hat{2}} + \Gamma^{\hat{2}}{}_{\hat{3}} \wedge \Gamma^{\hat{3}}{}_{\hat{2}} = 0 \\
\Omega^{\hat{3}}{}_{\hat{2}} &= d\Gamma^{\hat{3}}{}_{\hat{2}} + \Gamma^{\hat{3}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{2}} \\
&= \Gamma^{\hat{3}}{}_{\hat{0}} \wedge \Gamma^{\hat{0}}{}_{\hat{2}} + \Gamma^{\hat{3}}{}_{\hat{1}} \wedge \Gamma^{\hat{1}}{}_{\hat{2}} + \Gamma^{\hat{3}}{}_{\hat{2}} \wedge \Gamma^{\hat{2}}{}_{\hat{2}} + \Gamma^{\hat{3}}{}_{\hat{3}} \wedge \Gamma^{\hat{3}}{}_{\hat{2}} \\
&= \frac{1}{b}\frac{db}{dU}\omega^{\hat{3}} \wedge \frac{1}{a}\frac{da}{dU}\omega^{\hat{2}} + \frac{1}{b}\frac{db}{dU}\omega^{\hat{3}} \wedge \left(-\frac{1}{a}\frac{da}{dU}\omega^{\hat{2}}\right) \\
&= 0 \\
\Omega^{\hat{3}}{}_{\hat{3}} &= d\Gamma^{\hat{3}}{}_{\hat{3}} + \Gamma^{\hat{3}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{3}} = \Gamma^{\hat{3}}{}_{\hat{0}} \wedge \Gamma^{\hat{0}}{}_{\hat{3}} + \Gamma^{\hat{3}}{}_{\hat{1}} \wedge \Gamma^{\hat{1}}{}_{\hat{3}} + \Gamma^{\hat{3}}{}_{\hat{2}} \wedge \Gamma^{\hat{2}}{}_{\hat{3}} + \Gamma^{\hat{3}}{}_{\hat{3}} \wedge \Gamma^{\hat{3}}{}_{\hat{3}} = 0
\end{aligned}
$$

Summarized in a matrix:

$$
\Omega^{\hat{a}}{}_{\hat{b}} = \begin{Bmatrix}
0 & 0 & \frac{1}{a}\frac{d^2a}{dU^2}\left(\omega^{\hat{0}} \wedge \omega^{\hat{2}} + \omega^{\hat{1}} \wedge \omega^{\hat{2}}\right) & \frac{1}{b}\frac{d^2b}{dU^2}\left(\omega^{\hat{0}} \wedge \omega^{\hat{3}} + \omega^{\hat{1}} \wedge \omega^{\hat{3}}\right) \\
0 & 0 & \frac{1}{a}\frac{d^2a}{dU^2}\left(\omega^{\hat{0}} \wedge \omega^{\hat{2}} + \omega^{\hat{1}} \wedge \omega^{\hat{2}}\right) & \frac{1}{b}\frac{d^2b}{dU^2}\left(\omega^{\hat{0}} \wedge \omega^{\hat{3}} + \omega^{\hat{1}} \wedge \omega^{\hat{3}}\right) \\
S & AS & 0 & 0 \\
S & AS & 0 & 0
\end{Bmatrix}
$$

Where $\hat{a}$ refers to column and $\hat{b}$ to row

Now we can write down the independent elements of the Riemann tensor in the non-coordinate basis:

$$
\begin{aligned}
R^{\hat{2}}{}_{\hat{0}\hat{2}\hat{0}} &= -\frac{1}{a}\frac{d^2a}{dU^2} & R^{\hat{3}}{}_{\hat{0}\hat{3}\hat{0}} &= -\frac{1}{b}\frac{d^2b}{dU^2} \\
R^{\hat{2}}{}_{\hat{0}\hat{2}\hat{1}} &= -\frac{1}{a}\frac{d^2a}{dU^2} & R^{\hat{3}}{}_{\hat{0}\hat{3}\hat{1}} &= -\frac{1}{b}\frac{d^2b}{dU^2} \\
R^{\hat{2}}{}_{\hat{1}\hat{2}\hat{1}} &= -\frac{1}{a}\frac{d^2a}{dU^2} & R^{\hat{3}}{}_{\hat{1}\hat{3}\hat{1}} &= -\frac{1}{b}\frac{d^2b}{dU^2}
\end{aligned}
$$

14.5.4.1 [j]The Riemann tensor in the coordinate basis

The transformation:

$$
R_{abcd} = \Lambda^{\hat{e}}{}_{a}\Lambda^{\hat{f}}{}_{b}\Lambda^{\hat{g}}{}_{c}\Lambda^{\hat{h}}{}_{d}R_{\hat{e}\hat{f}\hat{g}\hat{h}}
$$

$$\Lambda^{\hat b}{}_a = \text{[17]} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & & \\ \frac{1}{2} & -\frac{1}{2} & & \\ & & a(U) & \\ & & & b(U) \end{Bmatrix}$$

$$\begin{aligned}
R_{0202} &= \Lambda^{\hat e}{}_0 \Lambda^{\hat f}{}_2 \Lambda^{\hat g}{}_0 \Lambda^{\hat h}{}_2 R_{\hat e \hat f \hat g \hat h} \\
&= \Lambda^{\hat e}{}_0 \left(\Lambda^{\hat 2}{}_2\right)^2 \Lambda^{\hat g}{}_0 R_{\hat e \hat 2 \hat g \hat 2} \\
&= \Lambda^{\hat 0}{}_0 \left(\Lambda^{\hat 2}{}_2\right)^2 \Lambda^{\hat g}{}_0 R_{\hat 0 \hat 2 \hat g \hat 2} + \Lambda^{\hat 1}{}_0 \left(\Lambda^{\hat 2}{}_2\right)^2 \Lambda^{\hat g}{}_0 R_{\hat 1 \hat 2 \hat g \hat 2} \\
&= \Lambda^{\hat 0}{}_0 \left(\Lambda^{\hat 2}{}_2\right)^2 \Lambda^{\hat 0}{}_0 R_{\hat 0 \hat 2 \hat 0 \hat 2} + \Lambda^{\hat 1}{}_0 \left(\Lambda^{\hat 2}{}_2\right)^2 \Lambda^{\hat 0}{}_0 R_{\hat 1 \hat 2 \hat 0 \hat 2} + \Lambda^{\hat 0}{}_0 \left(\Lambda^{\hat 2}{}_2\right)^2 \Lambda^{\hat 1}{}_0 R_{\hat 0 \hat 2 \hat 1 \hat 2} + \Lambda^{\hat 1}{}_0 \left(\Lambda^{\hat 2}{}_2\right)^2 \Lambda^{\hat 1}{}_0 R_{\hat 1 \hat 2 \hat 1 \hat 2} \\
&= \text{[18]} \frac{1}{4} \left(a(U)\right)^2 \left(R_{\hat 0 \hat 2 \hat 0 \hat 2} + 2R_{\hat 0 \hat 2 \hat 1 \hat 2} + R_{\hat 1 \hat 2 \hat 1 \hat 2}\right) \\
&= \frac{1}{4} \left(a(U)\right)^2 \left(R_{\hat 2 \hat 0 \hat 2 \hat 0} + 2R_{\hat 2 \hat 0 \hat 2 \hat 1} + R_{\hat 2 \hat 1 \hat 2 \hat 1}\right) \\
&= \frac{1}{4} \left(a(U)\right)^2 \eta_{\hat 2 \hat 2} \left(R^{\hat 2}{}_{\hat 0 \hat 2 \hat 0} + 2R^{\hat 2}{}_{\hat 0 \hat 2 \hat 1} + R^{\hat 2}{}_{\hat 1 \hat 2 \hat 1}\right) \\
&= -\frac{1}{4} \left(a(U)\right)^2 4 \left(-\frac{1}{a} \frac{d^2 a}{dU^2}\right) \\
&= a(U) \frac{d^2 a}{dU^2} \\
R_{0303} &= \Lambda^{\hat e}{}_0 \Lambda^{\hat f}{}_3 \Lambda^{\hat g}{}_0 \Lambda^{\hat h}{}_3 R_{\hat e \hat f \hat g \hat h} \\
&= \Lambda^{\hat e}{}_0 \left(\Lambda^{\hat 3}{}_3\right)^2 \Lambda^{\hat g}{}_0 R_{\hat e \hat 3 \hat g \hat 3} \\
&= \Lambda^{\hat 0}{}_0 \left(\Lambda^{\hat 3}{}_3\right)^2 \Lambda^{\hat g}{}_0 R_{\hat 0 \hat 3 \hat g \hat 3} + \Lambda^{\hat 1}{}_0 \left(\Lambda^{\hat 3}{}_3\right)^2 \Lambda^{\hat g}{}_0 R_{\hat 1 \hat 3 \hat g \hat 3} \\
&= \Lambda^{\hat 0}{}_0 \left(\Lambda^{\hat 3}{}_3\right)^2 \Lambda^{\hat 0}{}_0 R_{\hat 0 \hat 3 \hat 0 \hat 3} + \Lambda^{\hat 1}{}_0 \left(\Lambda^{\hat 3}{}_3\right)^2 \Lambda^{\hat 0}{}_0 R_{\hat 1 \hat 3 \hat 0 \hat 3} + \Lambda^{\hat 0}{}_0 \left(\Lambda^{\hat 3}{}_3\right)^2 \Lambda^{\hat 1}{}_0 R_{\hat 0 \hat 3 \hat 1 \hat 3} + \Lambda^{\hat 1}{}_0 \left(\Lambda^{\hat 3}{}_3\right)^2 \Lambda^{\hat 1}{}_0 R_{\hat 1 \hat 3 \hat 1 \hat 3} \\
&= \text{[19]} \frac{1}{4} \left(b(U)\right)^2 \left(R_{\hat 0 \hat 3 \hat 0 \hat 3} + 2R_{\hat 0 \hat 3 \hat 1 \hat 3} + R_{\hat 1 \hat 3 \hat 1 \hat 3}\right) \\
&= \frac{1}{4} \left(b(U)\right)^2 \left(R_{\hat 3 \hat 0 \hat 3 \hat 0} + 2R_{\hat 3 \hat 0 \hat 3 \hat 1} + R_{\hat 3 \hat 1 \hat 3 \hat 1}\right) \\
&= \frac{1}{4} \left(b(U)\right)^2 \eta_{\hat 3 \hat 3} \left(R^{\hat 3}{}_{\hat 0 \hat 3 \hat 0} + 2R^{\hat 3}{}_{\hat 0 \hat 3 \hat 1} + R^{\hat 3}{}_{\hat 1 \hat 3 \hat 1}\right) \\
&= -\frac{1}{4} \left(b(U)\right)^2 4 \left(-\frac{1}{b} \frac{d^2 b}{dU^2}\right) \\
&= b(U) \frac{d^2 b}{dU^2} \\
R_{0212} &= \Lambda^{\hat e}{}_0 \Lambda^{\hat f}{}_2 \Lambda^{\hat g}{}_1 \Lambda^{\hat h}{}_2 R_{\hat e \hat f \hat g \hat h} \\
&= \Lambda^{\hat e}{}_0 \left(\Lambda^{\hat 2}{}_2\right)^2 \Lambda^{\hat g}{}_1 R_{\hat e \hat 2 \hat g \hat 2} \\
&= \Lambda^{\hat 0}{}_0 \left(\Lambda^{\hat 2}{}_2\right)^2 \Lambda^{\hat g}{}_1 R_{\hat 0 \hat 2 \hat g \hat 2} + \Lambda^{\hat 1}{}_0 \left(\Lambda^{\hat 2}{}_2\right)^2 \Lambda^{\hat g}{}_1 R_{\hat 1 \hat 2 \hat g \hat 2} \\
&= \Lambda^{\hat 0}{}_0 \left(\Lambda^{\hat 2}{}_2\right)^2 \Lambda^{\hat 0}{}_1 R_{\hat 0 \hat 2 \hat 0 \hat 2} + \Lambda^{\hat 1}{}_0 \left(\Lambda^{\hat 2}{}_2\right)^2 \Lambda^{\hat 0}{}_1 R_{\hat 1 \hat 2 \hat 0 \hat 2} + \Lambda^{\hat 0}{}_0 \left(\Lambda^{\hat 2}{}_2\right)^2 \Lambda^{\hat 1}{}_1 R_{\hat 0 \hat 2 \hat 1 \hat 2} + \Lambda^{\hat 1}{}_0 \left(\Lambda^{\hat 2}{}_2\right)^2 \Lambda^{\hat 1}{}_1 R_{\hat 1 \hat 2 \hat 1 \hat 2} \\
&= 0
\end{aligned}$$

14.5.5 The Ricci tensor

$$R_{\hat a \hat b} = R^{\hat c}{}_{\hat a \hat c \hat b}$$

---

[17] $\omega^{\hat b} = \Lambda^{\hat b}{}_a dx^a$

[18] $R_{\hat 1 \hat 2 \hat 0 \hat 2} = R_{\hat 0 \hat 2 \hat 1 \hat 2}$

[19] $R_{\hat 1 \hat 3 \hat 0 \hat 3} = R_{\hat 0 \hat 3 \hat 1 \hat 3}$

$$
\begin{aligned}
R_{\hat{0}\hat{0}} &= R^{\hat{c}}{}_{\hat{0}\hat{c}\hat{0}} = R^{\hat{0}}{}_{\hat{0}\hat{0}\hat{0}} + R^{\hat{1}}{}_{\hat{0}\hat{1}\hat{0}} + R^{\hat{2}}{}_{\hat{0}\hat{2}\hat{0}} + R^{\hat{3}}{}_{\hat{0}\hat{3}\hat{0}} = -\left(\frac{1}{a}\frac{d^2a}{dU^2} + \frac{1}{b}\frac{d^2b}{dU^2}\right)\\
R_{\hat{1}\hat{0}} &= R^{\hat{c}}{}_{\hat{1}\hat{c}\hat{0}} = R^{\hat{0}}{}_{\hat{1}\hat{0}\hat{0}} + R^{\hat{1}}{}_{\hat{1}\hat{1}\hat{0}} + R^{\hat{2}}{}_{\hat{1}\hat{2}\hat{0}} + R^{\hat{3}}{}_{\hat{1}\hat{3}\hat{0}} = -\left(\frac{1}{a}\frac{d^2a}{dU^2} + \frac{1}{b}\frac{d^2b}{dU^2}\right)\\
R_{\hat{2}\hat{0}} &= R^{\hat{c}}{}_{\hat{2}\hat{c}\hat{0}} = R^{\hat{0}}{}_{\hat{2}\hat{0}\hat{0}} + R^{\hat{1}}{}_{\hat{2}\hat{1}\hat{0}} + R^{\hat{2}}{}_{\hat{2}\hat{2}\hat{0}} + R^{\hat{3}}{}_{\hat{2}\hat{3}\hat{0}} = 0\\
R_{\hat{3}\hat{0}} &= R^{\hat{c}}{}_{\hat{3}\hat{c}\hat{0}} = R^{\hat{0}}{}_{\hat{3}\hat{0}\hat{0}} + R^{\hat{1}}{}_{\hat{3}\hat{1}\hat{0}} + R^{\hat{2}}{}_{\hat{3}\hat{2}\hat{0}} + R^{\hat{3}}{}_{\hat{3}\hat{3}\hat{0}} = 0\\
R_{\hat{0}\hat{1}} &= R^{\hat{c}}{}_{\hat{0}\hat{c}\hat{1}} = R^{\hat{0}}{}_{\hat{0}\hat{0}\hat{1}} + R^{\hat{1}}{}_{\hat{0}\hat{1}\hat{1}} + R^{\hat{2}}{}_{\hat{0}\hat{2}\hat{1}} + R^{\hat{3}}{}_{\hat{0}\hat{3}\hat{1}} = -\left(\frac{1}{a}\frac{d^2a}{dU^2} + \frac{1}{b}\frac{d^2b}{dU^2}\right)\\
R_{\hat{1}\hat{1}} &= R^{\hat{c}}{}_{\hat{1}\hat{c}\hat{1}} = R^{\hat{0}}{}_{\hat{1}\hat{0}\hat{1}} + R^{\hat{1}}{}_{\hat{1}\hat{1}\hat{1}} + R^{\hat{2}}{}_{\hat{1}\hat{2}\hat{1}} + R^{\hat{3}}{}_{\hat{1}\hat{3}\hat{1}} = -\left(\frac{1}{a}\frac{d^2a}{dU^2} + \frac{1}{b}\frac{d^2b}{dU^2}\right)\\
R_{\hat{1}\hat{2}} &= R^{\hat{c}}{}_{\hat{1}\hat{c}\hat{2}} = R^{\hat{0}}{}_{\hat{1}\hat{0}\hat{2}} + R^{\hat{1}}{}_{\hat{1}\hat{1}\hat{2}} + R^{\hat{2}}{}_{\hat{1}\hat{2}\hat{2}} + R^{\hat{3}}{}_{\hat{1}\hat{3}\hat{2}} = 0\\
R_{\hat{1}\hat{3}} &= R^{\hat{c}}{}_{\hat{1}\hat{c}\hat{3}} = R^{\hat{0}}{}_{\hat{1}\hat{0}\hat{3}} + R^{\hat{1}}{}_{\hat{1}\hat{1}\hat{3}} + R^{\hat{2}}{}_{\hat{1}\hat{2}\hat{3}} + R^{\hat{3}}{}_{\hat{1}\hat{3}\hat{3}} = 0\\
R_{\hat{0}\hat{2}} &= R^{\hat{c}}{}_{\hat{0}\hat{c}\hat{2}} = R^{\hat{0}}{}_{\hat{0}\hat{0}\hat{2}} + R^{\hat{1}}{}_{\hat{0}\hat{1}\hat{2}} + R^{\hat{2}}{}_{\hat{0}\hat{2}\hat{2}} + R^{\hat{3}}{}_{\hat{0}\hat{3}\hat{2}} = 0\\
R_{\hat{1}\hat{2}} &= R^{\hat{c}}{}_{\hat{1}\hat{c}\hat{2}} = R^{\hat{0}}{}_{\hat{1}\hat{0}\hat{2}} + R^{\hat{1}}{}_{\hat{1}\hat{1}\hat{2}} + R^{\hat{2}}{}_{\hat{1}\hat{2}\hat{2}} + R^{\hat{3}}{}_{\hat{1}\hat{3}\hat{2}} = 0\\
R_{\hat{2}\hat{2}} &= R^{\hat{c}}{}_{\hat{2}\hat{c}\hat{2}} = R^{\hat{0}}{}_{\hat{2}\hat{0}\hat{2}} + R^{\hat{1}}{}_{\hat{2}\hat{1}\hat{2}} + R^{\hat{2}}{}_{\hat{2}\hat{2}\hat{2}} + R^{\hat{3}}{}_{\hat{2}\hat{3}\hat{2}} = \frac{1}{a}\frac{d^2a}{dU^2} - \frac{1}{a}\frac{d^2a}{dU^2} = 0\\
R_{\hat{3}\hat{2}} &= R^{\hat{c}}{}_{\hat{3}\hat{c}\hat{2}} = 0\\
R_{\hat{0}\hat{3}} &= R^{\hat{c}}{}_{\hat{0}\hat{c}\hat{3}} = R^{\hat{0}}{}_{\hat{0}\hat{0}\hat{3}} + R^{\hat{1}}{}_{\hat{0}\hat{1}\hat{3}} + R^{\hat{2}}{}_{\hat{0}\hat{2}\hat{3}} + R^{\hat{3}}{}_{\hat{0}\hat{3}\hat{3}} = 0\\
R_{\hat{1}\hat{3}} &= R^{\hat{c}}{}_{\hat{1}\hat{c}\hat{3}} = R^{\hat{0}}{}_{\hat{1}\hat{0}\hat{3}} + R^{\hat{1}}{}_{\hat{1}\hat{1}\hat{3}} + R^{\hat{2}}{}_{\hat{1}\hat{2}\hat{3}} + R^{\hat{3}}{}_{\hat{1}\hat{3}\hat{3}} = 0\\
R_{\hat{3}\hat{3}} &= R^{\hat{c}}{}_{\hat{3}\hat{c}\hat{3}} = R^{\hat{0}}{}_{\hat{3}\hat{0}\hat{3}} + R^{\hat{1}}{}_{\hat{3}\hat{1}\hat{3}} + R^{\hat{2}}{}_{\hat{3}\hat{2}\hat{3}} + R^{\hat{3}}{}_{\hat{3}\hat{3}\hat{3}} = \frac{1}{b}\frac{d^2b}{dU^2} - \frac{1}{b}\frac{d^2b}{dU^2} = 0
\end{aligned}
$$

Summarized in a matrix:

$$
R_{\hat{a}\hat{b}} = \begin{Bmatrix}
-\left(\frac{1}{a}\frac{d^2a}{dU^2} + \frac{1}{b}\frac{d^2b}{dU^2}\right) & -\left(\frac{1}{a}\frac{d^2a}{dU^2} + \frac{1}{b}\frac{d^2b}{dU^2}\right) & 0 & 0\\
-\left(\frac{1}{a}\frac{d^2a}{dU^2} + \frac{1}{b}\frac{d^2b}{dU^2}\right) & -\left(\frac{1}{a}\frac{d^2a}{dU^2} + \frac{1}{b}\frac{d^2b}{dU^2}\right) & 0 & 0\\
0 & 0 & 0 & 0\\
0 & 0 & 0 & 0
\end{Bmatrix}
$$

Where $\hat{a}$ refers to column and $\hat{b}$ to row

14.5.6 The Ricci scalar

$$
R = \eta^{\hat{a}\hat{b}}R_{\hat{a}\hat{b}} = R_{\hat{0}\hat{0}} - R_{\hat{1}\hat{1}} - R_{\hat{2}\hat{2}} - R_{\hat{3}\hat{3}} = -\left(\frac{1}{a}\frac{d^2a}{dU^2} + \frac{1}{b}\frac{d^2b}{dU^2}\right) - \left(-\left(\frac{1}{a}\frac{d^2a}{dU^2} + \frac{1}{b}\frac{d^2b}{dU^2}\right)\right) = 0
$$

14.5.7 The Einstein tensor

$$
G_{\hat{a}\hat{b}} = R_{\hat{a}\hat{b}} - \frac{1}{2}\eta_{\hat{a}\hat{b}}R = R_{\hat{a}\hat{b}}
$$

Summarized in a matrix:

$$G_{\hat{a}\hat{b}} = \begin{Bmatrix} -\left(\frac{1}{a}\frac{d^2a}{dU^2}+\frac{1}{b}\frac{d^2b}{dU^2}\right) & -\left(\frac{1}{a}\frac{d^2a}{dU^2}+\frac{1}{b}\frac{d^2b}{dU^2}\right) & 0 & 0 \\ -\left(\frac{1}{a}\frac{d^2a}{dU^2}+\frac{1}{b}\frac{d^2b}{dU^2}\right) & -\left(\frac{1}{a}\frac{d^2a}{dU^2}+\frac{1}{b}\frac{d^2b}{dU^2}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Bmatrix}$$

Where $\hat{a}$ refers to column and $\hat{b}$ to row

14.5.7.1 The Einstein tensor in the coordinate basis

The transformation:

$$G_{ab} = \Lambda^{\hat{c}}{}_a\Lambda^{\hat{d}}{}_b G_{\hat{c}\hat{d}}$$

$$\Lambda^{\hat{b}}{}_a =^{20} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & & \\ \frac{1}{2} & -\frac{1}{2} & & \\ & & a(U) & \\ & & & b(U) \end{Bmatrix}$$

$$\begin{aligned} G_{00} &= \Lambda^{\hat{c}}{}_0\Lambda^{\hat{d}}{}_0 G_{\hat{c}\hat{d}} \\ &= \Lambda^{\hat{0}}{}_0\Lambda^{\hat{d}}{}_0 G_{\hat{0}\hat{d}} + \Lambda^{\hat{1}}{}_0\Lambda^{\hat{d}}{}_0 G_{\hat{1}\hat{d}} + \Lambda^{\hat{2}}{}_0\Lambda^{\hat{d}}{}_0 G_{\hat{2}\hat{d}} + \Lambda^{\hat{3}}{}_0\Lambda^{\hat{d}}{}_0 G_{\hat{3}\hat{d}} \\ &= \Lambda^{\hat{0}}{}_0\Lambda^{\hat{0}}{}_0 G_{\hat{0}\hat{0}} + \Lambda^{\hat{0}}{}_0\Lambda^{\hat{1}}{}_0 G_{\hat{0}\hat{1}} + \Lambda^{\hat{1}}{}_0\Lambda^{\hat{0}}{}_0 G_{\hat{1}\hat{0}} + \Lambda^{\hat{1}}{}_0\Lambda^{\hat{1}}{}_0 G_{\hat{1}\hat{1}} \\ &=^{21} -\left(\frac{1}{a}\frac{d^2a}{dU^2}+\frac{1}{b}\frac{d^2b}{dU^2}\right) \end{aligned}$$

$$\begin{aligned} G_{10} &= \Lambda^{\hat{c}}{}_1\Lambda^{\hat{d}}{}_0 G_{\hat{c}\hat{d}} \\ &= \Lambda^{\hat{0}}{}_1\Lambda^{\hat{d}}{}_0 G_{\hat{0}\hat{d}} + \Lambda^{\hat{1}}{}_1\Lambda^{\hat{d}}{}_0 G_{\hat{1}\hat{d}} + \Lambda^{\hat{2}}{}_1\Lambda^{\hat{d}}{}_0 G_{\hat{2}\hat{d}} + \Lambda^{\hat{3}}{}_1\Lambda^{\hat{d}}{}_0 G_{\hat{3}\hat{d}} \\ &= \Lambda^{\hat{0}}{}_1\Lambda^{\hat{0}}{}_0 G_{\hat{0}\hat{0}} + \Lambda^{\hat{0}}{}_1\Lambda^{\hat{1}}{}_0 G_{\hat{0}\hat{1}} + \Lambda^{\hat{1}}{}_1\Lambda^{\hat{0}}{}_0 G_{\hat{1}\hat{0}} + +\Lambda^{\hat{1}}{}_1\Lambda^{\hat{1}}{}_0 G_{\hat{1}\hat{1}} \\ &= 0 \end{aligned}$$

$$G_{20} = \Lambda^{\hat{c}}{}_2\Lambda^{\hat{d}}{}_0 G_{\hat{c}\hat{d}} = \Lambda^{\hat{0}}{}_2\Lambda^{\hat{d}}{}_0 G_{\hat{0}\hat{d}} + \Lambda^{\hat{1}}{}_2\Lambda^{\hat{d}}{}_0 G_{\hat{1}\hat{d}} + \Lambda^{\hat{2}}{}_2\Lambda^{\hat{d}}{}_0 G_{\hat{2}\hat{d}} + \Lambda^{\hat{3}}{}_2\Lambda^{\hat{d}}{}_0 G_{\hat{3}\hat{d}} = 0$$

$$G_{30} = \Lambda^{\hat{c}}{}_3\Lambda^{\hat{d}}{}_0 G_{\hat{c}\hat{d}} = \Lambda^{\hat{0}}{}_3\Lambda^{\hat{d}}{}_0 G_{\hat{0}\hat{d}} + \Lambda^{\hat{1}}{}_3\Lambda^{\hat{d}}{}_0 G_{\hat{1}\hat{d}} + \Lambda^{\hat{2}}{}_3\Lambda^{\hat{d}}{}_0 G_{\hat{2}\hat{d}} + \Lambda^{\hat{3}}{}_3\Lambda^{\hat{d}}{}_0 G_{\hat{3}\hat{d}} = 0$$

$$\begin{aligned} G_{11} &= \Lambda^{\hat{c}}{}_1\Lambda^{\hat{d}}{}_1 G_{\hat{c}\hat{d}} \\ &= \Lambda^{\hat{0}}{}_1\Lambda^{\hat{d}}{}_1 G_{\hat{0}\hat{d}} + \Lambda^{\hat{1}}{}_1\Lambda^{\hat{d}}{}_1 G_{\hat{1}\hat{d}} + \Lambda^{\hat{2}}{}_1\Lambda^{\hat{d}}{}_1 G_{\hat{2}\hat{d}} + \Lambda^{\hat{3}}{}_1\Lambda^{\hat{d}}{}_1 G_{\hat{3}\hat{d}} \\ &= \Lambda^{\hat{0}}{}_1\Lambda^{\hat{0}}{}_1 G_{\hat{0}\hat{0}} + \Lambda^{\hat{0}}{}_1\Lambda^{\hat{1}}{}_1 G_{\hat{0}\hat{1}} + \Lambda^{\hat{1}}{}_1\Lambda^{\hat{0}}{}_1 G_{\hat{1}\hat{0}} + \Lambda^{\hat{1}}{}_1\Lambda^{\hat{1}}{}_1 G_{\hat{1}\hat{1}} \\ &= 0 \end{aligned}$$

$$G_{21} = \Lambda^{\hat{c}}{}_2\Lambda^{\hat{d}}{}_1 G_{\hat{c}\hat{d}} = \Lambda^{\hat{0}}{}_2\Lambda^{\hat{d}}{}_1 G_{\hat{0}\hat{d}} + \Lambda^{\hat{1}}{}_2\Lambda^{\hat{d}}{}_1 G_{\hat{1}\hat{d}} + \Lambda^{\hat{2}}{}_2\Lambda^{\hat{d}}{}_1 G_{\hat{2}\hat{d}} + \Lambda^{\hat{3}}{}_2\Lambda^{\hat{d}}{}_1 G_{\hat{3}\hat{d}} = 0$$

$$G_{31} = \Lambda^{\hat{c}}{}_3\Lambda^{\hat{d}}{}_1 G_{\hat{c}\hat{d}} = \Lambda^{\hat{0}}{}_3\Lambda^{\hat{d}}{}_1 G_{\hat{0}\hat{d}} + \Lambda^{\hat{1}}{}_3\Lambda^{\hat{d}}{}_1 G_{\hat{1}\hat{d}} + \Lambda^{\hat{2}}{}_3\Lambda^{\hat{d}}{}_1 G_{\hat{2}\hat{d}} + \Lambda^{\hat{3}}{}_3\Lambda^{\hat{d}}{}_1 G_{\hat{3}\hat{d}} = 0$$

$$G_{22} = \Lambda^{\hat{c}}{}_2\Lambda^{\hat{d}}{}_2 G_{\hat{c}\hat{d}} = \Lambda^{\hat{0}}{}_2\Lambda^{\hat{d}}{}_2 G_{\hat{0}\hat{d}} + \Lambda^{\hat{1}}{}_2\Lambda^{\hat{d}}{}_2 G_{\hat{1}\hat{d}} + \Lambda^{\hat{2}}{}_2\Lambda^{\hat{d}}{}_2 G_{\hat{2}\hat{d}} + \Lambda^{\hat{3}}{}_2\Lambda^{\hat{d}}{}_2 G_{\hat{3}\hat{d}} = 0$$

$$G_{32} = \Lambda^{\hat{c}}{}_3\Lambda^{\hat{d}}{}_2 G_{\hat{c}\hat{d}} = \Lambda^{\hat{0}}{}_3\Lambda^{\hat{d}}{}_2 G_{\hat{0}\hat{d}} + \Lambda^{\hat{1}}{}_3\Lambda^{\hat{d}}{}_2 G_{\hat{1}\hat{d}} + \Lambda^{\hat{2}}{}_3\Lambda^{\hat{d}}{}_2 G_{\hat{2}\hat{d}} + \Lambda^{\hat{3}}{}_3\Lambda^{\hat{d}}{}_2 G_{\hat{3}\hat{d}} = 0$$

$$G_{33} = \Lambda^{\hat{c}}{}_3\Lambda^{\hat{d}}{}_3 G_{\hat{c}\hat{d}} = \Lambda^{\hat{0}}{}_3\Lambda^{\hat{d}}{}_3 G_{\hat{0}\hat{d}} + \Lambda^{\hat{1}}{}_3\Lambda^{\hat{d}}{}_3 G_{\hat{1}\hat{d}} + \Lambda^{\hat{2}}{}_3\Lambda^{\hat{d}}{}_3 G_{\hat{2}\hat{d}} + \Lambda^{\hat{3}}{}_3\Lambda^{\hat{d}}{}_3 G_{\hat{3}\hat{d}} = 0$$

$$G_{ab} = \begin{Bmatrix} -\left(\frac{1}{a}\frac{d^2a}{dU^2}+\frac{1}{b}\frac{d^2b}{dU^2}\right) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Bmatrix}$$

Which leaves us with the single Einstein equation

---

[20] $\omega^{\hat{b}} = \Lambda^{\hat{b}}{}_a dx^a$

[21] $= -\frac{1}{2}\cdot\frac{1}{2}\left(\frac{1}{a}\frac{d^2a}{dU^2}+\frac{1}{b}\frac{d^2b}{dU^2}\right) - \frac{1}{2}\cdot\frac{1}{2}\left(\frac{1}{a}\frac{d^2a}{dU^2}+\frac{1}{b}\frac{d^2b}{dU^2}\right) - \frac{1}{2}\cdot\frac{1}{2}\left(\frac{1}{a}\frac{d^2a}{dU^2}+\frac{1}{b}\frac{d^2b}{dU^2}\right) - \frac{1}{2}\cdot\frac{1}{2}\left(\frac{1}{a}\frac{d^2a}{dU^2}+\frac{1}{b}\frac{d^2b}{dU^2}\right) =$

$$G_{00} = -\left(\frac{1}{a}\frac{d^2a}{dU^2} + \frac{1}{b}\frac{d^2b}{dU^2}\right)$$

14.5.8 The vacuum solution

$$G_{00} = 0$$

$$\Rightarrow \quad 0 = \frac{a''(U)}{a} + \frac{b''(U)}{b}$$

So a vacuum solution requires

$$\frac{a''(U)}{a} = -\frac{b''(U)}{b}$$

14.5.9 [k]Transformation of the Rosen line-element

Often the Rosen line-element is written

$$ds^2 = 2dudv - f^2(u)dx^2 - g^2(u)dy^2$$

This corresponds to the transformation

$$\begin{aligned} U &= \sqrt{2}u \\ V &= \sqrt{2}v \\ x &= x \\ y &= y \\ a(U) &= f(u) \\ b(U) &= g(u) \end{aligned}$$

$$\Rightarrow \quad g_{ab} = \begin{Bmatrix} & 1 & & \\ 1 & & & \\ & & -f^2(u) & \\ & & & -g^2(u) \end{Bmatrix}$$

$$\frac{da(U)}{dU} = \frac{df(u)}{dU} = \frac{du}{dU}\frac{df(u)}{du} = \frac{1}{\sqrt{2}}\frac{df(u)}{du}$$

$$\frac{db(U)}{dU} = \frac{dg(u)}{dU} = \frac{du}{dU}\frac{dg(u)}{du} = \frac{1}{\sqrt{2}}\frac{dh(u)}{du}$$

$$\frac{d^2a}{dU^2} = \frac{d}{dU}\left(\frac{1}{\sqrt{2}}\frac{df(u)}{du}\right) = \frac{1}{\sqrt{2}}\frac{du}{dU}\frac{d}{du}\left(\frac{df(u)}{du}\right) = \frac{1}{2}\frac{d^2f}{du^2}$$

$$\frac{d^2b}{dU^2} = \frac{d}{dU}\left(\frac{1}{\sqrt{2}}\frac{dg(u)}{du}\right) = \frac{1}{\sqrt{2}}\frac{du}{dU}\frac{d}{du}\left(\frac{dg(u)}{du}\right) = \frac{1}{2}\frac{d^2g}{du^2}$$

The independent elements of the Riemann tensor

$$R^{\hat{2}}{}_{\hat{0}\hat{2}\hat{0}} = R^{\hat{2}}{}_{\hat{0}\hat{2}\hat{1}} = R^{\hat{2}}{}_{\hat{1}\hat{2}\hat{1}} = -\frac{1}{a}\frac{d^2a}{dU^2} = -\frac{1}{2}\frac{1}{f}\frac{d^2f}{du^2}$$

$$R^{\hat{3}}{}_{\hat{0}\hat{3}\hat{0}} = R^{\hat{3}}{}_{\hat{0}\hat{3}\hat{1}} = R^{\hat{3}}{}_{\hat{1}\hat{3}\hat{1}} = -\frac{1}{b}\frac{d^2b}{dU^2} = -\frac{1}{2}\frac{1}{h}\frac{d^2g}{du^2}$$

The non-zero components of the Ricci tensor

$$R_{\hat{0}\hat{0}} = R_{\hat{1}\hat{0}} = R_{\hat{0}\hat{1}} = R_{\hat{1}\hat{1}} = -\left(\frac{1}{a}\frac{d^2a}{dU^2} + \frac{1}{b}\frac{d^2b}{dU^2}\right) = -\frac{1}{2}\left(\frac{1}{f}\frac{d^2f}{du^2} + \frac{1}{g}\frac{d^2g}{du^2}\right)$$

The Ricci scalar

$$R = 0$$

The non-zero components of the Einstein tensor

$$G_{00} = -\left(\frac{1}{a}\frac{d^2a}{dU^2} + \frac{1}{b}\frac{d^2b}{dU^2}\right) = -\frac{1}{2}\left(\frac{1}{f}\frac{d^2f}{du^2} + \frac{1}{g}\frac{d^2g}{du^2}\right)$$

With the vacuum solution

$$\frac{f''(u)}{f} = -\frac{g''(u)}{h}$$

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14.6 The Penrose Kahn metric (Colliding gravitational waves)

The line element:

$$ds^2 = 2dudv - (1-u)^2dx^2 - (1+u)^2dy^2$$

The metric tensor:

$$g_{ab} = \begin{Bmatrix} & 1 & & \\ 1 & & & \\ & & -(1-u)^2 & \\ & & & -(1+u)^2 \end{Bmatrix}$$

and its inverse:

$$g^{ab} = \begin{Bmatrix} & 1 & & \\ 1 & & & \\ & & \frac{-1}{(1-u)^2} & \\ & & & \frac{-1}{(1+u)^2} \end{Bmatrix}$$

Notice: The Penrose Kahn space-time is a Rosen space-time with

$$f(u) = 1-u$$
$$g(u) = 1+u$$

Because

$$f''(u) = -g''(u) = 0$$

The Kahn-Penrose space-time is a no-curvature vacuum solution

14.6.1 [l]The Christoffel symbols of the Kahn-Penrose space-time

The non-zero Christoffel symbols

$$\Gamma_{abc} = \frac{1}{2}(\partial_a g_{bc} + \partial_b g_{ca} - \partial_c g_{ab}) \qquad \Gamma^a{}_{bc} = g^{ad}\Gamma_{bcd}$$

$$\Gamma_{uxx} = \Gamma_{xux} = {}^{22} -\frac{1}{2}(\partial_u(1-u)^2) = 1-u \quad \Rightarrow \quad \Gamma^x{}_{xu} = \Gamma^x{}_{ux} = {}^{23} g^{xx}\Gamma_{xux} = -\frac{1}{1-u}$$

$$\Gamma_{xxu} = {}^{24}\frac{1}{2}(\partial_u(1-u)^2) = -(1-u) \quad \Rightarrow \quad \Gamma^v{}_{xx} = g^{vd}\Gamma_{xxd} = g^{vu}\Gamma_{xxu} = -(1-u)$$

$$\Gamma_{uyy} = \Gamma_{yuy} = {}^{25} -\frac{1}{2}(\partial_u(1+u)^2) = -(1+u) \quad \Rightarrow \quad \Gamma^y{}_{yu} = \Gamma^y{}_{uy} = {}^{26} g^{yy}\Gamma_{yuy} = {}^{27}\frac{1}{1+u}$$

$$\Gamma_{yyu} = {}^{28} -\frac{1}{2}(\partial_u g_{yy}) = \frac{1}{2}(\partial_u(1+u)^2) = 1+u \quad \Rightarrow \quad \Gamma^v{}_{yy} = g^{vd}\Gamma_{yyd} = g^{vu}\Gamma_{yyu} = 1+u$$

14.6.2 [m]The Ricci scalar of the Penrose Kahn metric

[29]The Ricci scalar:

$$R = g^{ab}R_{ab}$$

The Ricci tensor

$$R_{ab} = R^c{}_{acb}$$

---

[22] $= \frac{1}{2}(\partial_u g_{xx} + \partial_x g_{ux} - \partial_x g_{ux}) = \frac{1}{2}(\partial_u g_{xx}) =$

[23] $= g^{xd}\Gamma_{xud} =$

[24] $= \frac{1}{2}(\partial_x g_{xu} + \partial_x g_{xu} - \partial_u g_{xx}) = -\frac{1}{2}(\partial_u g_{xx}) =$

[25] $= \frac{1}{2}(\partial_u g_{yy} + \partial_y g_{uy} - \partial_y g_{uy}) = \frac{1}{2}(\partial_u g_{yy}) =$

[26] $= g^{yd}\Gamma_{yud} =$

[27] $= \frac{1}{(1+u)^2}(1+u) =$

[28] $= \frac{1}{2}(\partial_y g_{yu} + \partial_y g_{uy} - \partial_u g_{yy}) =$

[29] This calculation is purely instructive because we already know that the curvature is zero $(R = 0)$

Sum over $a = u, v, x, y$:

$$R = g^{ub}R_{ub} + g^{vb}R_{vb} + g^{xb}R_{xb} + g^{yb}R_{yb}$$

Sum over $b = u, v, x, y$:

$$= g^{uv}R_{uv} + g^{vu}R_{vu} + g^{xx}R_{xx} + g^{yy}R_{yy}$$
$$= g^{uv}R^{c}{}_{ucv} + g^{vu}R^{c}{}_{vcu} + g^{xx}R^{c}{}_{xcx} + g^{yy}R^{c}{}_{ycy}$$

$R_{abcd} = R_{cdab} = -R_{bacd} = -R_{abdc}$:

$$= [30]\, 4g^{uv}g^{xx}R_{xuxv} + 4g^{uv}g^{yy}R_{yuyv} + 2g^{xx}g^{yy}R_{yxyx}$$
$$= 4g^{uv}R^{x}{}_{uxv} + 4g^{uv}R^{y}{}_{uyv} + 2g^{xx}R^{y}{}_{xyx}$$

Notice we can rewrite this into to a general expression for a non-diagonal metric of the type:

$$g_{ab} = \begin{Bmatrix} & g_{12} & & \\ g_{12} & & & \\ & & g_{33} & \\ & & & g_{44} \end{Bmatrix}$$

We write

$$R = 4g^{12}R^{3}{}_{132} + 4g^{12}R^{4}{}_{142} + 2g^{33}R^{4}{}_{343}$$

Now we need to calculate the three elements in the Riemann tensor: $R^{x}{}_{uxv}; R^{y}{}_{uyv}; R^{y}{}_{xyx}$

$$R^{x}{}_{uxv} = \partial_x\Gamma^{x}{}_{uv} - \partial_v\Gamma^{x}{}_{ux} + \Gamma^{e}{}_{uv}\Gamma^{x}{}_{ex} - \Gamma^{e}{}_{ux}\Gamma^{x}{}_{ev} = 0$$
$$R^{y}{}_{uyv} = \partial_y\Gamma^{y}{}_{uv} - \partial_v\Gamma^{y}{}_{uy} + \Gamma^{e}{}_{uv}\Gamma^{y}{}_{ey} - \Gamma^{e}{}_{uy}\Gamma^{y}{}_{ev} = 0$$
$$R^{y}{}_{xyx} = \partial_y\Gamma^{y}{}_{xx} - \partial_x\Gamma^{y}{}_{xy} + \Gamma^{e}{}_{xx}\Gamma^{y}{}_{ey} - \Gamma^{e}{}_{xy}\Gamma^{y}{}_{ex} = 0$$
$$\Rightarrow \quad R = 0$$

14.7 [n]The Brinkmann metric (Plane gravitational waves)

The line element:

$$ds^2 = H(u,x,y)du^2 + 2dudv - dx^2 - dy^2$$

The metric tensor:

$$g_{ab} = \begin{Bmatrix} H(u,x,y) & 1 & & \\ 1 & & & \\ & & -1 & \\ & & & -1 \end{Bmatrix}$$

and its inverse:

$$g^{ab} = [31] \begin{Bmatrix} & 1 & & \\ 1 & -H(u,x,y) & & \\ & & -1 & \\ & & & -1 \end{Bmatrix}$$

14.7.1 The basis one forms

Finding the basis one forms is not so obvious, we write:

$$ds^2 = H(u,x,y)du^2 + 2dudv - dx^2 - dy^2 = \left(\omega^{\hat{u}}\right)^2 - \left(\omega^{\hat{v}}\right)^2 - \left(\omega^{\hat{x}}\right)^2 - \left(\omega^{\hat{y}}\right)^2$$
$$\Rightarrow \quad du[H(u,x,y)du + 2dv] - dx^2 - dy^2 = \left(\omega^{\hat{u}} + \omega^{\hat{v}}\right)\left(\omega^{\hat{u}} - \omega^{\hat{v}}\right) - \left(\omega^{\hat{x}}\right)^2 - \left(\omega^{\hat{y}}\right)^2$$
$$\Rightarrow \quad \omega^{\hat{u}} + \omega^{\hat{v}} = du$$
$$\omega^{\hat{u}} - \omega^{\hat{v}} = Hdu + 2dv$$
$$\omega^{\hat{x}} = dx$$

[30] Sum over $c = u, v, x, y$: $= g^{uv}R^{x}{}_{uxv} + g^{uv}R^{y}{}_{uyv} + g^{vu}R^{x}{}_{vxu} + g^{vu}R^{y}{}_{vyu} + g^{xx}R^{u}{}_{xux} + g^{xx}R^{v}{}_{xvx} + g^{xx}R^{y}{}_{xyx} + g^{yy}R^{u}{}_{yuy} + g^{yy}R^{v}{}_{yvy} + g^{yy}R^{x}{}_{yxy}$. $g^{uv} = g^{vu}$: $= g^{uv}g^{xx}R_{xuxv} + g^{uv}g^{yy}R_{yuyv} + g^{vu}g^{xx}R_{xvxu} + g^{vu}g^{yy}R_{yvyu} + g^{xx}g^{uv}R_{vxux} + g^{xx}g^{vu}R_{uxvx} + g^{xx}g^{yy}R_{yxyx} + g^{yy}g^{uv}R_{vyuy} + g^{yy}g^{vu}R_{uyvy} + g^{yy}g^{xx}R_{xyxy}$

[31] Checking: $\begin{Bmatrix} H(u,x,y) & 1 & & \\ 1 & & & \\ & & -1 & \\ & & & -1 \end{Bmatrix}\begin{Bmatrix} & 1 & & \\ 1 & -H(u,x,y) & & \\ & & -1 & \\ & & & -1 \end{Bmatrix} = \begin{Bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{Bmatrix}$

$$\omega^{\hat{y}} \quad = dy$$

$$\begin{aligned}
\omega^{\hat{u}} &= \frac{1}{2}(H+1)du + dv & du &= \omega^{\hat{u}} + \omega^{\hat{v}} \\
\omega^{\hat{v}} &= \frac{1}{2}(1-H)du - dv & dv &= \frac{1}{2}(1-H)\omega^{\hat{u}} - \frac{1}{2}(1+H)\omega^{\hat{v}} \\
\omega^{\hat{x}} &= dx & dx &= \omega^{\hat{x}} \\
\omega^{\hat{y}} &= dy & dy &= \omega^{\hat{y}}
\end{aligned}$$

$$\eta^{ij} = \begin{Bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{Bmatrix}$$

14.7.2 The orthonormal null tetrad

Now we can use the basis one-forms to construct a orthonormal null tetrad

$$\begin{pmatrix} l \\ n \\ m \\ \overline{m} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & 1 & -i \end{pmatrix} \begin{pmatrix} \omega^{\hat{u}} \\ \omega^{\hat{v}} \\ \omega^{\hat{x}} \\ \omega^{\hat{y}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \omega^{\hat{u}} + \omega^{\hat{v}} \\ \omega^{\hat{u}} - \omega^{\hat{v}} \\ \omega^{\hat{x}} + i\omega^{\hat{y}} \\ \omega^{\hat{x}} - i\omega^{\hat{y}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} du \\ Hdu + 2dv \\ dx + idy \\ dx - idy \end{pmatrix}$$

Written in terms of the coordinate basis

$$\begin{aligned}
l_a &= \frac{1}{\sqrt{2}}(1, \quad 0, \quad 0, \quad 0) \\
n_a &= \frac{1}{\sqrt{2}}(H, \quad 2, \quad 0, \quad 0) \\
m_a &= \frac{1}{\sqrt{2}}(0, \quad 0, \quad 1, \quad i) \\
\overline{m_a} &= \frac{1}{\sqrt{2}}(0, \quad 0, \quad 1, \quad -i)
\end{aligned}$$

Next we use the metric to rise the indices

$$\begin{aligned}
l^u &= g^{au} l_a = g^{vu} l_v = 1 \cdot 0 = 0 \\
l^v &= g^{av} l_a = g^{uv} l_u + g^{vv} l_v = 1 \cdot \left(\frac{1}{\sqrt{2}}\right) + (-H) \cdot 0 = \frac{1}{\sqrt{2}} \\
l^x &= l^y = 0 \\
n^u &= g^{au} n_a = g^{vu} n_v = 1 \cdot \left(\frac{2}{\sqrt{2}}\right) = \sqrt{2} \\
n^v &= g^{av} n_a = g^{uv} n_u + g^{vv} n_v = 1 \cdot \left(\frac{1}{\sqrt{2}} H\right) + (-H) \cdot \left(\frac{2}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}} H \\
n^x &= n^y = 0 \\
m^u &= m^v = 0 \\
m^x &= g^{ax} m_a = g^{xx} m_x = (-1) \cdot \frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}} \\
m^y &= g^{ay} m_a = g^{yy} m_y = (-1) \cdot i \frac{1}{\sqrt{2}} = -i \frac{1}{\sqrt{2}}
\end{aligned}$$

Collecting the results

$$\begin{aligned}
l_a &= \frac{1}{\sqrt{2}}(1, \quad 0, \quad 0, \quad 0) & l^a &= \frac{1}{\sqrt{2}}(0, \quad 1, \quad 0, \quad 0) \\
n_a &= \frac{1}{\sqrt{2}}(H, \quad 2, \quad 0, \quad 0) & n^a &= \frac{1}{\sqrt{2}}(2, \quad -H, \quad 0, \quad 0) \\
m_a &= \frac{1}{\sqrt{2}}(0, \quad 0, \quad 1, \quad i) & m^a &= \frac{1}{\sqrt{2}}(0, \quad 0, \quad -1, \quad -i)
\end{aligned}$$

$\overline{m_a} = \frac{1}{\sqrt{2}}(0, \quad 0, \quad 1, \quad -i) \quad \bar{m}^a = \frac{1}{\sqrt{2}}(0, \quad 0, \quad -1, \quad i)$

14.7.3 Christoffel symbols

The non-zero Christoffel symbols of first kind

$$\Gamma_{abc} = \frac{1}{2}(\partial_a g_{bc} + \partial_b g_{ca} - \partial_c g_{ab})$$
$$\Gamma_{uuu} = \frac{1}{2}(\partial_u g_{uu}) = \frac{1}{2}\frac{\partial H}{\partial u}$$
$$\Gamma_{xuu} = \Gamma_{uxu} = \frac{1}{2}(\partial_x g_{uu}) = \frac{1}{2}\frac{\partial H}{dx}$$
$$\Gamma_{uux} = -\frac{1}{2}(\partial_x g_{uu}) = -\frac{1}{2}\frac{\partial H}{dx}$$
$$\Gamma_{yuu} = \Gamma_{uyu} = \frac{1}{2}(\partial_y g_{uu}) = \frac{1}{2}\frac{\partial H}{dy}$$
$$\Gamma_{uuy} = -\frac{1}{2}(\partial_y g_{uu}) = -\frac{1}{2}\frac{\partial H}{dy}$$

$\Rightarrow$ The non-zero Christoffel symbols of second kind

$$\Gamma^a{}_{bc} = g^{ad}\Gamma_{bcd}$$
$$\Gamma^v{}_{uu} = g^{vd}\Gamma_{uud} = {}^{32}\Gamma_{uuu} = \frac{1}{2}\frac{\partial H}{\partial u}$$
$$\Gamma^v{}_{xu} = \Gamma^v{}_{ux} = g^{vd}\Gamma_{xud} = {}^{33}\Gamma_{xuu} = \frac{1}{2}\frac{\partial H}{dx}$$
$$\Gamma^x{}_{uu} = g^{xx}\Gamma_{uux} = \frac{1}{2}\frac{\partial H}{dx}$$
$$\Gamma^v{}_{yu} = g^{vd}\Gamma_{yud} = {}^{34}\Gamma_{yuu} = \frac{1}{2}\frac{\partial H}{dy}$$
$$\Gamma^y{}_{uu} = g^{yy}\Gamma_{uuy} = \frac{1}{2}\frac{\partial H}{dy}$$

14.7.4 The spin coefficients calculated from the null tetrad

$$\begin{aligned}
\pi &= -\nabla_b n_a \bar{m}^a l^b \\
&= -\nabla_v n_a \bar{m}^a l^v \\
&= -\nabla_v n_x \bar{m}^x l^v - \nabla_v n_y \bar{m}^y l^v \\
&= -(\partial_v n_x - \Gamma^c{}_{xv} n_c)\bar{m}^x l^v - (\partial_v n_y - \Gamma^c{}_{yv} n_c)\bar{m}^y l^v \\
&= 0 \\
\nu &= -\nabla_b n_a \bar{m}^a n^b \\
&= -\nabla_u n_a \bar{m}^a n^u - \nabla_v n_a \bar{m}^a n^v \\
&= -\nabla_u n_x \bar{m}^x n^u - \nabla_u n_y \bar{m}^y n^u - \nabla_v n_x \bar{m}^x n^v - \nabla_v n_y \bar{m}^y n^v \\
&= {}^{35}\Gamma^v{}_{xu} n_v \bar{m}^x n^u + \Gamma^v{}_{yu} n_v \bar{m}^y n^u \\
&= (\Gamma^v{}_{xu}\bar{m}^x + \Gamma^v{}_{yu}\bar{m}^y) n_v n^u \\
&= \left(\frac{1}{2}\frac{\partial H}{\partial x}\left(\frac{-1}{\sqrt{2}}\right) + \frac{1}{2}\frac{\partial H}{\partial y} i \frac{1}{\sqrt{2}}\right)\sqrt{2}\cdot\sqrt{2} \\
&= \frac{1}{\sqrt{2}}\left(-\frac{\partial H}{\partial x} + i\frac{\partial H}{\partial y}\right) \\
\lambda &= -\nabla_b n_a \bar{m}^a \bar{m}^b \\
&= -\nabla_x n_a \bar{m}^a \bar{m}^x - \nabla_y n_a \bar{m}^a \bar{m}^y \\
&= {}^{36} - \nabla_x n_x \bar{m}^x \bar{m}^x - \nabla_x n_y \bar{m}^y \bar{m}^x - \nabla_y n_x \bar{m}^x \bar{m}^y - \nabla_y n_y \bar{m}^y \bar{m}^y \\
&= 0 \\
\mu &= -\nabla_b n_a \bar{m}^a m^b = 0 \\
\kappa &= \nabla_b l_a m^a l^b \\
&= \nabla_v l_a m^a l^v \\
&= \nabla_v l_x m^x l^v + \nabla_v l_y m^y l^v
\end{aligned}$$

[32] $= g^{vu}\Gamma_{uuu} + g^{vv}\Gamma_{uuv} =$
[33] $= g^{vu}\Gamma_{xuu} + g^{vv}\Gamma_{xuv} =$
[34] $= g^{vu}\Gamma_{yuu} + g^{vv}\Gamma_{yuv} =$
[35] $= -(\partial_u n_x - \Gamma^c{}_{xu} n_c)\bar{m}^x n^u - (\partial_u n_y - \Gamma^c{}_{yu} n_c)\bar{m}^y n^u - (\partial_v n_x - \Gamma^c{}_{xv} n_c)\bar{m}^x n^v - (\partial_v n_y - \Gamma^c{}_{yv} n_c)\bar{m}^y n^v =$
[36] $= -(\partial_x n_x - \Gamma^c{}_{xx} n_c)\bar{m}^x \bar{m}^x - (\partial_x n_y - \Gamma^c{}_{xy} n_c)\bar{m}^y \bar{m}^x - (\partial_y n_x - \Gamma^c{}_{yx} n_c)\bar{m}^x \bar{m}^y - (\partial_y n_y - \Gamma^c{}_{yy} n_c)\bar{m}^y \bar{m}^y =$

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$$
\begin{aligned}
&= (\partial_v l_x - \Gamma^c{}_{vx} l_c) m^x l^v + (\partial_v l_y - \Gamma^c{}_{vy} l_c) m^y l^v \\
&= 0 \\
\tau &= \nabla_b l_a m^a n^b \\
&= \nabla_u l_a m^a n^u + \nabla_v l_a m^a n^v \\
&= \nabla_u l_x m^x n^u + \nabla_v l_x m^x n^v + \nabla_u l_y m^y n^u + \nabla_v l_y m^y n^v \\
&= {}^{37} 0 \\
\rho &= \nabla_b l_a m^a \overline{m}^b \\
&= \nabla_x l_a m^a \overline{m}^x + \nabla_y l_a m^a \overline{m}^y \\
&= {}^{38} \nabla_x l_x m^x \overline{m}^x + \nabla_y l_x m^x \overline{m}^y + \nabla_x l_y m^y \overline{m}^x + \nabla_y l_y m^y \overline{m}^y \\
&= 0 \\
\sigma &= \nabla_b l_a m^a m^b = 0 \\
\varepsilon &= \frac{1}{2}\left(\nabla_b l_a n^a l^b - \nabla_b m_a \overline{m}^a l^b\right) \\
&= \frac{1}{2}(\nabla_v l_a n^a l^v - \nabla_v m_a \overline{m}^a l^v) \\
&= {}^{39} \frac{1}{2}(\nabla_v l_u n^u l^v - \nabla_v m_x \overline{m}^x l^v) + \frac{1}{2}\left(\nabla_v l_v n^v l^v - \nabla_v m_y \overline{m}^y l^v\right) \\
&= 0 \\
\gamma &= \frac{1}{2}\left(\nabla_b l_a n^a n^b - \nabla_b m_a \overline{m}^a n^b\right) \\
&= \frac{1}{2}(\nabla_u l_a n^a n^u - \nabla_u m_a \overline{m}^a n^u) + \frac{1}{2}(\nabla_v l_a n^a n^v - \nabla_v m_a \overline{m}^a n^v) \\
&= {}^{40} 0 \\
\alpha &= \frac{1}{2}\left(\nabla_b l_a n^a \overline{m}^b - \nabla_b m_a \overline{m}^a \overline{m}^b\right) \\
&= \frac{1}{2}(\nabla_x l_a n^a \overline{m}^x - \nabla_x m_a \overline{m}^a \overline{m}^x) + \frac{1}{2}\left(\nabla_y l_a n^a \overline{m}^y - \nabla_y m_a \overline{m}^a \overline{m}^y\right) \\
&= {}^{41} 0 \\
\beta &= \frac{1}{2}\left(\nabla_b l_a n^a m^b - \nabla_b m_a \overline{m}^a m^b\right) = 0
\end{aligned}
$$

The non-zero spin-coefficient

$$
\nu = \frac{1}{\sqrt{2}}\left(-\frac{\partial H}{\partial x} + i\frac{\partial H}{\partial y}\right)
$$

---

[37] $= (\partial_u l_x - \Gamma^c{}_{ux} l_c) m^x n^u + (\partial_v l_x - \Gamma^c{}_{vx} l_c) m^x n^v + (\partial_u l_y - \Gamma^c{}_{uy} l_c) m^y n^u + (\partial_v l_y - \Gamma^c{}_{vy} l_c) \nabla_v l_y m^y n^v =$

[38] $= (\partial_x l_x - \Gamma^c{}_{xx} l_c) m^x \overline{m}^x + (\partial_y l_x - \Gamma^c{}_{yx} l_c) m^x \overline{m}^y + (\partial_x l_y - \Gamma^c{}_{xy} l_c) m^y \overline{m}^x + (\partial_y l_y - \Gamma^c{}_{yy} l_c) m^y \overline{m}^y =$

[39] $= \frac{1}{2}\left((\partial_v l_u - \Gamma^c{}_{vu} l_c) n^u l^v - (\partial_v m_x - \Gamma^c{}_{vx} m_c) \overline{m}^x l^v\right) + \frac{1}{2}\left((\partial_v l_v - \Gamma^c{}_{vv} l_c) n^v l^v - (\partial_v m_y - \Gamma^c{}_{vy} m_c) \overline{m}^y l^v\right) =$

[40] $= \frac{1}{2}(\nabla_u l_u n^u n^u - \nabla_u m_x \overline{m}^x n^u) + \frac{1}{2}(\nabla_v l_u n^u n^v - \nabla_v m_x \overline{m}^x n^v) + \frac{1}{2}\left(\nabla_u l_v n^v n^u - \nabla_u m_y \overline{m}^y n^u\right) + \frac{1}{2}\left(\nabla_v l_v n^v n^v - \nabla_v m_y \overline{m}^y n^v\right) = \frac{1}{2}\left((\partial_u l_u - \Gamma^c{}_{uu} l_c) n^u n^u - (\partial_u m_x - \Gamma^c{}_{ux} m_c) \overline{m}^x n^u\right) + \frac{1}{2}\left((\partial_v l_u - \Gamma^c{}_{vu} l_c) n^u n^v - (\partial_v m_x - \Gamma^c{}_{vx} m_c) \overline{m}^x n^v\right) + \frac{1}{2}\left((\partial_u l_v - \Gamma^c{}_{uv} l_c) n^v n^u - (\partial_u m_y - \Gamma^c{}_{uy} m_c) \overline{m}^y n^u\right) + \frac{1}{2}\left((\partial_v l_v - \Gamma^c{}_{vv} l_c) n^v n^v - (\partial_v m_y - \Gamma^c{}_{vy} m_c) \overline{m}^y n^v\right) =$

[41] $= \frac{1}{2}(\nabla_x l_u n^u \overline{m}^x - \nabla_x m_x \overline{m}^x \overline{m}^x) + \frac{1}{2}\left(\nabla_y l_u n^u \overline{m}^y - \nabla_y m_x \overline{m}^x \overline{m}^y\right) + \frac{1}{2}\left(\nabla_x l_v n^v \overline{m}^x - \nabla_x m_y \overline{m}^y \overline{m}^x\right) + \frac{1}{2}\left(\nabla_y l_v n^v \overline{m}^y - \nabla_y m_y \overline{m}^y \overline{m}^y\right) = \frac{1}{2}\left((\partial_x l_u - \Gamma^c{}_{xu} l_c) n^u \overline{m}^x - (\partial_x m_x - \Gamma^c{}_{xx} m_c) \overline{m}^x \overline{m}^x\right) + \frac{1}{2}\left((\partial_y l_u - \Gamma^c{}_{yu} l_c) n^u \overline{m}^y - (\partial_y m_x - \Gamma^c{}_{yx} m_c) \overline{m}^x \overline{m}^y\right) + \frac{1}{2}\left((\partial_x l_v - \Gamma^c{}_{xv} l_c) n^v \overline{m}^x - (\partial_x m_y - \Gamma^c{}_{xy} m_c) \overline{m}^y \overline{m}^x\right) + \frac{1}{2}\left((\partial_y l_v - \Gamma^c{}_{yv} l_c) n^v \overline{m}^y - (\partial_y m_y - \Gamma^c{}_{yy} m_c) \overline{m}^y \overline{m}^y\right) =$

14.7.5 The Weyl Scalars and Petrov classification

$$\begin{aligned}
\Psi_0 &= D\sigma - \delta\kappa - \sigma(\rho + \bar{\rho}) - \sigma(3\varepsilon - \bar{\varepsilon}) + \kappa(\pi - \bar{\pi} + \bar{\alpha} + 3\beta) \\
\Psi_1 &= D\beta - \delta\varepsilon - \sigma(\alpha + \pi) - \beta(\bar{\rho} - \bar{\varepsilon}) + \kappa(\mu + \gamma) + \varepsilon(\bar{\alpha} - \bar{\pi}) \\
\Psi_2 &= \bar{\delta}\tau - \Delta\rho - \rho\bar{\mu} - \sigma\lambda + \tau(\bar{\beta} - \alpha - \bar{\tau}) + \rho(\gamma + \bar{\gamma}) + \kappa\nu - 2\Lambda \\
\Psi_3 &= \bar{\delta}\gamma - \Delta\alpha + \nu(\rho + \varepsilon) - \lambda(\tau + \beta) + \alpha(\bar{\gamma} - \bar{\mu}) + \gamma(\bar{\beta} - \bar{\tau}) \\
\Psi_4 &= \bar{\delta}\nu - \Delta\lambda + \lambda(\mu + \bar{\mu}) - \lambda(3\gamma - \bar{\gamma}) + \nu(3\alpha + \bar{\beta} + \pi - \bar{\tau})
\end{aligned}$$

Where

$$\begin{aligned}
D &= l^a\nabla_a \\
\Delta &= n^a\nabla_a \\
\delta &= m^a\nabla_a \\
\bar{\delta} &= \bar{m}^a\nabla_a
\end{aligned}$$

We see that $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0$ and

$$\begin{aligned}
\Psi_4 &= \bar{\delta}\nu \\
&= \bar{m}^a\nabla_a\nu \\
&= \bar{m}^a\,\partial_a\left(\frac{1}{\sqrt{2}}\left(-\frac{\partial H}{\partial x} + i\frac{\partial H}{\partial y}\right)\right) \\
&= \frac{1}{\sqrt{2}}\left[\bar{m}^x\,\partial_x\left(-\frac{\partial H}{\partial x} + i\frac{\partial H}{\partial y}\right) + \bar{m}^y\partial_y\left(-\frac{\partial H}{\partial x} + i\frac{\partial H}{\partial y}\right)\right] \\
&= \frac{1}{\sqrt{2}}\left[\left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{\partial^2 H}{\partial x^2} + i\frac{\partial^2 H}{\partial x\partial y}\right) + \left(i\frac{1}{\sqrt{2}}\right)\left(-\frac{\partial^2 H}{\partial x\partial y} + i\frac{\partial^2 H}{\partial y^2}\right)\right] \\
&= \frac{1}{2}\left[\frac{\partial^2 H}{\partial x^2} - \frac{\partial^2 H}{\partial y^2} - 2i\frac{\partial^2 H}{\partial x\partial y}\right]
\end{aligned}$$

$\Psi_4 \neq 0$: This is a Petrov type N, which means there is a single principal null direction of multiplicity 4. This corresponds to transverse gravity waves.

14.7.6 The Ricci tensor

$$\begin{aligned}
\Phi_{22} &= \delta\nu - \Delta\mu - \mu^2 - \lambda\bar{\lambda} - \mu(\gamma + \bar{\gamma}) + \bar{\nu}\pi - \nu(\tau - 3\beta - \bar{\alpha}) \\
&= \delta\nu \\
&= m^a\nabla_a\nu \\
&= m^a\partial_a\left(\frac{1}{\sqrt{2}}\left(-\frac{\partial H}{\partial x} + i\frac{\partial H}{\partial y}\right)\right) \\
&= m^x\partial_x\left(\frac{1}{\sqrt{2}}\left(-\frac{\partial H}{\partial x} + i\frac{\partial H}{\partial y}\right)\right) + m^y\partial_y\left(\frac{1}{\sqrt{2}}\left(-\frac{\partial H}{\partial x} + i\frac{\partial H}{\partial y}\right)\right) \\
&= \left(-\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)\left(-\frac{\partial^2 H}{\partial x^2} + i\frac{\partial^2 H}{\partial x\partial y}\right) + \left(-i\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)\left(-\frac{\partial^2 H}{\partial x\partial y} + i\frac{\partial^2 H}{\partial y^2}\right) \\
&= \frac{1}{2}\left[\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2}\right]
\end{aligned}$$

$$\begin{aligned}
\Phi_{22} &= -\frac{1}{2}R_{ab}n^a n^b \\
&= -\frac{1}{2}R_{ub}n^u n^b - \frac{1}{2}R_{vb}n^v n^b \\
&= -\frac{1}{2}R_{uu}n^u n^u - \frac{1}{2}R_{vu}n^v n^u - \frac{1}{2}R_{uv}n^u n^v - \frac{1}{2}R_{vv}n^v n^v \\
&= -\frac{1}{2}R_{uu}\sqrt{2}\cdot\sqrt{2} - R_{uv}\sqrt{2}\cdot\left(-\frac{1}{\sqrt{2}}H\right) - \frac{1}{2}R_{vv}\left(-\frac{1}{\sqrt{2}}H\right)\cdot\left(-\frac{1}{\sqrt{2}}H\right) \\
&= -R_{uu} + HR_{uv} - \frac{H^2}{4}R_{vv}
\end{aligned}$$

$$= \frac{1}{2}\left[\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2}\right]$$

$$\Rightarrow \quad R_{uu} = {}^{42} -\frac{1}{2}\left[\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2}\right]$$

$$R_{uv} = R_{vv} = 0$$

14.7.7 The Weyl tensor calculated from the null tetrad

This calculation show that the spin coefficients and the Weyl scalars depend on the chosen null tetrad, and the Ricci tensor does not (of course).

The null tetrad

$$l_a = (1, \quad 0, \quad 0, \quad 0) \qquad l^a = (0, \quad 1, \quad 0, \quad 0)$$

$$n_a = \left(\frac{1}{2}H, \quad 1, \quad 0, \quad 0\right) \qquad n^a = \left(1, \quad -\frac{1}{2}H, \quad 0, \quad 0\right)$$

$$m_a = \frac{1}{\sqrt{2}}(0, \quad 0, \quad 1, \quad -i) \qquad m^a = \frac{1}{\sqrt{2}}(0, \quad 0, \quad -1, \quad i)$$

$$\overline{m_a} = \frac{1}{\sqrt{2}}(0, \quad 0, \quad 1, \quad i) \qquad \overline{m}^a = \frac{1}{\sqrt{2}}(0, \quad 0, \quad -1, \quad -i)$$

$$\begin{aligned}
\nu &= -\nabla_b n_a \overline{m}^a n^b \\
&= -\nabla_u n_a \overline{m}^a n^u - \nabla_v n_a \overline{m}^a n^v \\
&= -\nabla_u n_x \overline{m}^x n^u - \nabla_u n_y \overline{m}^y n^u - \nabla_v n_x \overline{m}^x n^v - \nabla_v n_y \overline{m}^y n^v \\
&= {}^{43}\Gamma^v{}_{xu} n_v \overline{m}^x n^u + \Gamma^v{}_{yu} n_v \overline{m}^y n^u \\
&= \left(\Gamma^v{}_{xu} \overline{m}^x + \Gamma^v{}_{yu} \overline{m}^y\right) n_v n^u \\
&= \left(\frac{1}{2}\frac{\partial H}{\partial x}\left(\frac{-1}{\sqrt{2}}\right) + \frac{1}{2}\frac{\partial H}{\partial y}\left(-i\frac{1}{\sqrt{2}}\right)\right) 1 \cdot 1 \\
&= -\frac{1}{2\sqrt{2}}\left(\frac{\partial H}{\partial x} + i\frac{\partial H}{\partial y}\right)
\end{aligned}$$

$$\begin{aligned}
\Psi_4 &= \bar{\delta}\nu \\
&= \overline{m}^a \nabla_a \nu \\
&= \overline{m}^a\, \partial_a \left(-\frac{1}{2\sqrt{2}}\left(\frac{\partial H}{\partial x} + i\frac{\partial H}{\partial y}\right)\right) \\
&= \left(-\frac{1}{2\sqrt{2}}\right)\left[\overline{m}^x\, \partial_{\mathrm{x}}\left(\frac{\partial H}{\partial x} + i\frac{\partial H}{\partial y}\right) + \overline{m}^y \partial_{\mathrm{y}}\left(\frac{\partial H}{\partial x} + i\frac{\partial H}{\partial y}\right)\right] \\
&= \left(-\frac{1}{2\sqrt{2}}\right)\left[\left(-\frac{1}{\sqrt{2}}\right)\left(\frac{\partial^2 H}{\partial x^2} + i\frac{\partial^2 H}{\partial x \partial y}\right) + \left(-i\frac{1}{\sqrt{2}}\right)\left(\frac{\partial^2 H}{\partial x \partial y} + i\frac{\partial^2 H}{\partial y^2}\right)\right] \\
&= \frac{1}{4}\left[\frac{\partial^2 H}{\partial x^2} - \frac{\partial^2 H}{\partial y^2} + 2i\frac{\partial^2 H}{\partial x \partial y}\right]
\end{aligned}$$

$$\begin{aligned}
\Phi_{22} &= \delta\nu \\
&= m^a \nabla_a \nu \\
&= m^a \partial_a \left(-\frac{1}{2\sqrt{2}}\left(\frac{\partial H}{\partial x} + i\frac{\partial H}{\partial y}\right)\right) \\
&= m^x \partial_x \left(-\frac{1}{2\sqrt{2}}\left(\frac{\partial H}{\partial x} + i\frac{\partial H}{\partial y}\right)\right) + m^y \partial_y \left(-\frac{1}{2\sqrt{2}}\left(\frac{\partial H}{\partial x} + i\frac{\partial H}{\partial y}\right)\right) \\
&= \left(-\frac{1}{2\sqrt{2}}\right)\left[\left(\frac{-1}{\sqrt{2}}\right)\left(\frac{\partial^2 H}{\partial x^2} + i\frac{\partial^2 H}{\partial x \partial y}\right) + \left(i\frac{1}{\sqrt{2}}\right)\left(\frac{\partial^2 H}{\partial x \partial y} + i\frac{\partial^2 H}{\partial y^2}\right)\right]
\end{aligned}$$

---

[42] According to the Cartan calculation further below the sign is wrong

[43] $= -(\partial_u n_x - \Gamma^c{}_{xu} n_c)\overline{m}^x n^u - (\partial_u n_y - \Gamma^c{}_{yu} n_c)\overline{m}^y n^u - (\partial_v n_x - \Gamma^c{}_{xv} n_c)\overline{m}^x n^v - (\partial_v n_y - \Gamma^c{}_{yv} n_c)\overline{m}^y n^v =$

$$
\begin{aligned}
&= \frac{1}{4}\left[\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2}\right] \\
\Phi_{22} &= {}^{44} -\frac{1}{2} R_{ab} n^a n^b \\
&= -\frac{1}{2} R_{uu} n^u n^u - \frac{1}{2} R_{vu} n^v n^u - \frac{1}{2} R_{uv} n^u n^v - \frac{1}{2} R_{vv} n^v n^v \\
&= -\frac{1}{2} R_{uu} 1 \cdot 1 - R_{vu} 1 \left(-\frac{1}{2} H\right) - \frac{1}{2} R_{vv} \left(-\frac{1}{2} H\right)^2 \\
&= -\frac{1}{2} R_{uu} + \frac{H}{2} R_{vu} - \frac{H^2}{8} R_{vv} \\
&= \frac{1}{4}\left[\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2}\right] \\
\Rightarrow \qquad R_{uu} &= {}^{45} -\frac{1}{2}\left[\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2}\right] \\
R_{uv} &= R_{vv} = 0
\end{aligned}
$$

14.7.8 Finding the Ricci tensor of the Brinkmann metric using Cartan's structure equation

14.7.8.1 Cartan's First Structure equation and the calculation of the curvature one-forms

$$
\begin{aligned}
d\omega^{\hat{a}} &= -\Gamma^{\hat{a}}{}_{\hat{b}} \wedge \omega^{\hat{b}} \\
\Gamma^{\hat{a}}{}_{\hat{b}} &= \Gamma^{\hat{a}}{}_{\hat{b}\hat{c}} \omega^{\hat{c}} \\
d\omega^{\hat{u}} &= d\left(\frac{1}{2}(H(u,x,y)+1)du + dv\right) \\
&= \frac{1}{2}\left(\frac{\partial H}{\partial x} dx \wedge du + \frac{\partial H}{\partial y} dy \wedge du\right) \\
&= \frac{1}{2}\left(\frac{\partial H}{\partial x} \omega^{\hat{x}} \wedge \left(\omega^{\hat{u}} + \omega^{\hat{v}}\right) + \frac{\partial H}{\partial y} \omega^{\hat{y}} \wedge \left(\omega^{\hat{u}} + \omega^{\hat{v}}\right)\right) \\
d\omega^{\hat{v}} &= d\left(\frac{1}{2}(1-H)du - dv\right) \\
&= -\frac{1}{2}\left(\frac{\partial H}{\partial x} dx \wedge du + \frac{\partial H}{\partial y} dy \wedge du\right) \\
&= -\frac{1}{2}\left(\frac{\partial H}{\partial x} \omega^{\hat{x}} \wedge \left(\omega^{\hat{u}} + \omega^{\hat{v}}\right) + \frac{\partial H}{\partial y} \omega^{\hat{y}} \wedge \left(\omega^{\hat{u}} + \omega^{\hat{v}}\right)\right) \\
d\omega^{\hat{x}} &= 0 \\
d\omega^{\hat{y}} &= 0
\end{aligned}
$$

The curvature one-forms summarized in a matrix:

$$
\Gamma^{\hat{a}}{}_{\hat{b}} = \begin{pmatrix}
0 & 0 & \frac{1}{2}\frac{\partial H}{\partial x}\left(\omega^{\hat{u}} + \omega^{\hat{v}}\right)(A) & \frac{1}{2}\frac{\partial H}{\partial y}\left(\omega^{\hat{u}} + \omega^{\hat{v}}\right)(B) \\
0 & 0 & \frac{1}{2}\frac{\partial H}{\partial x}\left(\omega^{\hat{u}} + \omega^{\hat{v}}\right)(A) & \frac{1}{2}\frac{\partial H}{\partial y}\left(\omega^{\hat{u}} + \omega^{\hat{v}}\right)(B) \\
\frac{1}{2}\frac{\partial H}{\partial x}\left(\omega^{\hat{u}} + \omega^{\hat{v}}\right)(A) & -\frac{1}{2}\frac{\partial H}{\partial x}\left(\omega^{\hat{u}} + \omega^{\hat{v}}\right)(-A) & 0 & 0 \\
\frac{1}{2}\frac{\partial H}{\partial y}\left(\omega^{\hat{u}} + \omega^{\hat{v}}\right)(B) & -\frac{1}{2}\frac{\partial H}{\partial y}\left(\omega^{\hat{u}} + \omega^{\hat{v}}\right)(-B) & 0 & 0
\end{pmatrix}
$$

[44] $n^a = \left(1, \quad -\frac{1}{2}H, \quad 0, \quad 0\right)$

[45] According to the Cartan calculation further below the sign is wrong

Where $\hat{a}$ refers to column and $\hat{b}$ to row.
A and B will be used later in order to make the calculations easier.

14.7.8.2 The curvature two forms

$$\Omega^{\hat{a}}{}_{\hat{b}} = d\Gamma^{\hat{a}}{}_{\hat{b}} + \Gamma^{\hat{a}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{b}} = \frac{1}{2} R^{\hat{a}}{}_{\hat{b}\hat{c}\hat{d}} \omega^{\hat{c}} \wedge \omega^{\hat{d}}$$

First we will calculate

$$\begin{aligned}
A \wedge A &= \frac{1}{2}\frac{\partial H}{\partial x}\left(\omega^{\hat{u}} + \omega^{\hat{v}}\right) \wedge \frac{1}{2}\frac{\partial H}{\partial x}\left(\omega^{\hat{u}} + \omega^{\hat{v}}\right) \\
&= \left(\frac{1}{2}\frac{\partial H}{\partial x}\right)^2 \left(\omega^{\hat{u}} \wedge \omega^{\hat{u}} + \omega^{\hat{u}} \wedge \omega^{\hat{v}} + \omega^{\hat{v}} \wedge \omega^{\hat{u}} + \omega^{\hat{v}} \wedge \omega^{\hat{v}}\right) \\
&= \left(\frac{1}{2}\frac{\partial H}{\partial x}\right)^2 \left(\omega^{\hat{u}} \wedge \omega^{\hat{v}} + \omega^{\hat{v}} \wedge \omega^{\hat{u}}\right) \\
&= \left(\frac{1}{2}\frac{\partial H}{\partial x}\right)^2 \left(\omega^{\hat{u}} \wedge \omega^{\hat{v}} - \omega^{\hat{u}} \wedge \omega^{\hat{v}}\right) \\
&= 0 \\
B \wedge B &= \frac{1}{2}\frac{\partial H}{\partial y}\left(\omega^{\hat{u}} + \omega^{\hat{v}}\right) \wedge \frac{1}{2}\frac{\partial H}{\partial y}\left(\omega^{\hat{u}} + \omega^{\hat{v}}\right) = 0 \\
dA &= d\left(\frac{1}{2}\frac{\partial H}{\partial x}\left(\omega^{\hat{u}} + \omega^{\hat{v}}\right)\right) \\
&= d\left(\frac{1}{2}\frac{\partial H(u,x,y)}{\partial x} du\right) \\
&= \frac{1}{2}\frac{\partial^2 H(u,x,y)}{\partial x^2} dx \wedge du + \frac{1}{2}\frac{\partial^2 H(u,x,y)}{\partial x \partial y} dy \wedge du \\
&= \frac{1}{2}\frac{\partial^2 H(u,x,y)}{\partial x^2} \omega^{\hat{x}} \wedge \left(\omega^{\hat{u}} + \omega^{\hat{v}}\right) + \frac{1}{2}\frac{\partial^2 H(u,x,y)}{\partial x \partial y} \omega^{\hat{y}} \wedge \left(\omega^{\hat{u}} + \omega^{\hat{v}}\right) \\
dB &= d\left(\frac{1}{2}\frac{\partial H}{\partial y}\left(\omega^{\hat{u}} + \omega^{\hat{v}}\right)\right) \\
&= d\left(\frac{1}{2}\frac{\partial H(u,x,y)}{\partial y} du\right) \\
&= \frac{1}{2}\frac{\partial^2 H}{\partial x \partial y} dx \wedge du + \frac{1}{2}\frac{\partial^2 H}{\partial y^2} dy \wedge du \\
&= \frac{1}{2}\frac{\partial^2 H}{\partial x \partial y} \omega^{\hat{x}} \wedge \left(\omega^{\hat{u}} + \omega^{\hat{v}}\right) + \frac{1}{2}\frac{\partial^2 H}{\partial y^2} \omega^{\hat{y}} \wedge \left(\omega^{\hat{u}} + \omega^{\hat{v}}\right)
\end{aligned}$$

Now we are ready to calculate the curvature two-forms

<u>$\Omega^{\hat{u}}{}_{\hat{u}}$:</u>

$$\begin{aligned}
d\Gamma^{\hat{u}}{}_{\hat{u}} &= 0 \\
\Gamma^{\hat{u}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{u}} &= \Gamma^{\hat{u}}{}_{\hat{u}} \wedge \Gamma^{\hat{u}}{}_{\hat{u}} + \Gamma^{\hat{u}}{}_{\hat{v}} \wedge \Gamma^{\hat{v}}{}_{\hat{u}} + \Gamma^{\hat{u}}{}_{\hat{x}} \wedge \Gamma^{\hat{x}}{}_{\hat{u}} + \Gamma^{\hat{u}}{}_{\hat{y}} \wedge \Gamma^{\hat{y}}{}_{\hat{u}} \\
&= \Gamma^{\hat{u}}{}_{\hat{x}} \wedge \Gamma^{\hat{x}}{}_{\hat{u}} + \Gamma^{\hat{u}}{}_{\hat{y}} \wedge \Gamma^{\hat{y}}{}_{\hat{u}} \\
&= A \wedge A + B \wedge B \\
&= 0 \\
\Rightarrow \quad \Omega^{\hat{u}}{}_{\hat{u}} &= d\Gamma^{\hat{u}}{}_{\hat{u}} + \Gamma^{\hat{u}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{u}} = 0
\end{aligned}$$

<u>$\Omega^{\hat{v}}{}_{\hat{u}}$:</u>

$$\begin{aligned}
d\Gamma^{\hat{v}}{}_{\hat{u}} &= 0 \\
\Gamma^{\hat{v}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{u}} &= \Gamma^{\hat{v}}{}_{\hat{u}} \wedge \Gamma^{\hat{u}}{}_{\hat{u}} + \Gamma^{\hat{v}}{}_{\hat{v}} \wedge \Gamma^{\hat{v}}{}_{\hat{u}} + \Gamma^{\hat{v}}{}_{\hat{x}} \wedge \Gamma^{\hat{x}}{}_{\hat{u}} + \Gamma^{\hat{v}}{}_{\hat{y}} \wedge \Gamma^{\hat{y}}{}_{\hat{u}} \\
&= \Gamma^{\hat{v}}{}_{\hat{x}} \wedge \Gamma^{\hat{x}}{}_{\hat{u}} + \Gamma^{\hat{v}}{}_{\hat{y}} \wedge \Gamma^{\hat{y}}{}_{\hat{u}} \\
&= (-A) \wedge (A) + (-B) \wedge (B) \\
&= 0 \\
\Rightarrow \quad \Omega^{\hat{v}}{}_{\hat{u}} &= 0
\end{aligned}$$

$$\begin{aligned}
\underline{\Omega^{\hat{x}}{}_{\hat{u}}}: \quad & d\Gamma^{\hat{x}}{}_{\hat{u}} &&= dA = \frac{1}{2}\frac{\partial^2 H}{\partial x^2}\omega^{\hat{x}} \wedge \left(\omega^{\hat{u}} + \omega^{\hat{v}}\right) + \frac{1}{2}\frac{\partial^2 H}{\partial x \partial y}\omega^{\hat{y}} \wedge \left(\omega^{\hat{u}} + \omega^{\hat{v}}\right) \\
& \Gamma^{\hat{x}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{u}} &&= \Gamma^{\hat{x}}{}_{\hat{u}} \wedge \Gamma^{\hat{u}}{}_{\hat{u}} + \Gamma^{\hat{x}}{}_{\hat{v}} \wedge \Gamma^{\hat{v}}{}_{\hat{u}} + \Gamma^{\hat{x}}{}_{\hat{x}} \wedge \Gamma^{\hat{x}}{}_{\hat{u}} + \Gamma^{\hat{x}}{}_{\hat{y}} \wedge \Gamma^{\hat{y}}{}_{\hat{u}} = 0 \\
\Rightarrow \quad & \Omega^{\hat{x}}{}_{\hat{u}} &&= \frac{1}{2}\frac{\partial^2 H}{\partial x^2}\omega^{\hat{x}} \wedge \left(\omega^{\hat{u}} + \omega^{\hat{v}}\right) + \frac{1}{2}\frac{\partial^2 H}{\partial x \partial y}\omega^{\hat{y}} \wedge \left(\omega^{\hat{u}} + \omega^{\hat{v}}\right) \\
\underline{\Omega^{\hat{y}}{}_{\hat{u}}}: \quad & \Gamma^{\hat{y}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{u}} &&= \Gamma^{\hat{y}}{}_{\hat{u}} \wedge \Gamma^{\hat{u}}{}_{\hat{u}} + \Gamma^{\hat{y}}{}_{\hat{v}} \wedge \Gamma^{\hat{v}}{}_{\hat{u}} + \Gamma^{\hat{y}}{}_{\hat{x}} \wedge \Gamma^{\hat{x}}{}_{\hat{u}} + \Gamma^{\hat{y}}{}_{\hat{y}} \wedge \Gamma^{\hat{y}}{}_{\hat{u}} = 0 \\
\Rightarrow \quad & \Omega^{\hat{y}}{}_{\hat{u}} &&= \frac{1}{2}\frac{\partial^2 H}{\partial x \partial y}\omega^{\hat{x}} \wedge \left(\omega^{\hat{u}} + \omega^{\hat{v}}\right) + \frac{1}{2}\frac{\partial^2 H}{\partial y^2}\omega^{\hat{y}} \wedge \left(\omega^{\hat{u}} + \omega^{\hat{v}}\right) \\
\underline{\Omega^{\hat{v}}{}_{\hat{v}}} \quad & d\Gamma^{\hat{v}}{}_{\hat{v}} &&= 0 \\
& \Gamma^{\hat{v}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{v}} &&= \Gamma^{\hat{v}}{}_{\hat{u}} \wedge \Gamma^{\hat{u}}{}_{\hat{v}} + \Gamma^{\hat{v}}{}_{\hat{v}} \wedge \Gamma^{\hat{v}}{}_{\hat{v}} + \Gamma^{\hat{v}}{}_{\hat{x}} \wedge \Gamma^{\hat{x}}{}_{\hat{v}} + \Gamma^{\hat{v}}{}_{\hat{y}} \wedge \Gamma^{\hat{y}}{}_{\hat{v}} \\
& &&= \Gamma^{\hat{v}}{}_{\hat{x}} \wedge \Gamma^{\hat{x}}{}_{\hat{v}} + \Gamma^{\hat{v}}{}_{\hat{y}} \wedge \Gamma^{\hat{y}}{}_{\hat{v}} \\
& &&= -(A) \wedge (A) - (B) \wedge (B) \\
& &&= 0 \\
\Rightarrow \quad & \Omega^{\hat{v}}{}_{\hat{v}} &&= 0 \\
\underline{\Omega^{\hat{x}}{}_{\hat{v}}}: \quad & d\Gamma^{\hat{x}}{}_{\hat{v}} &&= dA = \frac{1}{2}\frac{\partial^2 H}{\partial x^2}\omega^{\hat{x}} \wedge \left(\omega^{\hat{u}} + \omega^{\hat{v}}\right) + \frac{1}{2}\frac{\partial^2 H}{\partial x \partial y}\omega^{\hat{y}} \wedge \left(\omega^{\hat{u}} + \omega^{\hat{v}}\right) \\
& \Gamma^{\hat{x}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{v}} &&= \Gamma^{\hat{x}}{}_{\hat{u}} \wedge \Gamma^{\hat{u}}{}_{\hat{v}} + \Gamma^{\hat{x}}{}_{\hat{v}} \wedge \Gamma^{\hat{v}}{}_{\hat{v}} + \Gamma^{\hat{x}}{}_{\hat{x}} \wedge \Gamma^{\hat{x}}{}_{\hat{v}} + \Gamma^{\hat{x}}{}_{\hat{y}} \wedge \Gamma^{\hat{y}}{}_{\hat{v}} = 0 \\
\Rightarrow \quad & \Omega^{\hat{x}}{}_{\hat{v}} &&= \frac{1}{2}\frac{\partial^2 H}{\partial x^2}\omega^{\hat{x}} \wedge \left(\omega^{\hat{u}} + \omega^{\hat{v}}\right) + \frac{1}{2}\frac{\partial^2 H}{\partial x \partial y}\omega^{\hat{y}} \wedge \left(\omega^{\hat{u}} + \omega^{\hat{v}}\right) \\
\underline{\Omega^{\hat{y}}{}_{\hat{v}}}: \quad & d\Gamma^{\hat{y}}{}_{\hat{v}} &&= dB = \frac{1}{2}\frac{\partial^2 H}{\partial x \partial y}\omega^{\hat{x}} \wedge \left(\omega^{\hat{u}} + \omega^{\hat{v}}\right) + \frac{1}{2}\frac{\partial^2 H}{\partial y^2}\omega^{\hat{y}} \wedge \left(\omega^{\hat{u}} + \omega^{\hat{v}}\right) \\
& \Gamma^{\hat{y}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{v}} &&= \Gamma^{\hat{y}}{}_{\hat{u}} \wedge \Gamma^{\hat{u}}{}_{\hat{v}} + \Gamma^{\hat{y}}{}_{\hat{v}} \wedge \Gamma^{\hat{v}}{}_{\hat{v}} + \Gamma^{\hat{y}}{}_{\hat{x}} \wedge \Gamma^{\hat{x}}{}_{\hat{v}} + \Gamma^{\hat{y}}{}_{\hat{y}} \wedge \Gamma^{\hat{y}}{}_{\hat{v}} = 0 \\
\Rightarrow \quad & \Omega^{\hat{y}}{}_{\hat{v}} &&= \frac{1}{2}\frac{\partial^2 H}{\partial x \partial y}\omega^{\hat{x}} \wedge \left(\omega^{\hat{u}} + \omega^{\hat{v}}\right) + \frac{1}{2}\frac{\partial^2 H}{\partial y^2}\omega^{\hat{y}} \wedge \left(\omega^{\hat{u}} + \omega^{\hat{v}}\right) \\
\underline{\Omega^{\hat{x}}{}_{\hat{x}}}: \quad & d\Gamma^{\hat{x}}{}_{\hat{x}} &&= 0 \\
& \Gamma^{\hat{x}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{x}} &&= \Gamma^{\hat{x}}{}_{\hat{u}} \wedge \Gamma^{\hat{u}}{}_{\hat{x}} + \Gamma^{\hat{x}}{}_{\hat{v}} \wedge \Gamma^{\hat{v}}{}_{\hat{x}} + \Gamma^{\hat{x}}{}_{\hat{x}} \wedge \Gamma^{\hat{x}}{}_{\hat{x}} + \Gamma^{\hat{x}}{}_{\hat{y}} \wedge \Gamma^{\hat{y}}{}_{\hat{x}} = (A) \wedge (A) - (A) \wedge (A) = 0 \\
\Rightarrow \quad & \Omega^{\hat{x}}{}_{\hat{x}} &&= 0 \\
\underline{\Omega^{\hat{y}}{}_{\hat{x}}}: \quad & \Gamma^{\hat{y}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{x}} &&= \Gamma^{\hat{y}}{}_{\hat{u}} \wedge \Gamma^{\hat{u}}{}_{\hat{x}} + \Gamma^{\hat{y}}{}_{\hat{v}} \wedge \Gamma^{\hat{v}}{}_{\hat{x}} + \Gamma^{\hat{y}}{}_{\hat{x}} \wedge \Gamma^{\hat{x}}{}_{\hat{x}} + \Gamma^{\hat{y}}{}_{\hat{y}} \wedge \Gamma^{\hat{y}}{}_{\hat{x}} = (B) \wedge (A) - (B) \wedge (A) = 0 \\
\Rightarrow \quad & \Omega^{\hat{y}}{}_{\hat{x}} &&= 0 \\
\Rightarrow \quad & \Omega^{\hat{y}}{}_{\hat{y}} &&= 0
\end{aligned}$$

Summarized in a matrix:

$$\Omega^{\hat{a}}{}_{\hat{b}} = \begin{Bmatrix}
0 & 0 & \left[\frac{1}{2}\frac{\partial^2 H}{\partial x^2}\omega^{\hat{x}} + \frac{1}{2}\frac{\partial^2 H}{\partial x \partial y}\omega^{\hat{y}}\right] \wedge \left(\omega^{\hat{u}} + \omega^{\hat{v}}\right) & \left[\frac{1}{2}\frac{\partial^2 H}{\partial x \partial y}\omega^{\hat{x}} + \frac{1}{2}\frac{\partial^2 H}{\partial y^2}\omega^{\hat{y}}\right] \wedge \left(\omega^{\hat{u}} + \omega^{\hat{v}}\right) \\
0 & 0 & \left[\frac{1}{2}\frac{\partial^2 H}{\partial x^2}\omega^{\hat{x}} + \frac{1}{2}\frac{\partial^2 H}{\partial x \partial y}\omega^{\hat{y}}\right] \wedge \left(\omega^{\hat{u}} + \omega^{\hat{v}}\right) & \left[\frac{1}{2}\frac{\partial^2 H}{\partial x \partial y}\omega^{\hat{x}} + \frac{1}{2}\frac{\partial^2 H}{\partial y^2}\omega^{\hat{y}}\right] \wedge \left(\omega^{\hat{u}} + \omega^{\hat{v}}\right) \\
S & AS & 0 & 0 \\
S & AS & 0 & 0
\end{Bmatrix}$$

Now we can write down the independent elements of the Riemann tensor in the non-coordinate basis:

$$R^{\hat{x}}{}_{\hat{u}\hat{x}\hat{u}}(C) = \frac{1}{2}\frac{\partial^2 H}{\partial x^2} \qquad\qquad R^{\hat{y}}{}_{\hat{u}\hat{y}\hat{u}}(E) = \frac{1}{2}\frac{\partial^2 H}{\partial y^2}$$

$$\mathrm{R}^{\hat{x}}{}_{\hat{u}\hat{x}\hat{v}}(C) = \frac{1}{2}\frac{\partial^2 H}{\partial x^2} \qquad \mathrm{R}^{\hat{y}}{}_{\hat{u}\hat{y}\hat{v}}(E) = \frac{1}{2}\frac{\partial^2 H}{\partial y^2}$$

$$\mathrm{R}^{\hat{x}}{}_{\hat{u}\hat{y}\hat{u}}(D) = \frac{1}{2}\frac{\partial^2 H}{\partial x \partial y} \qquad \mathrm{R}^{\hat{y}}{}_{\hat{v}\hat{y}\hat{v}}(E) = \frac{1}{2}\frac{\partial^2 H}{\partial y^2}$$

$$\mathrm{R}^{\hat{x}}{}_{\hat{u}\hat{y}\hat{v}}(D) = \frac{1}{2}\frac{\partial^2 H}{\partial x \partial y} \qquad \mathrm{R}^{\hat{x}}{}_{\hat{v}\hat{y}\hat{u}}(D) = \frac{1}{2}\frac{\partial^2 H}{\partial x \partial y}$$

$$\mathrm{R}^{\hat{x}}{}_{\hat{v}\hat{x}\hat{v}}(C) = \frac{1}{2}\frac{\partial^2 H}{\partial x^2} \qquad \mathrm{R}^{\hat{x}}{}_{\hat{v}\hat{y}\hat{v}}(D) = \frac{1}{2}\frac{\partial^2 H}{\partial x \partial y}$$

Where C,D and E will be used later, to make the calculations easier

14.7.8.3 The Ricci tensor

$$R_{\hat{a}\hat{b}} = R^{\hat{c}}{}_{\hat{a}\hat{c}\hat{b}}$$

$$\begin{aligned}
R_{\hat{u}\hat{u}} &= R^{\hat{c}}{}_{\hat{u}\hat{c}\hat{u}} \\
&= R^{\hat{u}}{}_{\hat{u}\hat{u}\hat{u}} + R^{\hat{v}}{}_{\hat{u}\hat{v}\hat{u}} + R^{\hat{x}}{}_{\hat{u}\hat{x}\hat{u}} + R^{\hat{y}}{}_{\hat{u}\hat{y}\hat{u}} \\
&= C + E \\
&= \frac{1}{2}\frac{\partial^2 H}{\partial x^2} + \frac{1}{2}\frac{\partial^2 H}{\partial y^2}
\end{aligned}$$

$$\begin{aligned}
R_{\hat{v}\hat{u}} &= R^{\hat{c}}{}_{\hat{v}\hat{c}\hat{u}} \\
&= R^{\hat{u}}{}_{\hat{v}\hat{u}\hat{u}} + R^{\hat{v}}{}_{\hat{v}\hat{v}\hat{u}} + R^{\hat{x}}{}_{\hat{v}\hat{x}\hat{u}} + R^{\hat{y}}{}_{\hat{v}\hat{y}\hat{u}} \\
&= C + E \\
&= \frac{1}{2}\frac{\partial^2 H}{\partial x^2} + \frac{1}{2}\frac{\partial^2 H}{\partial y^2}
\end{aligned}$$

$$R_{\hat{x}\hat{u}} = 0$$

$$R_{\hat{y}\hat{u}} = 0$$

$$\begin{aligned}
R_{\hat{v}\hat{v}} &= R^{\hat{c}}{}_{\hat{v}\hat{c}\hat{v}} \\
&= R^{\hat{u}}{}_{\hat{v}\hat{u}\hat{v}} + R^{\hat{v}}{}_{\hat{v}\hat{v}\hat{v}} + R^{\hat{x}}{}_{\hat{v}\hat{x}\hat{v}} + R^{\hat{y}}{}_{\hat{v}\hat{y}\hat{v}} \\
&= C + E \\
&= \frac{1}{2}\frac{\partial^2 H}{\partial x^2} + \frac{1}{2}\frac{\partial^2 H}{\partial y^2}
\end{aligned}$$

$$R_{\hat{x}\hat{v}} = 0$$

$$R_{\hat{y}\hat{v}} = 0$$

$$R_{\hat{x}\hat{x}} = R^{\hat{c}}{}_{\hat{x}\hat{c}\hat{x}} = R^{\hat{u}}{}_{\hat{x}\hat{u}\hat{x}} + R^{\hat{v}}{}_{\hat{x}\hat{v}\hat{x}} + R^{\hat{x}}{}_{\hat{x}\hat{x}\hat{x}} + R^{\hat{y}}{}_{\hat{x}\hat{y}\hat{x}} = -C + C = 0$$

$$R_{\hat{y}\hat{x}} = R^{\hat{c}}{}_{\hat{y}\hat{c}\hat{x}} = R^{\hat{u}}{}_{\hat{y}\hat{u}\hat{x}} + R^{\hat{v}}{}_{\hat{y}\hat{v}\hat{x}} + R^{\hat{x}}{}_{\hat{y}\hat{x}\hat{x}} + R^{\hat{y}}{}_{\hat{y}\hat{y}\hat{x}} = -D + D = 0$$

$$R_{\hat{y}\hat{y}} = R^{\hat{c}}{}_{\hat{y}\hat{c}\hat{y}} = R^{\hat{u}}{}_{\hat{y}\hat{u}\hat{y}} + R^{\hat{v}}{}_{\hat{y}\hat{v}\hat{y}} + R^{\hat{x}}{}_{\hat{y}\hat{x}\hat{y}} + R^{\hat{y}}{}_{\hat{y}\hat{y}\hat{y}} = -E + E = 0$$

Summarized in a matrix:

$$R_{\hat{a}\hat{b}} = \begin{pmatrix} \frac{1}{2}\frac{\partial^2 H}{\partial x^2} + \frac{1}{2}\frac{\partial^2 H}{\partial y^2} & \frac{1}{2}\frac{\partial^2 H}{\partial x^2} + \frac{1}{2}\frac{\partial^2 H}{\partial y^2} & 0 & 0 \\ \frac{1}{2}\frac{\partial^2 H}{\partial x^2} + \frac{1}{2}\frac{\partial^2 H}{\partial y^2} & \frac{1}{2}\frac{\partial^2 H}{\partial x^2} + \frac{1}{2}\frac{\partial^2 H}{\partial y^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Where $\hat{a}$ refers to column and $\hat{b}$ to row

14.7.8.4 The Ricci tensor in the coordinate basis

The transformation:

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$$\begin{aligned}
R_{ab} &= \Lambda^{\hat{c}}{}_a \Lambda^{\hat{d}}{}_b R_{\hat{c}\hat{d}} \\
R_{uu} &= \Lambda^{\hat{c}}{}_u \Lambda^{\hat{d}}{}_u R_{\hat{c}\hat{d}} \\
&= \Lambda^{\hat{u}}{}_u \Lambda^{\hat{d}}{}_u R_{\hat{u}\hat{d}} + \Lambda^{\hat{v}}{}_u \Lambda^{\hat{d}}{}_u R_{\hat{v}\hat{d}} \\
&= \Lambda^{\hat{u}}{}_u \Lambda^{\hat{u}}{}_u R_{\hat{u}\hat{u}} + \Lambda^{\hat{u}}{}_u \Lambda^{\hat{v}}{}_u R_{\hat{u}\hat{v}} + \Lambda^{\hat{v}}{}_u \Lambda^{\hat{u}}{}_u R_{\hat{v}\hat{u}} + \Lambda^{\hat{v}}{}_u \Lambda^{\hat{v}}{}_u R_{\hat{v}\hat{v}} \\
&= {}^{46}(C+E) \\
&= \frac{1}{2}\frac{\partial^2 H}{\partial x^2} + \frac{1}{2}\frac{\partial^2 H}{\partial y^2} \\
R_{uv} &= \Lambda^{\hat{c}}{}_u \Lambda^{\hat{d}}{}_v R_{\hat{c}\hat{d}} \\
&= \Lambda^{\hat{u}}{}_u \Lambda^{\hat{d}}{}_v R_{\hat{u}\hat{d}} + \Lambda^{\hat{v}}{}_u \Lambda^{\hat{d}}{}_v R_{\hat{v}\hat{d}} \\
&= \Lambda^{\hat{u}}{}_u \Lambda^{\hat{u}}{}_v R_{\hat{u}\hat{u}} + \Lambda^{\hat{u}}{}_u \Lambda^{\hat{v}}{}_v R_{\hat{u}\hat{v}} + \Lambda^{\hat{v}}{}_u \Lambda^{\hat{u}}{}_v R_{\hat{v}\hat{u}} + \Lambda^{\hat{v}}{}_u \Lambda^{\hat{v}}{}_v R_{\hat{v}\hat{v}} \\
&= {}^{47}0 \\
R_{vv} &= \Lambda^{\hat{c}}{}_v \Lambda^{\hat{d}}{}_v R_{\hat{c}\hat{d}} \\
&= \Lambda^{\hat{u}}{}_v \Lambda^{\hat{d}}{}_v R_{\hat{u}\hat{d}} + \Lambda^{\hat{v}}{}_v \Lambda^{\hat{d}}{}_v R_{\hat{v}\hat{d}} \\
&= \Lambda^{\hat{u}}{}_v \Lambda^{\hat{u}}{}_v R_{\hat{u}\hat{u}} + \Lambda^{\hat{u}}{}_v \Lambda^{\hat{v}}{}_v R_{\hat{u}\hat{v}} + \Lambda^{\hat{v}}{}_v \Lambda^{\hat{u}}{}_v R_{\hat{v}\hat{u}} + \Lambda^{\hat{v}}{}_v \Lambda^{\hat{v}}{}_v R_{\hat{v}\hat{v}} \\
&= 1\cdot 1(C+E) + 1(-1)(C+E) + (-1)1(C+E) + (-1)(-1)(C+E) \\
&= 0 \\
R_{xx} &= R_{yy} = R_{xy} = 0 \\
\Rightarrow \quad R &= g^{ab}R_{ab} = 0 \cdot R_{uu} = 0
\end{aligned}$$

Summarized in a matrix:

$$R_{ab} = \begin{Bmatrix} \frac{1}{2}\frac{\partial^2 H}{\partial x^2} + \frac{1}{2}\frac{\partial^2 H}{\partial y^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Bmatrix}$$

Where a refers to column and b to row

$$\Rightarrow \quad G_{uu} = \frac{1}{2}\frac{\partial^2 H}{\partial x^2} + \frac{1}{2}\frac{\partial^2 H}{\partial y^2}$$

14.8 °The Aichelburg-Sexl Solution – The passing of a black hole

The line element

$$ds^2 = 4\mu \log(x^2+y^2)\, du^2 + 2dudr - \mathrm{dx}^2 - dy^2$$

Comparing with the Brinkmann metric

$$ds^2 = H(u,x,y)du^2 + 2dudv - dx^2 - dy^2$$

We see that we can copy the results from the Brinkmann calculations with $H(u,x,y) = 4\mu\log(x^2+y^2)$

The only non-zero spin-coefficient is:

$$\begin{aligned}
\nu &= -\frac{1}{2\sqrt{2}}\left(\frac{\partial H}{\partial x} + i\frac{\partial H}{\partial y}\right) \\
&= -\frac{1}{2\sqrt{2}}\left(\frac{\partial(4\mu\log(x^2+y^2))}{\partial x} + i\frac{\partial(4\mu\log(x^2+y^2))}{\partial y}\right)
\end{aligned}$$

---

[46] $= \left(\frac{1}{2}(H+1)\right)^2(C+E) + \frac{1}{2}(H+1)\frac{1}{2}(1-H)(C+E) + \frac{1}{2}(1-H)\frac{1}{2}(H+1)(C+E) + \left(\frac{1}{2}(1-H)\right)^2(C+E) =$

[47] $= \frac{1}{2}(H+1)\cdot 1(C+E) + \frac{1}{2}(H+1)(-1)(C+E) + \frac{1}{2}(1-H)\cdot 1(C+E) + \frac{1}{2}(1-H)(-1)(C+E) =$

$$= -\frac{4\mu}{2\sqrt{2}}\left(\frac{2x}{x^2+y^2} + i\frac{2y}{x^2+y^2}\right)$$
$$= -2\sqrt{2}\mu\left(\frac{x}{x^2+y^2} + i\frac{y}{x^2+y^2}\right)$$

14.9 Coordinate transformations

14.9.1 PTransformation of the Brinkmann line-element – a vacuum solution

The Brinkmann line-element

$$ds^2 = H(U,Y,Z)dU^2 + dUdV - dY^2 - dZ^2$$

We can use the following coordinate transformation and show that the Brinkmann space-time is a Rosen spacetime.

$$\begin{aligned} U &= u \\ V &= v + y^2 ff' + z^2 gg' \\ Y &= fy \\ Z &= gz \end{aligned}$$

where

$$\begin{aligned} f &= f(u) = f(U) \\ g &= g(u) = g(U) \end{aligned}$$

$$\begin{aligned} \Rightarrow\quad dU &= du \\ dV &= dv + 2yff'dy + y^2[(f')^2 + ff'']du + 2zgg'dz + z^2[(g')^2 + g'']du \\ &= dv + 2yff'dy + 2zgg'dz + (y^2[(f')^2 + ff''] + z^2[(g')^2 + g''])du \\ dY &= fdy + yf'du \\ dZ &= gdz + zg'du \end{aligned}$$

$$\begin{aligned} \Rightarrow\quad dU^2 &= du^2 \\ dUdV &= dudv + 2yff'dudy + 2zgg'dudz + (y^2[(f')^2 + ff''] + z^2[(g')^2 + g''])du^2 \\ dY^2 &= (fdy + yf'du)^2 = f^2dy^2 + y^2(f')^2du^2 + 2yff'dudy \\ dZ^2 &= (gdz + zg'du)^2 = g^2dz^2 + z^2(g')^2du^2 + 2zgg'dudz \end{aligned}$$

$$\begin{aligned} \Rightarrow\quad ds^2 &= H(U,Y,Z)dU^2 + dUdV - dY^2 - dZ^2 \\ &= {}^{48}H(U,Y,Z)du^2 + dudv + (y^2ff'' + z^2g'')du^2 - f^2dy^2 - g^2dz^2 \\ &= [H(U,Y,Z) + y^2ff'' + z^2gg'']du^2 + dudv - f^2dy^2 - g^2dz^2 \end{aligned}$$

Which equals the Rosen line-element if

$$\begin{aligned} ds^2 &= dudv - f^2dy^2 - g^2dz^2 \\ H(U,Y,Z) &= -y^2ff'' - z^2gg'' \end{aligned}$$

<u>In the vacuum case</u>

$$\frac{f''}{f} = -\frac{g''}{g} = h(u)$$

$$\Rightarrow\quad H(U,Y,Z) = -y^2f^2h(u) + z^2g^2h(u) = h(U)(Z^2 - Y^2)$$

$$\Rightarrow\quad ds^2 = h(U)(Z^2 - Y^2)dU^2 + dUdV - dY^2 - dZ^2$$

14.9.1.1 The inverse transformation and the transformation matrices

<u>The inverse transformation</u>

$$\begin{aligned} u &= U \\ v &= V - y^2ff' - z^2gg' = V - \left(\frac{Y}{f}\right)^2 ff' - \left(\frac{Z}{g}\right)^2 gg' = V - Y^2\frac{f'}{f} - Z^2\frac{g'}{g} \\ y &= \frac{Y}{f} \end{aligned}$$

[48] $= H(U,Y,Z)du^2 + dudv + 2yff'dudy + 2zgg'dudz + (y^2[(f')^2 + ff''] + z^2[(g')^2 + gg''])du^2 - (f^2dy^2 + y^2(f')^2du^2 + 2yff'dudy) - (g^2dz^2 + z^2(g')^2du^2 + 2zgg'dudz) =$

$$
\begin{aligned}
z &= \frac{Z}{g} \\
du &= dU \\
dv &= dV - 2Y\frac{f'}{f}dY - Y^2\frac{f''}{f}dU + Y^2\frac{(f')^2}{f^2}dU - 2Z\frac{g'}{g}dZ - Z^2\frac{g''}{g}dU + Z^2\frac{(g')^2}{g^2}dU \\
&= dV - 2Y\frac{f'}{f}dY - 2Z\frac{g'}{g}dZ - \left(Y^2\left[\frac{f''}{f} - \frac{(f')^2}{f^2}\right] + Z^2\left[\frac{g''}{g} - \frac{(g')^2}{g^2}\right]\right)dU \\
dy &= \frac{1}{f}dY - \frac{Yf'}{f^2}dU \\
dz &= \frac{1}{g}dZ - \frac{Zg'}{g^2}dU
\end{aligned}
$$

<u>The transformation matrices</u>

$$
\begin{Bmatrix} dU \\ dV \\ dY \\ dZ \end{Bmatrix} = \begin{Bmatrix} 1 & 0 & 0 & 0 \\ y^2[(f')^2 + ff''] + z^2[(g')^2 + g''] & 1 & 2yff' & 2zgg' \\ yf' & 0 & f & 0 \\ zg' & 0 & 0 & g \end{Bmatrix} \begin{Bmatrix} du \\ dv \\ dy \\ dz \end{Bmatrix}
$$

$$
\begin{Bmatrix} du \\ dv \\ dy \\ dz \end{Bmatrix} = \begin{Bmatrix} 1 & 0 & 0 & 0 \\ Y^2\left[\frac{f''}{f} - \frac{(f')^2}{f^2}\right] + Z^2\left[\frac{g''}{g} - \frac{(g')^2}{g^2}\right] & 1 & -2Y\frac{f'}{f} & -2Z\frac{g'}{g} \\ -\frac{Yf'}{f^2} & 0 & \frac{1}{f} & 0 \\ -\frac{Zg'}{g^2} & 0 & 0 & \frac{1}{g} \end{Bmatrix} \begin{Bmatrix} dU \\ dV \\ dY \\ dZ \end{Bmatrix}
$$

$$
= \begin{Bmatrix} 1 & 0 & 0 & 0 \\ y^2[ff'' - (f')^2] + z^2[gg'' - (g')^2] & 1 & -2yf' & -2zg' \\ -\frac{yf'}{f} & 0 & \frac{1}{f} & 0 \\ -\frac{zg'}{g} & 0 & 0 & \frac{1}{g} \end{Bmatrix} \begin{Bmatrix} dU \\ dV \\ dY \\ dZ \end{Bmatrix}
$$

14.9.2 [q]Colliding gravity waves

The metric of a plane gravitational wave

$$ds^2 = \delta(u)(X^2 - Y^2)du^2 + 2dudr - dX^2 - dY^2$$

is a special case (vacuum-solution) of the Brinkmann space-time with $H(u, X, Y) = \delta(u)(X^2 - Y^2)$. [r]It can be written in terms of the null coordinates u and v by using the following coordinate transformation

$$
\begin{aligned}
u &= u \\
r &= v - \frac{1}{2}\Theta(u)(1-u)x^2 + \frac{1}{2}\Theta(u)(1+u)y^2 \\
X &= \big(1 - u\Theta(u)\big)x \\
Y &= \big(1 + u\Theta(u)\big)y \\
du &= du \\
dr &= \frac{\partial r}{\partial v}dv + \frac{\partial r}{\partial u}du + \frac{\partial r}{\partial x}dx + \frac{\partial r}{\partial y}dy
\end{aligned}
$$

$$= {}^{49}dv - \frac{1}{2}\big(\delta(u)(x^2-y^2) - \Theta(u)(x^2+y^2)\big)du - \Theta(u)(1-u)xdx + \Theta(u)(1+u)ydy$$

$$dX = \big(-\Theta(u) - u\Theta'(u)\big)xdu + \big(1-u\Theta(u)\big)dx = -\Theta(u)xdu + \big(1-u\Theta(u)\big)dx$$

$$dY = \big(\Theta(u) + u\Theta'(u)\big)ydu + \big(1+u\Theta(u)\big)dy = \Theta(u)ydu + \big(1+u\Theta(u)\big)dy$$

$$X^2 - Y^2 = \big(1-u\Theta(u)\big)^2x^2 - \big(1+u\Theta(u)\big)^2y^2 = \big(1+u^2\Theta^2(u)\big)(x^2-y^2) - 2u\Theta(u)(x^2+y^2)$$

$$\delta(u)(X^2-Y^2) = \delta(u)\big(1+u^2\Theta^2(u)\big)(x^2-y^2) - \delta(u)2u\Theta(u)(x^2+y^2) = \delta(u)(x^2-y^2)$$

$$dX^2 = \big(-\Theta(u)xdu + \big(1-u\Theta(u)\big)dx\big)^2$$

$$= \Theta^2(u)x^2du^2 + \big(1-u\Theta(u)\big)^2dx^2 - 2\Theta(u)\big(1-u\Theta(u)\big)xdudx$$

$$dY^2 = \big(\Theta(u)ydu + \big(1+u\Theta(u)\big)dy\big)^2$$

$$= \Theta^2(u)y^2du^2 + \big(1+u\Theta(u)\big)^2dy^2 + 2\Theta(u)\big(1+u\Theta(u)\big)ydudy$$

$$dX^2 + dY^2 = {}^{50}\Theta^2(u)(x^2+y^2)du^2 + \big(1-u\Theta(u)\big)^2dx^2 - 2\Theta(u)\big(1-u\Theta(u)\big)xdudx + \big(1+u\Theta(u)\big)^2dy^2 + 2\Theta(u)\big(1+u\Theta(u)\big)ydudy$$

$$ds^2 = \delta(u)(X^2-Y^2)du^2 + 2dudr - dX^2 - dY^2$$

$$= {}^{51\ 52}2dudv - \big(1-u\Theta(u)\big)^2dx^2 - \big(1+u\Theta(u)\big)^2dy^2$$

14.10 [5]Impulsive gravitational wave Region III

The line element:

$$ds^2 = 2dudv - [1-v\Theta(v)]^2dx^2 - [1+v\Theta(v)]^2dy^2$$

Notice: This space-time is a Rosen space-time with

$$f(v) = 1 - v\Theta(v)$$
$$g(v) = 1 + v\Theta(v)$$

We have

$$f'(v) = -\Theta(v) - v\delta(v) = -\Theta(v)$$
$$g'(v) = \Theta(v) + v\delta(v) = \Theta(v)$$
$$\Rightarrow \quad f''(v) = -\delta(v)$$
$$g''(v) = \delta(v)$$
$$R_{0202} = f(v)f''(v) = \big(1-v\Theta(v)\big)\big(-\delta(v)\big) = -\delta(v)$$
$$R_{0303} = g(v)g''(v) = \big(1+v\Theta(v)\big)\delta(v) = \delta(v)$$

Because

$$f''(v) = -g''(v) = -\delta(v)$$

This space-time is a no-curvature vacuum solution with a non-vanishing Riemann tensor

14.10.1 The Christoffel symbols

To find the Christoffel symbols we calculate the geodesic from the Euler-Lagrange equation

---

[49] $= dv - \left(\frac{1}{2}\big(\Theta'(u)(1-u) - \Theta(u)\big)x^2 - \frac{1}{2}\big(\Theta'(u)(1+u) + \Theta(u)\big)y^2\right)du - \Theta(u)(1-u)xdx + \Theta(u)(1+u)ydy = dv - \left(\frac{1}{2}\big(\delta(u) - \Theta(u)\big)x^2 - \frac{1}{2}\big(\delta(u) + \Theta(u)\big)y^2\right)du - \Theta(u)(1-u)xdx + \Theta(u)(1+u)ydy\ [Notice{:}\, u\Theta'(u) = u\delta(u) = 0] =$

[50] $= \Theta^2(u)x^2du^2 + \big(1-u\Theta(u)\big)^2dx^2 - 2\Theta(u)\big(1-u\Theta(u)\big)xdudx + \Theta^2(u)y^2du^2 + \big(1+u\Theta(u)\big)^2dy^2 + 2\Theta(u)\big(1+u\Theta(u)\big)ydudy =$

[51] $= \delta(u)(x^2-y^2)du^2 + 2du\left(dv - \frac{1}{2}\big(\delta(u)(x^2-y^2) - \Theta(u)(x^2+y^2)\big)du - \Theta(u)(1-u)xdx + \Theta(u)(1+u)ydy\right) - \left(\Theta^2(u)(x^2+y^2)du^2 + \big(1-u\Theta(u)\big)^2dx^2 - 2\Theta(u)\big(1-u\Theta(u)\big)xdudx + \big(1+u\Theta(u)\big)^2dy^2 + 2\Theta(u)\big(1+u\Theta(u)\big)ydudy\right) = 2dudv + 2du(-\Theta(u)(1-u)xdx + \Theta(u)(1+u)ydy) - \left(\big(1-u\Theta(u)\big)^2dx^2 - 2\Theta(u)\big(1-u\Theta(u)\big)xdudx + \big(1+u\Theta(u)\big)^2dy^2 + 2\Theta(u)\big(1+u\Theta(u)\big)ydudy\right) =$

[52] $\Theta^2(u) = \Theta(u)$

$$0 = \frac{d}{ds}\left(\frac{\partial F}{\partial \dot{x}^a}\right) - \frac{\partial F}{\partial x^a}$$

Where

$$F = 2\dot{u}\dot{v} - [1 - v\Theta(v)]^2\dot{x}^2 - [1 + v\Theta(v)]^2\dot{y}^2$$

$\underline{x^a = u:}$

$$\frac{\partial F}{\partial u} = 0$$
$$\frac{\partial F}{\partial \dot{u}} = 2\dot{v}$$
$$\frac{d}{ds}\left(\frac{\partial F}{\partial \dot{u}}\right) = 2\ddot{v}$$
$$\Rightarrow \quad 0 = \ddot{v}$$

$\underline{x^a = v:}$

$$\frac{\partial F}{\partial v} = 2\Theta(v)[1 - v\Theta(v)]\dot{x}^2 - 2\Theta(v)[1 + v\Theta(v)]\dot{y}^2$$
$$\frac{\partial F}{\partial \dot{v}} = 2\dot{u}$$
$$\frac{d}{ds}\left(\frac{\partial F}{\partial \dot{v}}\right) = 2\ddot{u}$$
$$\Rightarrow \quad 0 = \ddot{u} - \Theta(v)[1 - v\Theta(v)]\dot{x}^2 + \Theta(v)[1 + v\Theta(v)]\dot{y}^2$$

$\underline{x^a = x:}$

$$\frac{\partial F}{\partial x} = 0$$
$$\frac{\partial F}{\partial \dot{x}} = -2[1 - v\Theta(v)]^2\dot{x}$$
$$\frac{d}{ds}\left(\frac{\partial F}{\partial \dot{x}}\right) = -2\left(-\dot{v}\Theta(v) - v\dot{v}\frac{d\Theta(v)}{dv}\right)2[1 - v\Theta(v)]\dot{x} - 2[1 - v\Theta(v)]^2\ddot{x}$$
$$= {}^{53}4\big(\Theta(v) + v\delta(v)\big)[1 - v\Theta(v)]\dot{x}\dot{v} - 2[1 - v\Theta(v)]^2\ddot{x}$$
$$\Rightarrow \quad 0 = {}^{54}[1 - v\Theta(v)]^2\ddot{x} - 2\Theta(v)[1 - v\Theta(v)]\dot{x}\dot{v}$$
$$\Leftrightarrow \quad 0 = \ddot{x} - \frac{2\Theta(v)}{[1 - v\Theta(v)]}\dot{x}\dot{v}$$

$\underline{x^a = y:}$

$$\frac{\partial F}{\partial y} = 0$$
$$\frac{\partial F}{\partial \dot{y}} = -2[1 + v\Theta(v)]^2\dot{y}$$
$$\frac{d}{ds}\left(\frac{\partial F}{\partial \dot{y}}\right) = -2\left(\dot{v}\Theta(v) + v\dot{v}\frac{d\Theta(v)}{dv}\right)2[1 + v\Theta(v)]\dot{y} - 2[1 + v\Theta(v)]^2\ddot{y}$$
$$= -4\big(\Theta(v) + v\delta(v)\big)[1 + v\Theta(v)]\dot{y}\dot{v} - 2[1 + v\Theta(v)]^2\ddot{y}$$
$$\Rightarrow \quad 0 = [1 + v\Theta(v)]^2\ddot{y} + 2\Theta(v)[1 + \Theta(v)]\dot{y}\dot{v}$$
$$\Leftrightarrow \quad 0 = \ddot{y} + \frac{2\Theta(v)}{[1 + v\Theta(v)]}\dot{y}\dot{v}$$

Collecting the results

$$0 = \ddot{v}$$
$$0 = \ddot{u} - \Theta(v)[1 - v\Theta(v)]\dot{x}^2 + \Theta(v)[1 + v\Theta(v)]\dot{y}^2$$

---

[53] $\frac{d\Theta(v)}{dv} = \delta(v)$

[54] $v\delta(v) = 0$

$$0 = \ddot{x} - \frac{2\Theta(v)}{[1 - v\Theta(v)]}\dot{x}\dot{v}$$
$$0 = \ddot{y} + \frac{2\Theta(v)}{[1 + v\Theta(v)]}\dot{y}\dot{v}$$

We can now find the Christoffel symbols:

$$\Gamma^u{}_{xx} = -\Theta(v)[1 - v\Theta(v)]$$
$$\Gamma^u{}_{yy} = \Theta(v)[1 + v\Theta(v)]$$
$$\Gamma^x{}_{vx} = -\frac{\Theta(v)}{[1 - v\Theta(v)]}$$
$$\Gamma^y{}_{yv} = \frac{\Theta(v)}{[1 + v\Theta(v)]}$$

14.10.2 The Petrov type

The line element

$$ds^2 = 2dudv - [1 - v\Theta(v)]^2 dx^2 - [1 + v\Theta(v)]^2 dy^2$$

The metric tensor:

$$g_{ab} = \begin{Bmatrix} & 1 & & \\ 1 & & & \\ & & -[1 - v\Theta(v)]^2 & \\ & & & -[1 + v\Theta(v)]^2 \end{Bmatrix}$$

and its inverse:

$$g^{ab} = \begin{Bmatrix} & 1 & & \\ 1 & & & \\ & & -\frac{1}{[1 - v\Theta(v)]^2} & \\ & & & -\frac{1}{[1 + v\Theta(v)]^2} \end{Bmatrix}$$

<u>The basis one forms</u>

Finding the basis one forms is not so obvious, we write:

$$\begin{aligned} ds^2 &= 2dudv - [1 - v\Theta(v)]^2 dx^2 - [1 + v\Theta(v)]^2 dy^2 \\ &= \left(\omega^{\hat{u}}\right)^2 - \left(\omega^{\hat{v}}\right)^2 - \left(\omega^{\hat{x}}\right)^2 - \left(\omega^{\hat{y}}\right)^2 \\ &= \left(\omega^{\hat{u}} + \omega^{\hat{v}}\right)\left(\omega^{\hat{u}} - \omega^{\hat{v}}\right) - \left(\omega^{\hat{x}}\right)^2 - \left(\omega^{\hat{y}}\right)^2 \end{aligned}$$

$$\Rightarrow \quad \sqrt{2}du = \left(\omega^{\hat{u}} + \omega^{\hat{v}}\right)$$
$$\sqrt{2}dv = \left(\omega^{\hat{u}} - \omega^{\hat{v}}\right)$$

$$\omega^{\hat{u}} = \frac{1}{\sqrt{2}}(du + dv) \qquad du = \frac{1}{\sqrt{2}}\left(\omega^{\hat{u}} + \omega^{\hat{v}}\right)$$
$$\omega^{\hat{v}} = \frac{1}{\sqrt{2}}(du - dv) \qquad dv = \frac{1}{\sqrt{2}}\left(\omega^{\hat{u}} - \omega^{\hat{v}}\right)$$
$$\Rightarrow \quad \omega^{\hat{x}} = \left(1 - v\Theta(v)\right)dx \qquad dx = \frac{1}{1 - v\Theta(v)}\omega^{\hat{x}}$$
$$\omega^{\hat{y}} = \left(1 + v\Theta(v)\right)dy \qquad dy = \frac{1}{1 + v\Theta(v)}\omega^{\hat{y}}$$

$$\eta^{ij} = \begin{Bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{Bmatrix}$$

<u>The orthonormal null tetrad</u>

Now we can use the basis one-forms to construct a orthonormal null tetrad

$$\begin{pmatrix} l \\ n \\ m \\ \overline{m} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & 1 & -i \end{pmatrix}\begin{pmatrix} \omega^{\hat{u}} \\ \omega^{\hat{v}} \\ \omega^{\hat{x}} \\ \omega^{\hat{y}} \end{pmatrix}$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix} \omega^{\hat{u}} + \omega^{\hat{v}} \\ \omega^{\hat{u}} - \omega^{\hat{v}} \\ \omega^{\hat{x}} + i\omega^{\hat{y}} \\ \omega^{\hat{x}} - i\omega^{\hat{y}} \end{pmatrix}$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2}du \\ \sqrt{2}dv \\ \big(1 - \nu\Theta(\nu)\big)dx + i\big(1 + \nu\Theta(\nu)\big)dy \\ \big(1 - \nu\Theta(\nu)\big)dx - i\big(1 + \nu\Theta(\nu)\big)dy \end{pmatrix}$$

Written in terms of the coordinate basis

$$l_a = (1, \quad 0, \quad 0, \quad 0)$$
$$n_a = (0, \quad 1, \quad 0, \quad 0)$$
$$m_a = \frac{1}{\sqrt{2}}\Big(0, \quad 0, \quad \big(1 - \nu\Theta(\nu)\big), \quad i\big(1 + \nu\Theta(\nu)\big)\Big)$$
$$\overline{m_a} = \frac{1}{\sqrt{2}}\Big(0, \quad 0, \quad \big(1 - \nu\Theta(\nu)\big), \quad -i\big(1 + \nu\Theta(\nu)\big)\Big)$$

Next we use the metric to rise the indices

$$l^u = g^{au}l_a = g^{vu}l_v = 1 \cdot 0 = 0$$
$$l^v = g^{av}l_a = g^{uv}l_u = 1 \cdot 1 = 1$$
$$l^x = l^y = 0$$
$$n^u = g^{au}n_a = g^{vu}n_v = 1 \cdot 1 = 1$$
$$n^v = g^{av}n_a = g^{uv}n_u = 1 \cdot 0 = 0$$
$$n^x = n^y = 0$$
$$m^v = m^u = 0$$
$$m^x = g^{xx}m_x = -\frac{1}{[1 - \nu\Theta(\nu)]^2} \cdot \frac{1}{\sqrt{2}} \cdot \big(1 - \nu\Theta(\nu)\big) = -\frac{1}{\sqrt{2}}\frac{1}{\big(1 - \nu\Theta(\nu)\big)}$$
$$m^y = g^{yy}m_y = -\frac{1}{[1 + \nu\Theta(\nu)]^2}\frac{1}{\sqrt{2}} \cdot i\big(1 + \nu\Theta(\nu)\big) = -i\frac{1}{\sqrt{2}}\frac{1}{\big(1 + \nu\Theta(\nu)\big)}$$

Collecting the results

$$l_a = (1, \quad 0, \quad 0, \quad 0)$$
$$l^a = (0, \quad 1, \quad 0, \quad 0)$$
$$n_a = (0, \quad 1, \quad 0, \quad 0)$$
$$n^a = (1, \quad 0, \quad 0, \quad 0)$$
$$m_a = \frac{1}{\sqrt{2}}\Big(0, \quad 0, \quad \big(1 - \nu\Theta(\nu)\big), \quad i\big(1 + \nu\Theta(\nu)\big)\Big)$$
$$m^a = \frac{1}{\sqrt{2}}\left(0, \quad 0, \quad -\frac{1}{\big(1 - \nu\Theta(\nu)\big)}, \quad -i\frac{1}{\big(1 + \nu\Theta(\nu)\big)}\right)$$
$$\overline{m_a} = \frac{1}{\sqrt{2}}\Big(0, \quad 0, \quad \big(1 - \nu\Theta(\nu)\big), \quad -i\big(1 + \nu\Theta(\nu)\big)\Big)$$
$$\overline{m}^a = \frac{1}{\sqrt{2}}\left(0, \quad 0, \quad -\frac{1}{\big(1 - \nu\Theta(\nu)\big)}, \quad i\frac{1}{\big(1 + \nu\Theta(\nu)\big)}\right)$$

The spin coefficients calculated from the orthonormal tetrad

$$\pi = -\nabla_b n_a \overline{m}^a l^b \qquad \kappa = \nabla_b l_a m^a l^b \qquad \varepsilon = \frac{1}{2}\big(\nabla_b l_a n^a l^b - \nabla_b m_a \overline{m}^a l^b\big)$$
$$\nu = -\nabla_b n_a \overline{m}^a n^b \qquad \tau = \nabla_b l_a m^a n^b \qquad \gamma = \frac{1}{2}\big(\nabla_b l_a n^a n^b - \nabla_b m_a \overline{m}^a n^b\big)$$

$$\lambda = -\nabla_b n_a \bar{m}^a \bar{m}^b \quad \rho = \nabla_b l_a m^a \bar{m}^b \quad \alpha = \frac{1}{2}\left(\nabla_b l_a n^a \bar{m}^b - \nabla_b m_a \bar{m}^a \bar{m}^b\right)$$

$$\mu = -\nabla_b n_a \bar{m}^a m^b \quad \sigma = \nabla_b l_a m^a m^b \quad \beta = \frac{1}{2}\left(\nabla_b l_a n^a m^b - \nabla_b m_a \bar{m}^a m^b\right)$$

Calculating the spin-coefficients

$$\pi = -\nabla_b n_a \bar{m}^a l^b = -\nabla_v n_a \bar{m}^a l^v = -\nabla_v n_x \bar{m}^x l^v - \nabla_v n_y \bar{m}^y l^v = 0$$

$$\nu = -\nabla_b n_a \bar{m}^a n^b = -\nabla_u n_a \bar{m}^a n^u = -\nabla_u n_x \bar{m}^x n^u - \nabla_u n_y \bar{m}^y n^u = 0$$

$$\begin{aligned}
\lambda &= -\nabla_b n_a \bar{m}^a \bar{m}^b \\
&= -\nabla_x n_a \bar{m}^a \bar{m}^x - \nabla_y n_a \bar{m}^a \bar{m}^y \\
&= -\nabla_x n_x \bar{m}^x \bar{m}^x - \nabla_y n_x \bar{m}^x \bar{m}^y - \nabla_x n_y \bar{m}^y \bar{m}^x - \nabla_y n_y \bar{m}^y \bar{m}^y \\
&= {}^{55}\Gamma^u{}_{xx} n_u \bar{m}^x \bar{m}^x + \Gamma^u{}_{yy} n_u \bar{m}^y \bar{m}^y \\
&= 0
\end{aligned}$$

$$\mu = -\nabla_b n_a \bar{m}^a m^b = \Gamma^u{}_{xx} n_u \bar{m}^x m^x + \Gamma^u{}_{yy} n_u \bar{m}^y m^y = 0$$

$$\kappa = \nabla_b l_a m^a l^b = \nabla_v l_a m^a l^v = \nabla_v l_x m^x l^v + \nabla_v l_y m^y l^v = 0$$

$$\tau = \nabla_b l_a m^a n^b = \nabla_u l_a m^a n^u = \nabla_u l_x m^x n^u + \nabla_u l_y m^u n^v = 0$$

$$\begin{aligned}
\rho &= \nabla_b l_a m^a \bar{m}^b \\
&= \nabla_x l_a m^a \bar{m}^x + \nabla_y l_a m^a \bar{m}^y \\
&= \nabla_x l_x m^x \bar{m}^x + \nabla_y l_x m^x \bar{m}^y + \nabla_x l_y m^y \bar{m}^x + \nabla_y l_y m^y \bar{m}^y \\
&= {}^{56} - \left(\Gamma^u{}_{xx} l_u m^x \bar{m}^x + \Gamma^u{}_{yy} l_u m^y \bar{m}^y\right) \\
&= {}^{57} \frac{\Theta(\nu)}{2}\left(\frac{1}{\left(1-\nu\Theta(\nu)\right)} - \frac{1}{\left(1+\nu\Theta(\nu)\right)}\right) \\
&= {}^{58} \frac{\nu\Theta(\nu)}{\left(1+\nu\Theta(\nu)\right)\left(1-\nu\Theta(\nu)\right)}
\end{aligned}$$

$$\begin{aligned}
\sigma &= \nabla_b l_a m^a m^b \\
&= \frac{\Theta(\nu)}{2}\left(\frac{1}{\left(1-\nu\Theta(\nu)\right)} + \frac{1}{\left(1+\nu\Theta(\nu)\right)}\right) \\
&= \frac{\Theta(\nu)}{\left(1+\nu\Theta(\nu)\right)\left(1-\nu\Theta(\nu)\right)}
\end{aligned}$$

$$\begin{aligned}
\varepsilon &= \frac{1}{2}\left(\nabla_b l_a n^a l^b - \nabla_b m_a \bar{m}^a l^b\right) \\
&= \frac{1}{2}\left(\nabla_v l_a n^a l^v - \nabla_v m_a \bar{m}^a l^v\right) \\
&= \frac{1}{2}\left(\nabla_v l_u n^u l^v - \nabla_v m_x \bar{m}^x l^v - \nabla_v m_y \bar{m}^y l^v\right) \\
&= -\frac{1}{2}\left((\partial_v m_x - \Gamma^c{}_{vx} m_c)\bar{m}^x + \left(\partial_v m_y - \Gamma^c{}_{vy} m_c\right)\bar{m}^y\right) \\
&= -\frac{1}{2}\left(\left(\partial_v\left(1-\nu\Theta(\nu)\right) - \Gamma^x_{vx} m_x\right)\bar{m}^x n^v + \left(\partial_v\left(i\left(1+\nu\Theta(\nu)\right)\right) - \Gamma^y_{vy} m_y\right)\bar{m}^y n^v\right)
\end{aligned}$$

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[55] $= -(\partial_x n_x - \Gamma^c{}_{xx} n_c)\bar{m}^x \bar{m}^x - \left(\partial_y n_x - \Gamma^c{}_{yx} n_c\right)\bar{m}^x \bar{m}^y - \left(\partial_x n_y - \Gamma^c{}_{xy} n_c\right)\bar{m}^y \bar{m}^x - \left(\partial_y n_y - \Gamma^c{}_{yy} n_c\right)\bar{m}^y \bar{m}^y =$

[56] $= (\partial_x l_x - \Gamma^c{}_{xx} l_c) m^x \bar{m}^x + \left(\partial_y l_x - \Gamma^c{}_{yx} l_c\right) m^x \bar{m}^y + \left(\partial_x l_y - \Gamma^c{}_{xy} l_c\right) m^y \bar{m}^x + \left(\partial_y l_y - \Gamma^c{}_{yy} l_c\right) m^y \bar{m}^y =$

[57] $= -\left(-\Theta(\nu)[1-\nu\Theta(\nu)]\frac{1}{2}\left(-\frac{1}{(1-\nu\Theta(\nu))}\right)\left(-\frac{1}{(1-\nu\Theta(\nu))}\right) + \Theta(\nu)[1+\nu\Theta(\nu)]\frac{1}{2}\left(-i\frac{1}{(1+\nu\Theta(\nu))}\right)\left(i\frac{1}{(1+\nu\Theta(\nu))}\right)\right) =$

[58] $\Theta^2(\nu) = \Theta(\nu)$

$$= {}^{59}0$$

$$\gamma = \frac{1}{2}\left(\nabla_b l_a n^a n^b - \nabla_b m_a \bar{m}^a n^b\right)$$
$$= \frac{1}{2}\left(\nabla_u l_a n^a n^u - \nabla_u m_a \bar{m}^a n^u\right)$$
$$= \frac{1}{2}\left(\nabla_u l_u n^u n^u - \nabla_u m_x \bar{m}^x n^u - \nabla_u m_y \bar{m}^y n^u\right)$$
$$= 0$$

$$\alpha = \frac{1}{2}\left(\nabla_b l_a n^a \bar{m}^b - \nabla_b m_a \bar{m}^a \bar{m}^b\right)$$
$$= \frac{1}{2}\left(\nabla_x l_a n^a \bar{m}^x - \nabla_x m_a \bar{m}^a \bar{m}^x\right) + \frac{1}{2}\left(\nabla_y l_a n^a \bar{m}^y - \nabla_y m_a \bar{m}^a \bar{m}^y\right)$$
$$= {}^{60}0$$

$$\beta = \frac{1}{2}\left(\nabla_b l_a n^a m^b - \nabla_b m_a \bar{m}^a m^b\right) = 0$$

Collecting the results:

$$\pi = 0 \quad \kappa = 0 \quad \varepsilon = 0$$
$$\nu = 0 \quad \tau = 0 \quad \gamma = 0$$
$$\lambda = 0 \quad \rho = \frac{\nu\Theta(\nu)}{\left(1+\nu\Theta(\nu)\right)\left(1-\nu\Theta(\nu)\right)} \quad \alpha = 0$$
$$\mu = 0 \quad \sigma = \frac{\Theta(\nu)}{\left(1+\nu\Theta(\nu)\right)\left(1-\nu\Theta(\nu)\right)} \quad \beta = 0$$

The Weyl Scalars and Petrov classification

$$\Psi_0 = D\sigma - \delta\kappa - \sigma(\rho+\bar{\rho}) - \sigma(3\varepsilon - \bar{\varepsilon}) + \kappa(\pi - \bar{\pi} + \bar{\alpha} + 3\beta)$$
$$\Psi_1 = D\beta - \delta\varepsilon - \sigma(\alpha+\pi) - \beta(\bar{\rho} - \bar{\varepsilon}) + \kappa(\mu+\gamma) + \varepsilon(\bar{\alpha} - \bar{\pi})$$
$$\Psi_2 = \bar{\delta}\tau - \Delta\rho - \rho\bar{\mu} - \sigma\lambda + \tau\left(\bar{\beta} - \alpha - \bar{\tau}\right) + \rho(\gamma+\bar{\gamma}) + \kappa\nu - 2\Lambda$$
$$\Psi_3 = \bar{\delta}\gamma - \Delta\alpha + \nu(\rho+\varepsilon) - \lambda(\tau+\beta) + \alpha(\bar{\gamma} - \bar{\mu}) + \gamma\left(\bar{\beta} - \bar{\tau}\right)$$
$$\Psi_4 = \bar{\delta}\nu - \Delta\lambda + \lambda(\mu+\bar{\mu}) - \lambda(3\gamma - \bar{\gamma}) + \nu\left(3\alpha + \bar{\beta} + \pi - \bar{\tau}\right)$$

Where

$$D = l^a\nabla_a$$
$$\Delta = n^a\nabla_a$$
$$\delta = m^a\nabla_a$$
$$\bar{\delta} = \bar{m}^a\nabla_a$$

$$\Psi_0 = D\sigma - \delta\kappa - \sigma(\rho+\bar{\rho}) - \sigma(3\varepsilon - \bar{\varepsilon}) + \kappa(\pi - \bar{\pi} + \bar{\alpha} + 3\beta)$$
$$= D\sigma - 2\sigma\rho$$
$$= {}^{61}l^\nu\,\partial_\nu\left(\frac{\Theta(\nu)}{\left(1-\nu\Theta(\nu)\right)\left(1+\nu\Theta(\nu)\right)}\right) - 2\frac{\nu\Theta(\nu)}{\left(1-\nu^2\Theta(\nu)\right)^2}$$
$$= \partial_\nu\left(\frac{\Theta(\nu)}{1-\nu^2\Theta(\nu)}\right) - 2\frac{\nu\Theta(\nu)}{\left(1-\nu^2\Theta(\nu)\right)^2}$$

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[59] $= -\frac{1}{2}\left(\left(-\Theta(\nu) - \nu\delta(\nu) - \left(-\frac{\Theta(\nu)}{[1-\nu\Theta(\nu)]}\right)\left(1-\nu\Theta(\nu)\right)\right)\bar{m}^x n^\nu + \left(i\left(\Theta(\nu)+\nu\delta(\nu)\right) - \frac{\Theta(\nu)}{[1+\nu\Theta(\nu)]}i\left(1+\nu\Theta(\nu)\right)\right)\bar{m}^y n^\nu\right) =$

[60] $= \frac{1}{2}\left(\nabla_x l_u n^u \bar{m}^x - \nabla_x m_x \bar{m}^x \bar{m}^x - \nabla_x m_y \bar{m}^y \bar{m}^x + \nabla_y l_u n^u \bar{m}^y - \nabla_y m_x \bar{m}^x \bar{m}^y - \nabla_y m_y \bar{m}^y \bar{m}^y\right) =$

[61] $= l^a\nabla_a\left(\frac{\Theta(\nu)}{(1+\nu\Theta(\nu))(1-\nu\Theta(\nu))}\right) - 2\left(\frac{\Theta(\nu)}{(1+\nu\Theta(\nu))(1-\nu\Theta(\nu))}\right)\frac{\nu\Theta(\nu)}{(1+\nu\Theta(\nu))(1-\nu\Theta(\nu))} =$

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$$= \frac{\delta(v)\left(1 - v^2\Theta(v)\right) - \Theta(v)\left(-2v\Theta(v) - v^2\delta(v)\right)}{\left(1 - v^2\Theta(v)\right)^2} - 2\frac{v\Theta(v)}{\left(1 - v^2\Theta(v)\right)^2}$$

$$= \delta(v)$$

$$\Psi_1 = 0$$

$$\Psi_2 = \bar{\delta}\tau - \Delta\rho - \rho\bar{\mu} - \sigma\lambda + \tau\left(\bar{\beta} - \alpha - \bar{\tau}\right) + \rho(\gamma + \bar{\gamma}) + \kappa\nu - 2\Lambda$$

$$= -\Delta\rho$$

$$= -n^a\nabla_a\rho$$

$$= -n^u\nabla_u\rho$$

$$= 0$$

$$\Psi_3 = 0$$

$$\Psi_4 = 0$$

[t]$\Psi_0 \neq 0$: This is a Petrov type N, which means there is a single principal null direction of multiplicity 4. This corresponds to transverse gravity waves in region III.

14.11 [u]Two interacting waves

The line element:

$$ds^2 = 2dudv - \cos^2 av\, dx^2 - \cosh^2 av\, dy^2$$

<u>Notice:</u> This space-time is a Rosen space-time with

$$f(v) = \cos^2 av$$

$$g(v) = \cosh^2 av$$

We have

$$f'(v) = -2a\cos(av)\sin(av)$$

$$g'(v) = 2a\cosh(av)\sinh(av)$$

$$\Rightarrow \quad f''(v) = 2a^2\sin^2(av) - 2a^2\cos^2(av)$$

$$g''(v) = 2a^2\sinh^2(av) + 2a^2\cosh^2(av) = -2a^2\sin^2(iav) + 2a^2\cos^2(iav)$$

Because

$$f''(v) \neq -h''(v)$$

this is not a vacuum solution. If we change the line-element

$$ds^2 = 2dudv - \cos^2 av\, dx^2 - \cosh^2 bv\, dy^2$$

this is a vacuum solution if

$$a = ib$$

14.11.1 The Christoffel symbols

To find the Christoffel symbols we calculate the geodesic from the Euler-Lagrange equation

$$0 = \frac{d}{ds}\left(\frac{\partial F}{\partial \dot{x}^a}\right) - \frac{\partial F}{\partial x^a}$$

Where

$$F = 2\dot{u}\dot{v} - \cos^2 av\, \dot{x}^2 - \cosh^2 av\, \dot{y}^2$$

<u>$x^a = v$:</u>

$$\frac{\partial F}{\partial v} = 2a\cos av\sin av\, \dot{x}^2 - 2a\cosh av\sinh av\, \dot{y}^2$$

$$\frac{\partial F}{\partial \dot{v}} = 2\dot{u}$$

$$\frac{d}{ds}\left(\frac{\partial F}{\partial \dot{v}}\right) = 2\ddot{u}$$

$$\Rightarrow \quad 0 = \ddot{u} - a\cos av\sin av\, \dot{x}^2 + a\cosh av\sinh av\, \dot{y}^2$$

<u>$x^a = u$:</u>

$$\frac{\partial F}{\partial u} = 0$$

$$\frac{\partial F}{\partial \dot{u}} = 2\dot{v}$$

$$\frac{d}{ds}\left(\frac{\partial F}{\partial \dot{u}}\right) = 2\ddot{v}$$

$$\Rightarrow \quad 0 = \ddot{v}$$

$\underline{x^a = x:}$

$$\frac{\partial F}{\partial x} = 0$$

$$\frac{\partial F}{\partial \dot{x}} = -2\cos^2 av\,\dot{x}$$

$$\frac{d}{ds}\left(\frac{\partial F}{\partial \dot{x}}\right) = 4a\cos av \sin av\,\dot{v}\dot{x} - 2\cos^2 av\,\ddot{x}$$

$$\Rightarrow \quad 0 = 2a\cos av \sin av\,\dot{v}\dot{x} - \cos^2 av\,\ddot{x}$$

$$\Leftrightarrow \quad 0 = \ddot{x} - 2a\tan av\,\dot{v}\dot{x}$$

$\underline{x^a = y:}$

$$\frac{\partial F}{\partial y} = 0$$

$$\frac{\partial F}{\partial \dot{y}} = -2\cosh^2 av\,\dot{y}$$

$$\frac{d}{ds}\left(\frac{\partial F}{\partial \dot{y}}\right) = -4a\cosh av \sinh av\,\dot{v}\dot{y} - 2\cos^2 av\,\ddot{y}$$

$$\Rightarrow \quad 0 = -2a\cosh av \sinh av\,\dot{v}\dot{y} - \cosh^2 av\,\ddot{y}$$

$$\Leftrightarrow \quad 0 = \ddot{y} + 2a\tanh av\,\dot{v}\dot{y}$$

<u>Collecting the results</u>

$$0 = \ddot{v}$$

$$0 = \ddot{u} - a\cos av \sin av\,\dot{x}^2 + a\cosh av \sinh av\,\dot{y}^2$$

$$0 = \ddot{x} - 2a\tan av\,\dot{v}\dot{x}$$

$$0 = \ddot{y} + 2a\tanh av\,\dot{v}\dot{y}$$

We can now find the Christoffel symbols:

$$\Gamma^u{}_{xx} = -a\cos av \sin av$$

$$\Gamma^u{}_{yy} = a\cosh av \sinh av$$

$$\Gamma^x{}_{xv} = -a\tan av$$

$$\Gamma^x{}_{vx} = -a\tan av$$

$$\Gamma^y{}_{yv} = a\tanh av$$

$$\Gamma^y{}_{vy} = a\tanh av$$

14.11.2 The Petrov type

The line element

$$ds^2 = 2dudv - \cos^2 av\,dx^2 - \cosh^2 av\,dy^2$$

The metric tensor:

$$g_{ab} = \begin{Bmatrix} & 1 & & \\ 1 & & & \\ & & -\cos^2 av & \\ & & & -\cosh^2 av \end{Bmatrix}$$

and its inverse:

$$g^{ab} = \begin{Bmatrix} & 1 & & \\ 1 & & & \\ & & -\frac{1}{\cos^2 av} & \\ & & & -\frac{1}{\cosh^2 av} \end{Bmatrix}$$

<u>The basis one forms</u>
Finding the basis one forms is not so obvious, we write:

$$\begin{aligned} ds^2 &= 2dudv - \cos^2 av\, dx^2 - \cosh^2 av\, dy^2 \\ &= \left(\omega^{\hat{u}}\right)^2 - \left(\omega^{\hat{v}}\right)^2 - \left(\omega^{\hat{x}}\right)^2 - \left(\omega^{\hat{y}}\right)^2 \\ &= \left(\omega^{\hat{u}} + \omega^{\hat{v}}\right)\left(\omega^{\hat{u}} - \omega^{\hat{v}}\right) - \left(\omega^{\hat{x}}\right)^2 - \left(\omega^{\hat{y}}\right)^2 \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad \sqrt{2}du &= \left(\omega^{\hat{u}} + \omega^{\hat{v}}\right) \\ \sqrt{2}dv &= \left(\omega^{\hat{u}} - \omega^{\hat{v}}\right) \\ \omega^{\hat{x}} &= \cos av\, dx \\ \omega^{\hat{y}} &= \cosh av\, dy \end{aligned}$$

$$\begin{aligned} \omega^{\hat{u}} &= \frac{1}{\sqrt{2}}(du + dv) & du &= \frac{1}{\sqrt{2}}\left(\omega^{\hat{u}} + \omega^{\hat{v}}\right) \\ \omega^{\hat{v}} &= \frac{1}{\sqrt{2}}(du - dv) & dv &= \frac{1}{\sqrt{2}}\left(\omega^{\hat{u}} - \omega^{\hat{v}}\right) \\ \omega^{\hat{x}} &= \cos av\, dx & dx &= \frac{1}{\cos av}\omega^{\hat{x}} \\ \omega^{\hat{y}} &= \cosh av\, dy & dy &= \frac{1}{\cosh av}\omega^{\hat{y}} \end{aligned}$$

$$\eta^{ij} = \begin{Bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{Bmatrix}$$

14.11.3 The orthonormal null tetrad

Now we can use the basis one-forms to construct a orthonormal null tetrad

$$\begin{pmatrix} l \\ n \\ m \\ \overline{m} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & 1 & -i \end{pmatrix}\begin{pmatrix} \omega^{\hat{u}} \\ \omega^{\hat{v}} \\ \omega^{\hat{x}} \\ \omega^{\hat{y}} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \omega^{\hat{u}} + \omega^{\hat{v}} \\ \omega^{\hat{u}} - \omega^{\hat{v}} \\ \omega^{\hat{x}} + i\omega^{\hat{y}} \\ \omega^{\hat{x}} - i\omega^{\hat{y}} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2}du \\ \sqrt{2}dv \\ \cos av\, dx + i\cosh av\, dy \\ \cos av\, dx - i\cosh av\, dy \end{pmatrix}$$

Written in terms of the coordinate basis

$$\begin{aligned} l_a &= (1, \quad 0, \quad 0, \quad 0) \\ n_a &= (0, \quad 1, \quad 0, \quad 0) \\ m_a &= \frac{1}{\sqrt{2}}(0, \quad 0, \quad \cos av\,, \quad i\cosh av) \\ \overline{m_a} &= \frac{1}{\sqrt{2}}(0, \quad 0, \quad \cos av\,, \quad -i\cosh av) \end{aligned}$$

Next we use the metric to rise the indices

$$\begin{aligned} l^u &= g^{au}l_a = g^{vu}l_v = 1\cdot 0 = 0 \\ l^v &= g^{av}l_a = g^{uv}l_u = 1\cdot 1 = 1 \\ l^x &= l^y = 0 \\ n^u &= g^{au}n_a = g^{vu}n_v = 1\cdot 1 = 1 \\ n^v &= g^{av}n_a = g^{uv}n_u = 1\cdot 0 = 0 \\ n^x &= n^y = 0 \end{aligned}$$

$$m^v = m^u = 0$$
$$m^x = g^{xx}m_x = -\frac{1}{\cos^2 av}\cdot\frac{1}{\sqrt{2}}\cdot\cos av = -\frac{1}{\sqrt{2}}\frac{1}{\cos av}$$
$$m^y = g^{yy}m_y = -\frac{1}{\cosh^2 av}\frac{1}{\sqrt{2}}\cdot i\cosh av = -i\frac{1}{\sqrt{2}}\frac{1}{\cosh av}$$

<u>Collecting the results</u>

$$l_a = (1, \quad 0, \quad 0, \quad 0)$$
$$l^a = (0, \quad 1, \quad 0, \quad 0)$$
$$n_a = (0, \quad 1, \quad 0, \quad 0)$$
$$n^a = (1, \quad 0, \quad 0, \quad 0)$$
$$m_a = \frac{1}{\sqrt{2}}(0, \quad 0, \quad \cos av, \quad i\cosh av)$$
$$m^a = \frac{1}{\sqrt{2}}\left(0, \quad 0, \quad -\frac{1}{\cos av}, \quad -i\frac{1}{\cosh av}\right)$$
$$\overline{m_a} = \frac{1}{\sqrt{2}}(0, \quad 0, \quad \cos av, \quad -i\cosh av)$$
$$\overline{m}^a = \frac{1}{\sqrt{2}}\left(0, \quad 0, \quad -\frac{1}{\cos av}, \quad i\frac{1}{\cosh av}\right)$$

Calculating the spin-coefficients

$$\pi = -\nabla_b n_a \overline{m}^a l^b = -\nabla_v n_a \overline{m}^a l^v = -\nabla_v n_x \overline{m}^x l^v - \nabla_v n_y \overline{m}^y l^v = 0$$
$$\nu = -\nabla_b n_a \overline{m}^a n^b = -\nabla_u n_a \overline{m}^a n^u = -\nabla_u n_x \overline{m}^x n^u - \nabla_u n_y \overline{m}^y n^u = 0$$
$$\begin{aligned}\lambda &= -\nabla_b n_a \overline{m}^a \overline{m}^b \\ &= -\nabla_x n_a \overline{m}^a \overline{m}^x - \nabla_y n_a \overline{m}^a \overline{m}^y \\ &= -\nabla_x n_x \overline{m}^x \overline{m}^x - \nabla_y n_x \overline{m}^x \overline{m}^y - \nabla_x n_y \overline{m}^y \overline{m}^x - \nabla_y n_y \overline{m}^y \overline{m}^y \\ &= 0\end{aligned}$$
$$\mu = -\nabla_b n_a \overline{m}^a m^b = 0$$
$$\kappa = \nabla_b l_a m^a l^b = \nabla_v l_a m^a l^v = \nabla_v l_x m^x l^v + \nabla_v l_y m^y l^v = 0$$
$$\tau = \nabla_b l_a m^a n^b = \nabla_u l_a m^a n^u = \nabla_u l_x m^x n^u + \nabla_u l_y m^y n^u = 0$$
$$\begin{aligned}\rho &= \nabla_b l_a m^a \overline{m}^b \\ &= \nabla_x l_a m^a \overline{m}^x + \nabla_y l_a m^a \overline{m}^y \\ &= \nabla_x l_x m^x \overline{m}^x + \nabla_y l_x m^x \overline{m}^y + \nabla_x l_y m^y \overline{m}^x + \nabla_y l_y m^y \overline{m}^y \\ &=^{62} -\left(\Gamma^u{}_{xx} l_u m^x \overline{m}^x + \Gamma^u{}_{yy} l_u m^y \overline{m}^y\right) \\ &=^{63} \frac{a}{2}(\tan av - \tanh av)\end{aligned}$$
$$\sigma = \nabla_b l_a m^a m^b = -\left(\Gamma^u{}_{xx} l_u m^x m^x + \Gamma^u{}_{yy} l_u m^y m^y\right) = \frac{a}{2}(\tan av + \tanh av)$$
$$\begin{aligned}\varepsilon &= \frac{1}{2}\left(\nabla_b l_a n^a l^b - \nabla_b m_a \overline{m}^a l^b\right) \\ &= \frac{1}{2}\left(\nabla_v l_a n^a l^v - \nabla_v m_a \overline{m}^a l^v\right) \\ &= \frac{1}{2}\left(\nabla_v l_u n^u l^v - \nabla_v m_x \overline{m}^x l^v - \nabla_v m_y \overline{m}^y l^v\right) \\ &= -\frac{1}{2}\left((\partial_v m_x - \Gamma^c{}_{vx} m_c)\overline{m}^x l^v + (\partial_v m_y - \Gamma^c{}_{vy} m_c)\overline{m}^y l^v\right)\end{aligned}$$

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[62] $= (\partial_x l_x - \Gamma^c{}_{xx} l_c) m^x \overline{m}^x + (\partial_y l_x - \Gamma^c{}_{yx} l_c) m^x \overline{m}^y + (\partial_x l_y - \Gamma^c{}_{xy} l_c) m^y \overline{m}^x + (\partial_y l_y - \Gamma^c{}_{yy} l_c) m^y \overline{m}^y =$

[63] $= -\left(-a\cos av \sin av\left(-\frac{1}{\sqrt{2}\cos av}\right)^2 + a\cosh av \sinh av\left(-i\frac{1}{\sqrt{2}\cosh av}\right)\left(i\frac{1}{\sqrt{2}\cosh av}\right)\right) =$

$$= -\frac{1}{2}\left(\left(\partial_\nu \frac{1}{\sqrt{2}}\cos av - \Gamma^x_{\nu x} m_x\right)\bar{m}^x l^\nu + \left(\partial_\nu i\frac{1}{\sqrt{2}}\cosh av - \Gamma^y_{\nu y} m_y\right)\bar{m}^y l^\nu\right)$$

$$= {}^{64}0$$

$$\gamma = \frac{1}{2}(\nabla_b l_a n^a n^b - \nabla_b m_a \bar{m}^a n^b)$$

$$= \frac{1}{2}(\nabla_u l_a n^a n^u - \nabla_u m_a \bar{m}^a n^u)$$

$$= \frac{1}{2}(\nabla_u l_u n^u n^u - \nabla_u m_x \bar{m}^x n^u - \nabla_u m_y \bar{m}^y n^u)$$

$$= 0$$

$$\alpha = \frac{1}{2}(\nabla_b l_a n^a \bar{m}^b - \nabla_b m_a \bar{m}^a \bar{m}^b)$$

$$= \frac{1}{2}(\nabla_x l_a n^a \bar{m}^x - \nabla_x m_a \bar{m}^a \bar{m}^x) + \frac{1}{2}(\nabla_y l_a n^a \bar{m}^y - \nabla_y m_a \bar{m}^a \bar{m}^y)$$

$$= 0$$

$$\beta = \frac{1}{2}(\nabla_b l_a n^a m^b - \nabla_b m_a \bar{m}^a m^b) = 0$$

Collecting the results

$$\begin{array}{llllll}
\pi & = 0 & \kappa & = 0 & \varepsilon & = 0 \\
\nu & = 0 & \tau & = 0 & \gamma & = 0 \\
\lambda & = 0 & \rho & = \frac{a}{2}(\tan av - \tanh av) & \alpha & = 0 \\
\mu & = 0 & \sigma & = \frac{a}{2}(\tan av + \tanh av) & \beta & = 0
\end{array}$$

14.11.4 The Weyl Scalars and Petrov classification

$$\Psi_0 = D\sigma - \delta\kappa - \sigma(\rho + \bar{\rho}) - \sigma(3\varepsilon - \bar{\varepsilon}) + \kappa(\pi - \bar{\pi} + \bar{\alpha} + 3\beta)$$

$$\Psi_1 = D\beta - \delta\varepsilon - \sigma(\alpha + \pi) - \beta(\bar{\rho} - \bar{\varepsilon}) + \kappa(\mu + \gamma) + \varepsilon(\bar{\alpha} - \bar{\pi})$$

$$\Psi_2 = \bar{\delta}\tau - \Delta\rho - \rho\bar{\mu} - \sigma\lambda + \tau(\bar{\beta} - \alpha - \bar{\tau}) + \rho(\gamma + \bar{\gamma}) + \kappa\nu - 2\Lambda$$

$$\Psi_3 = \bar{\delta}\gamma - \Delta\alpha + \nu(\rho + \varepsilon) - \lambda(\tau + \beta) + \alpha(\bar{\gamma} - \bar{\mu}) + \gamma(\bar{\beta} - \bar{\tau})$$

$$\Psi_4 = \bar{\delta}\nu - \Delta\lambda + \lambda(\mu + \bar{\mu}) - \lambda(3\gamma - \bar{\gamma}) + \nu(3\alpha + \bar{\beta} + \pi - \bar{\tau})$$

where

$$D = l^a \nabla_a$$

$$\Delta = n^a \nabla_a$$

$$\delta = m^a \nabla_a$$

$$\bar{\delta} = \bar{m}^a \nabla_a$$

$$\Psi_0 = D\sigma - \sigma(\rho + \bar{\rho})$$

$$= D\sigma - 2\sigma\rho$$

$$= l^a \nabla_a \sigma - 2\sigma\rho$$

$$= l^\nu \partial_\nu \sigma - 2\sigma\rho$$

$$= l^\nu \partial_\nu \left(\frac{a}{2}(\tan av + \tanh av)\right) - 2\left(\frac{a}{2}(\tan av + \tanh av)\right)\left(\frac{a}{2}(\tan av - \tanh av)\right)$$

$$= \left(\frac{a^2}{2}(1 + \tan^2 av + 1 - \tanh^2 av)\right) - \left(\frac{a^2}{2}(\tan^2 av - \tanh^2 av)\right)$$

$$= a^2$$

$$\Psi_1 = 0$$

$$\Psi_2 = -\Delta\rho = -n^a \nabla_a \rho = -n^u \partial_u \rho = 0$$

$$\Psi_3 = 0$$

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[64] $= -\frac{1}{2}\left(\left(-a\frac{1}{\sqrt{2}}\sin av - a\tan av\right)\frac{1}{\sqrt{2}}\cos av\right)\bar{m}^x + \left(ia\frac{1}{\sqrt{2}}\sinh av - a\tanh av\, i\frac{1}{\sqrt{2}}\cosh av\right)\bar{m}^y =$

$$\Psi_4 = 0$$

$\Psi_0 \neq 0$: This is a Petrov type N, which means there is a single principal null direction (n^a) of multiplicity 4.

14.12 [v]Collision of a gravitational wave with an electromagnetic wave

The line element in region $v \geq 0$:

$$ds^2 = 2dudv - \cos^2 av\,(dx^2 + dy^2)$$

The metric tensor:

$$g_{ab} = \begin{Bmatrix} & 1 & & \\ 1 & & & \\ & & -\cos^2 av & \\ & & & -\cos^2 av \end{Bmatrix}$$

and its inverse:

$$g^{ab} = \begin{Bmatrix} & 1 & & \\ 1 & & & \\ & & -\frac{1}{\cos^2 av} & \\ & & & -\frac{1}{\cos^2 av} \end{Bmatrix}$$

Notice: This space-time is a Rosen space-time with

$$f(v) = g(v) = \cos^2 av$$

Because

$$f''(v) \neq -g''(v)$$

this is not a vacuum solution.

14.12.1 The Christoffel symbols

To find the Christoffel symbols we calculate the geodesic from the Euler-Lagrange equation

$$0 = \frac{d}{ds}\left(\frac{\partial F}{\partial \dot{x}^a}\right) - \frac{\partial F}{\partial x^a}$$

Where

$$F = 2\dot{u}\dot{v} - \cos^2 av\,(\dot{x}^2 + \dot{y}^2)$$

$x^a = u$:

$$\frac{\partial F}{\partial u} = 0$$

$$\frac{\partial F}{\partial \dot{u}} = 2\dot{v}$$

$$\frac{d}{ds}\left(\frac{\partial F}{\partial \dot{u}}\right) = 2\ddot{v}$$

$$\Rightarrow \quad 0 = 2\ddot{v}$$

$x^a = v$:

$$\frac{\partial F}{\partial v} = 2a\cos av \sin av\,(\dot{x}^2 + \dot{y}^2)$$

$$\frac{\partial F}{\partial \dot{v}} = 2\dot{u}$$

$$\frac{d}{ds}\left(\frac{\partial F}{\partial \dot{v}}\right) = 2\ddot{u}$$

$$\Rightarrow \quad 0 = \ddot{u} - a\cos av \sin av\,(\dot{x}^2 + \dot{y}^2)$$

$x^a = x$:

$$\frac{\partial F}{\partial x} = 0$$

$$\frac{\partial F}{\partial \dot{x}} = -2\cos^2 av\,\dot{x}$$

$$\frac{d}{ds}\left(\frac{\partial F}{\partial \dot{x}}\right) = 4a\cos av\sin av\,\dot{v}\dot{x} - 2\cos^2 av\,\ddot{x}$$
$$\Rightarrow \quad 0 = 2a\cos av\sin av\,\dot{v}\dot{x} - \cos^2 av\,\ddot{x}$$
$$\Leftrightarrow \quad 0 = \ddot{x} - 2a\tan av\,\dot{v}\dot{x}$$

$\underline{x^a = y:}$

$$\frac{\partial F}{\partial y} = 0$$
$$\frac{\partial F}{\partial \dot{y}} = -2\cos^2 av\,\dot{y}$$
$$\frac{d}{ds}\left(\frac{\partial F}{\partial \dot{y}}\right) = 4a\cos av\sin av\,\dot{v}\dot{y} - 2\cos^2 av\,\ddot{y}$$
$$\Rightarrow \quad 0 = 2a\cos av\sin av\,\dot{v}\dot{y} - \cos^2 av\,\ddot{y}$$
$$\Leftrightarrow \quad 0 = \ddot{y} - 2a\tan av\,\dot{v}\dot{y}$$

Collecting the results

$$0 = \ddot{v}$$
$$0 = \ddot{u} - a\cos av\sin av\,(\dot{x}^2 + \dot{y}^2)$$
$$0 = \ddot{x} - 2a\tan av\,\dot{v}\dot{x}$$
$$0 = \ddot{y} - 2a\tan av\,\dot{v}\dot{y}$$

We can now find the Christoffel symbols:

$$\Gamma^u{}_{xx} = -a\cos av\sin av \qquad \Gamma^u{}_{yy} = -a\cos av\sin av$$
$$\Gamma^x{}_{xv} = -a\tan av \qquad \Gamma^x{}_{vx} = -a\tan av$$
$$\Gamma^y{}_{yv} = -a\tan av \qquad \Gamma^y{}_{vy} = -a\tan av$$

14.12.2 The basis one forms:

$$\begin{aligned} ds^2 &= 2dudv - \cos^2 av\,(dx^2 + dy^2) \\ &= \left(\omega^{\hat{u}}\right)^2 - \left(\omega^{\hat{v}}\right)^2 - \left(\omega^{\hat{x}}\right)^2 - \left(\omega^{\hat{y}}\right)^2 \\ &= \left(\omega^{\hat{u}} + \omega^{\hat{v}}\right)\left(\omega^{\hat{u}} - \omega^{\hat{v}}\right) - \left(\omega^{\hat{x}}\right)^2 - \left(\omega^{\hat{y}}\right)^2 \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} \sqrt{2}du &= \left(\omega^{\hat{u}} + \omega^{\hat{v}}\right) \\ \sqrt{2}dv &= \left(\omega^{\hat{u}} - \omega^{\hat{v}}\right) \\ \omega^{\hat{x}} &= \cos av\,dx \\ \omega^{\hat{y}} &= \cos av\,dy \end{aligned}$$

$$\omega^{\hat{u}} = \frac{1}{\sqrt{2}}(du + dv) \qquad du = \frac{1}{\sqrt{2}}\left(\omega^{\hat{u}} + \omega^{\hat{v}}\right)$$
$$\omega^{\hat{v}} = \frac{1}{\sqrt{2}}(du - dv) \qquad dv = \frac{1}{\sqrt{2}}\left(\omega^{\hat{u}} - \omega^{\hat{v}}\right)$$
$$\omega^{\hat{x}} = \cos av\,dx \qquad dx = \frac{1}{\cos av}\omega^{\hat{x}}$$
$$\omega^{\hat{y}} = \cos av\,dy \qquad dy = \frac{1}{\cos av}\omega^{\hat{y}}$$

$$\eta^{ij} = \begin{Bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{Bmatrix}$$

14.12.3 The orthonormal null tetrad:

Now we can use the basis one-forms to construct a orthonormal null tetrad

$$\begin{pmatrix} l \\ n \\ m \\ \bar{m} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & 1 & -i \end{pmatrix}\begin{pmatrix} \omega^{\hat{u}} \\ \omega^{\hat{v}} \\ \omega^{\hat{x}} \\ \omega^{\hat{y}} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \omega^{\hat{u}} + \omega^{\hat{v}} \\ \omega^{\hat{u}} - \omega^{\hat{v}} \\ \omega^{\hat{x}} + i\omega^{\hat{y}} \\ \omega^{\hat{x}} - i\omega^{\hat{y}} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2}du \\ \sqrt{2}dv \\ \cos av\,dx + i\cos av\,dy \\ \cos av\,dx - i\cos av\,dy \end{pmatrix}$$

Written in terms of the coordinate basis

$$\begin{aligned} l_a &= (1, \quad 0, \quad 0, \quad 0) \\ n_a &= (0, \quad 1, \quad 0, \quad 0) \\ m_a &= \frac{1}{\sqrt{2}}(0, \quad 0, \quad \cos av\,, \quad i\cos av) \\ \overline{m_a} &= \frac{1}{\sqrt{2}}(0, \quad 0, \quad \cos av\,, \quad -i\cos av) \end{aligned}$$

Next we use the metric to rise the indices

$$\begin{aligned} l^u &= g^{au}l_a = g^{vu}l_v = 1\cdot 0 = 0 \\ l^v &= g^{av}l_a = g^{uv}l_u = 1\cdot 1 = 1 \\ l^x &= l^y = 0 \\ n^u &= g^{au}n_a = g^{vu}n_v = 1\cdot 1 = 1 \\ n^v &= g^{av}n_a = g^{uv}n_u = 1\cdot 0 = 0 \\ n^x &= n^y = 0 \\ m^v &= m^u = 0 \\ m^x &= g^{xx}m_x = -\frac{1}{\cos^2 av}\cdot\frac{1}{\sqrt{2}}\cdot\cos av = -\frac{1}{\sqrt{2}}\frac{1}{\cos av} \\ m^y &= g^{yy}m_y = -\frac{1}{\cos^2 av}\frac{1}{\sqrt{2}}\cdot i\cos av = -i\frac{1}{\sqrt{2}}\frac{1}{\cos av} \end{aligned}$$

Collecting the results

$$\begin{aligned} l_a &= (1, \quad 0, \quad 0, \quad 0) & l^a &= (0, \quad 1, \quad 0, \quad 0) \\ n_a &= (0, \quad 1, \quad 0, \quad 0) & n^a &= (1, \quad 0, \quad 0, \quad 0) \\ m_a &= \frac{1}{\sqrt{2}}(0, \quad 0, \quad \cos av\,, \quad i\cos av) & m^a &= \frac{1}{\sqrt{2}}\left(0, \quad 0, \quad -\frac{1}{\cos av}, \quad -i\frac{1}{\cos av}\right) \\ \overline{m_a} &= \frac{1}{\sqrt{2}}(0, \quad 0, \quad \cos av\,, \quad -i\cos av) & \overline{m}^a &= \frac{1}{\sqrt{2}}\left(0, \quad 0, \quad -\frac{1}{\cos av}, \quad i\frac{1}{\cos av}\right) \end{aligned}$$

14.12.4 The spin coefficients calculated from the orthonormal tetrad

$$\begin{aligned} \pi &= -\nabla_b n_a \overline{m}^a l^b = -\nabla_v n_a \overline{m}^a l^v = -\nabla_v n_x \overline{m}^x l^v - \nabla_v n_y \overline{m}^y l^v = 0 \\ \nu &= -\nabla_b n_a \overline{m}^a n^b = -\nabla_u n_a \overline{m}^a n^u = -\nabla_u n_x \overline{m}^x n^u - \nabla_u n_y \overline{m}^y n^u = 0 \\ \lambda &= -\nabla_b n_a \overline{m}^a \overline{m}^b \\ &= -\nabla_x n_a \overline{m}^a \overline{m}^x - \nabla_y n_a \overline{m}^a \overline{m}^y \\ &= -\nabla_x n_x \overline{m}^x \overline{m}^x - \nabla_y n_x \overline{m}^x \overline{m}^y - \nabla_x n_y \overline{m}^y \overline{m}^x - \nabla_y n_y \overline{m}^y \overline{m}^y \\ &= 0 \\ \mu &= -\nabla_b n_a \overline{m}^a m^b = 0 \\ \kappa &= \nabla_b l_a m^a l^b = \nabla_v l_a m^a l^v = \nabla_v l_x m^x l^v + \nabla_v l_y m^y l^v = 0 \\ \tau &= \nabla_b l_a m^a n^b = \nabla_u l_a m^a n^u = \nabla_u l_x m^x n^u + \nabla_u l_y m^y n^u = 0 \\ \rho &= \nabla_b l_a m^a \overline{m}^b \\ &= \nabla_x l_a m^a \overline{m}^x + \nabla_y l_a m^a \overline{m}^y \\ &= \nabla_x l_x m^x \overline{m}^x + \nabla_y l_x m^x \overline{m}^y + \nabla_x l_y m^y \overline{m}^x + \nabla_y l_y m^y \overline{m}^y \\ &=^{65} -\left(\Gamma^u{}_{xx} l_u m^x \overline{m}^x + \Gamma^u{}_{yy} l_u m^y \overline{m}^y\right) \\ &= -\left(-a\cos av\sin av\left(-\frac{1}{\sqrt{2}\cos av}\right)^2 - a\cos av\sin av\left(-i\frac{1}{\sqrt{2}\cos av}\right)\left(i\frac{1}{\sqrt{2}\cos av}\right)\right) \\ &= a\tan av \\ \sigma &= \nabla_b l_a m^a m^b \\ &= -\left(\Gamma^u{}_{xx} l_u m^x m^x + \Gamma^u{}_{yy} l_u m^y m^y\right) \end{aligned}$$

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[65] $= (\partial_x l_x - \Gamma^c{}_{xx} l_c) m^x \overline{m}^x + (\partial_y l_x - \Gamma^c{}_{yx} l_c) m^x \overline{m}^y + (\partial_x l_y - \Gamma^c{}_{xy} l_c) m^y \overline{m}^x + (\partial_y l_y - \Gamma^c{}_{yy} l_c) m^y \overline{m}^y =$

$$= -\left(-a\cos av\sin av\left(-\frac{1}{\sqrt{2}\cos av}\right)^2 - a\cos av\sin av\left(-i\frac{1}{\sqrt{2}\cos av}\right)^2\right)$$
$$= 0$$
$$\varepsilon = \frac{1}{2}\left(\nabla_b l_a n^a l^b - \nabla_b m_a \overline{m}^a l^b\right)$$
$$= \frac{1}{2}\left(\nabla_v l_a n^a l^v - \nabla_v m_a \overline{m}^a l^v\right)$$
$$= \frac{1}{2}\left(\nabla_v l_u n^u l^v - \nabla_v m_x \overline{m}^x l^v - \nabla_v m_y \overline{m}^y l^v\right)$$
$$= -\frac{1}{2}\left((\partial_v m_x - \Gamma^c_{vx} m_c)\overline{m}^x l^v + (\partial_v m_y - \Gamma^c_{vy} m_c)\overline{m}^y l^v\right)$$
$$= -\frac{1}{2}\left(\left(\partial_v \frac{1}{\sqrt{2}}\cos av - \Gamma^x_{vx} m_x\right)\overline{m}^x l^v + \left(\partial_v i\frac{1}{\sqrt{2}}\cosh av - \Gamma^y_{vy} m_y\right)\overline{m}^y l^v\right)$$
= [66] 0
$$\gamma = \frac{1}{2}\left(\nabla_b l_a n^a n^b - \nabla_b m_a \overline{m}^a n^b\right)$$
$$= \frac{1}{2}\left(\nabla_u l_a n^a n^u - \nabla_u m_a \overline{m}^a n^u\right)$$
$$= \frac{1}{2}\left(\nabla_u l_u n^u n^u - \nabla_u m_x \overline{m}^x n^u - \nabla_u m_y \overline{m}^y n^u\right)$$
$$= 0$$
$$\alpha = \frac{1}{2}\left(\nabla_b l_a n^a \overline{m}^b - \nabla_b m_a \overline{m}^a \overline{m}^b\right)$$
$$= \frac{1}{2}\left(\nabla_x l_a n^a \overline{m}^x - \nabla_x m_a \overline{m}^a \overline{m}^x\right) + \frac{1}{2}\left(\nabla_y l_a n^a \overline{m}^y - \nabla_y m_a \overline{m}^a \overline{m}^y\right)$$
$$= 0$$
$$\beta = \frac{1}{2}\left(\nabla_b l_a n^a m^b - \nabla_b m_a \overline{m}^a m^b\right) = 0$$

The only non-zero spin-coefficient is $\rho = a\tan av$. This means that $-Re(\rho) \neq 0$ and there is expansion (or pure focusing=divergence).

14.13 [w]The Nariai spacetime

The line element:

$$ds^2 = -\Lambda v^2 du^2 + 2dudv - \frac{1}{\Omega^2}(dx^2 + dy^2)$$
$$= \frac{1}{\Lambda v^2}dv^2 - \left(\sqrt{\Lambda}vdu - \frac{1}{\sqrt{\Lambda}v}dv\right)^2 - \frac{1}{\Omega^2}(dx^2 + dy^2)$$
$$\Omega = 1 + \frac{\Lambda}{4}(x^2 + y^2)$$

The metric tensor

$$g_{ab} = \begin{Bmatrix} 1 & \frac{1}{-\Lambda v^2} & & \\ & & -\frac{1}{\Omega^2} & \\ & & & -\frac{1}{\Omega^2} \end{Bmatrix}$$

and its inverse:

[66] $= -\frac{1}{2}\left(\left(-a\frac{1}{\sqrt{2}}\sin av - (-a\tan av)\frac{1}{\sqrt{2}}\cos av\right)\overline{m}^x + \left(-ia\frac{1}{\sqrt{2}}\sin av + a\tan av\, i\frac{1}{\sqrt{2}}\cos av\right)\overline{m}^y\right) =$

$$g^{ab} = \begin{Bmatrix} \Lambda v^2 & 1 & & \\ 1 & & & \\ & & -\Omega^2 & \\ & & & -\Omega^2 \end{Bmatrix}$$

14.13.1 The Christoffel symbols

$$\partial_v(g_{uu}) = \partial_v(-\Lambda v^2) = -2\Lambda v$$
$$\partial_x(g_{xx}) = \partial_x(g_{yy}) = \partial_x\left(-\frac{1}{\Omega^2}\right) = 2\frac{\partial_x(\Omega)}{\Omega^3} = \frac{\Lambda x}{\Omega^3}$$
$$\partial_y(g_{xx}) = \partial_y(g_{yy}) = \partial_y\left(-\frac{1}{\Omega^2}\right) = 2\frac{\partial_y(\Omega)}{\Omega^3} = \frac{\Lambda y}{\Omega^3}$$

$$\Gamma_{abc} = \frac{1}{2}(\partial_a g_{bc} + \partial_b g_{ac} - \partial_c g_{ab}) \qquad \Gamma^a{}_{bc} = g^{ad}\Gamma_{bcd}$$
$$\Gamma_{uuv} = -\frac{1}{2}(\partial_v g_{uu}) = \Lambda v \Rightarrow \Gamma^v{}_{uu} = g^{vd}\Gamma_{uud} = g^{vv}\Gamma_{uuv} + g^{vu}\Gamma_{uuu} = {}^{67}\Lambda^2 v^3$$
$$\Gamma^u{}_{uu} = g^{ud}\Gamma_{uud} = g^{uu}\Gamma_{uuu} + g^{uv}\Gamma_{uuv} = \Lambda v$$
$$\Gamma_{vuu} = \Gamma_{uvu} = \frac{1}{2}(\partial_v g_{uu}) = -\Lambda v \Rightarrow \Gamma^u{}_{vu} = \Gamma^u{}_{uv} = g^{ud}\Gamma_{uvd} = {}^{68}0$$
$$\Gamma^v{}_{vu} = \Gamma^v{}_{uv} = g^{vd}\Gamma_{uvd} = {}^{69} - \Lambda v$$
$$\Gamma_{xxx} = \frac{1}{2}(\partial_x g_{xx}) = \frac{1}{2}\frac{\Lambda x}{\Omega^3} \Rightarrow \Gamma^x{}_{xx} = g^{xx}\Gamma_{xxx} = (-\Omega^2)\frac{1}{2}\frac{\Lambda x}{\Omega^3} = -\frac{1}{2}\frac{\Lambda x}{\Omega}$$
$$\Gamma_{xxy} = -\frac{1}{2}\partial_y(g_{xx}) = -\frac{1}{2}\frac{\Lambda y}{\Omega^3} \Rightarrow \Gamma^y{}_{xx} = g^{yy}\Gamma_{xxy} = (-\Omega^2)\left(-\frac{1}{2}\frac{\Lambda y}{\Omega^3}\right) = \frac{1}{2}\frac{\Lambda y}{\Omega}$$
$$\Gamma_{yxx} = \Gamma_{xyx} = \frac{1}{2}\partial_y(g_{xx}) = \frac{1}{2}\frac{\Lambda y}{\Omega^3} \Rightarrow \Gamma^x{}_{yx} = \Gamma^x{}_{xy} = g^{xx}\Gamma_{yxx} = {}^{70} - \frac{1}{2}\frac{\Lambda y}{\Omega}$$
$$\Gamma_{yyy} = \frac{1}{2}(\partial_y g_{yy}) = \frac{1}{2}\frac{\Lambda y}{\Omega^3} \Rightarrow \Gamma^y{}_{yy} = g^{yy}\Gamma_{yyy} = (-\Omega^2)\frac{1}{2}\frac{\Lambda y}{\Omega^3} = -\frac{1}{2}\frac{\Lambda y}{\Omega}$$
$$\Gamma_{yyx} = -\frac{1}{2}\partial_x(g_{yy}) = -\frac{1}{2}\frac{\Lambda x}{\Omega^3} \Rightarrow \Gamma^x{}_{yy} = g^{xx}\Gamma_{yyx} = (-\Omega^2)\left(-\frac{1}{2}\frac{\Lambda x}{\Omega^3}\right) = \frac{1}{2}\frac{\Lambda x}{\Omega}$$
$$\Gamma_{xyy} = \Gamma_{yxy} = \frac{1}{2}\partial_x(g_{yy}) = \frac{1}{2}\frac{\Lambda x}{\Omega^3} \Rightarrow \Gamma^y{}_{xy} = \Gamma^y{}_{yx} = g^{yy}\Gamma_{xyy} = (-\Omega^2)\frac{1}{2}\frac{\Lambda x}{\Omega^3} = -\frac{1}{2}\frac{\Lambda x}{\Omega}$$

Collecting the results we find the non-zero Christoffel symbols

$$\Gamma^v{}_{uv} = -\Gamma^u{}_{uu} = -\Lambda v$$
$$\Gamma^v{}_{uu} = \Lambda^2 v^3$$
$$\Gamma^x{}_{xx} = \Gamma^y{}_{yx} = -\Gamma^x{}_{yy} = -\frac{\Lambda x}{2\Omega}$$
$$\Gamma^x{}_{xy} = -\Gamma^y{}_{xx} = \Gamma^y{}_{yy} = -\frac{\Lambda y}{2\Omega}$$

14.13.2 The basis one forms

$$\frac{1}{\Lambda v^2}dv^2 - \left(\sqrt{\Lambda}v du - \frac{1}{\sqrt{\Lambda}v}dv\right)^2 - \frac{1}{\Omega^2}(dx^2 + dy^2) = \left(\omega^{\hat{v}}\right)^2 - \left(\omega^{\hat{u}}\right)^2 - \left(\omega^{\hat{x}}\right)^2 - \left(\omega^{\hat{y}}\right)^2$$

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[67] $= \Lambda v^2(\Lambda v) =$
[68] $= g^{uu}\Gamma_{uvu} + g^{uv}\Gamma_{uvv} =$
[69] $= g^{vv}\Gamma_{uvv} + g^{vu}\Gamma_{uvu} =$
[70] $= (-\Omega^2)\frac{1}{2}\frac{\Lambda y}{\Omega^3} =$

$$\omega^{\hat{v}} = \frac{1}{\sqrt{\Lambda}v}dv \qquad dv = \sqrt{\Lambda}v\omega^{\hat{v}}$$
$$\omega^{\hat{u}} = \sqrt{\Lambda}vdu - \frac{1}{\sqrt{\Lambda}v}dv \qquad du = \frac{1}{\sqrt{\Lambda}v}\left(\omega^{\hat{v}} + \omega^{\hat{u}}\right) \qquad \eta^{ij} = \begin{Bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{Bmatrix}$$
$$\omega^{\hat{x}} = \frac{1}{\Omega}dx \qquad dx = \Omega\omega^{\hat{x}}$$
$$\omega^{\hat{y}} = \frac{1}{\Omega}dy \qquad dy = \Omega\omega^{\hat{y}}$$

14.13.3 Cartan's First Structure equation and the calculation of the curvature one-forms

$$d\omega^{\hat{a}} = -\Gamma^{\hat{a}}{}_{\hat{b}} \wedge \omega^{\hat{b}}$$
$$d\omega^{\hat{v}} = \frac{1}{\sqrt{\Lambda}v}dv = 0$$
$$d\omega^{\hat{u}} = d\left(\sqrt{\Lambda}vdu - \frac{1}{\sqrt{\Lambda}v}dv\right) = \sqrt{\Lambda}dv \wedge du = \sqrt{\Lambda}\left(\sqrt{\Lambda}v\omega^{\hat{v}}\right) \wedge \left(\frac{1}{\sqrt{\Lambda}v}\left(\omega^{\hat{v}} + \omega^{\hat{u}}\right)\right) = \sqrt{\Lambda}\omega^{\hat{v}} \wedge \omega^{\hat{u}}$$
$$= -\sqrt{\Lambda}\omega^{\hat{u}} \wedge \omega^{\hat{v}}$$
$$d\omega^{\hat{x}} = d\left(\frac{1}{\Omega}dx\right) = d\left(\frac{1}{1+\frac{\Lambda}{4}(x^2+y^2)}dx\right) = -\frac{1}{2}\frac{y\Lambda}{\Omega^2}dy \wedge dx = \frac{1}{2}y\Lambda\omega^{\hat{x}} \wedge \omega^{\hat{y}}$$
$$d\omega^{\hat{y}} = d\left(\frac{1}{\Omega}dy\right) = d\left(\frac{1}{1+\frac{\Lambda}{4}(x^2+y^2)}dy\right) = -\frac{1}{2}\frac{x\Lambda}{\Omega^2}dx \wedge dy = \frac{1}{2}x\Lambda\omega^{\hat{y}} \wedge \omega^{\hat{x}}$$

<u>The curvature one-forms summarized in a matrix:</u>

$$\Gamma^{\hat{a}}{}_{\hat{b}} = \begin{Bmatrix} 0 & \sqrt{\Lambda}\omega^{\hat{u}} & 0 & 0 \\ \sqrt{\Lambda}\omega^{\hat{u}} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}\Lambda\left(x\omega^{\hat{y}} - y\omega^{\hat{x}}\right) \\ 0 & 0 & \frac{1}{2}\Lambda\left(x\omega^{\hat{y}} - y\omega^{\hat{x}}\right) & 0 \end{Bmatrix}$$

Where $\hat{a}$ refers to column and $\hat{b}$ to row

14.13.4 The curvature two forms:

$$\Omega^{\hat{a}}{}_{\hat{b}} = d\Gamma^{\hat{a}}{}_{\hat{b}} + \Gamma^{\hat{a}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{b}} = \frac{1}{2}R^{\hat{a}}{}_{\hat{b}\hat{c}\hat{d}}\omega^{\hat{c}} \wedge \omega^{\hat{d}}$$
$$\Omega^{\hat{v}}{}_{\hat{v}} = d\Gamma^{\hat{v}}{}_{\hat{v}} + \Gamma^{\hat{v}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{v}} = \Gamma^{\hat{v}}{}_{\hat{u}} \wedge \Gamma^{\hat{u}}{}_{\hat{v}} = 0$$
$$\Omega^{\hat{v}}{}_{\hat{u}} = d\Gamma^{\hat{v}}{}_{\hat{u}} + \Gamma^{\hat{v}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{u}} = d\left(\sqrt{\Lambda}\omega^{\hat{u}}\right) + \Gamma^{\hat{v}}{}_{\hat{u}} \wedge \Gamma^{\hat{u}}{}_{\hat{u}} = \sqrt{\Lambda}d\omega^{\hat{u}} = \Lambda\omega^{\hat{v}} \wedge \omega^{\hat{u}} = \Omega^{\hat{u}}{}_{\hat{v}}$$
$$\Omega^{\hat{v}}{}_{\hat{x}} = \Omega^{\hat{v}}{}_{\hat{y}} = 0$$
$$\Omega^{\hat{u}}{}_{\hat{u}} = d\Gamma^{\hat{u}}{}_{\hat{u}} + \Gamma^{\hat{u}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{u}} = \Gamma^{\hat{u}}{}_{\hat{v}} \wedge \Gamma^{\hat{v}}{}_{\hat{u}} = 0$$
$$\Omega^{\hat{u}}{}_{\hat{x}} = \Omega^{\hat{u}}{}_{\hat{y}}$$
$$\Omega^{\hat{x}}{}_{\hat{x}} = d\Gamma^{\hat{x}}{}_{\hat{x}} + \Gamma^{\hat{x}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{x}} = \Gamma^{\hat{x}}{}_{\hat{y}} \wedge \Gamma^{\hat{y}}{}_{\hat{x}} = -\left(\frac{1}{2}\Lambda\left(x\omega^{\hat{y}} - y\omega^{\hat{x}}\right)\right) \wedge \left(\frac{1}{2}\Lambda\left(x\omega^{\hat{y}} - y\omega^{\hat{x}}\right)\right)$$
$$= -\left(\frac{1}{2}\Lambda\left(x\omega^{\hat{y}}\right)\right) \wedge \left(\frac{1}{2}\Lambda\left(-y\omega^{\hat{x}}\right)\right) - \left(\frac{1}{2}\Lambda\left(-y\omega^{\hat{x}}\right)\right) \wedge \left(\frac{1}{2}\Lambda\left(x\omega^{\hat{y}}\right)\right)$$
$$= \frac{1}{4}\Lambda^2xy\left(\omega^{\hat{y}} \wedge \omega^{\hat{x}} + \omega^{\hat{x}} \wedge \omega^{\hat{y}}\right) = 0$$
$$\Omega^{\hat{x}}{}_{\hat{y}} = d\Gamma^{\hat{x}}{}_{\hat{y}} + \Gamma^{\hat{x}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{y}} = d\left(\frac{1}{2}\Lambda\left(x\omega^{\hat{y}} - y\omega^{\hat{x}}\right)\right) + \Gamma^{\hat{x}}{}_{\hat{y}} \wedge \Gamma^{\hat{y}}{}_{\hat{y}} = d\left(\frac{1}{2}\frac{\Lambda}{\Omega}(xdy - ydx)\right)$$

$$= \left(\frac{1}{2}\frac{\Lambda}{\Omega} - \frac{1}{4}\frac{\Lambda^2}{\Omega^2}x^2\right)dx \wedge dy + \left(-\frac{1}{2}\frac{\Lambda}{\Omega} + \frac{1}{4}\frac{\Lambda^2}{\Omega^2}y^2\right)dy \wedge dx$$
$$= \left(\frac{1}{2}\Lambda\Omega - \frac{1}{4}\Lambda^2x^2\right)\omega^{\hat{x}} \wedge \omega^{\hat{y}} + \left(-\frac{1}{2}\Lambda\Omega + \frac{1}{4}\Lambda^2y^2\right)\omega^{\hat{y}} \wedge \omega^{\hat{x}}$$
$$= \left(\Lambda\Omega - \frac{1}{4}\Lambda^2(x^2+y^2)\right)\omega^{\hat{x}} \wedge \omega^{\hat{y}} = \Lambda\left(\Omega - \frac{\Lambda}{4}(x^2+y^2)\right)\omega^{\hat{x}} \wedge \omega^{\hat{y}} = \Lambda\omega^{\hat{x}} \wedge \omega^{\hat{y}}$$
$$\Omega^{\hat{y}}{}_{\hat{x}} = d\Gamma^{\hat{y}}{}_{\hat{x}} = -d\Gamma^{\hat{x}}{}_{\hat{y}} = \Lambda\omega^{\hat{y}} \wedge \omega^{\hat{x}}$$

Summarized in a matrix:

$$\Omega^{\hat{a}}{}_{\hat{b}} = \begin{Bmatrix} 0 & -\Lambda\omega^{\hat{u}} \wedge \omega^{\hat{v}} & 0 & 0 \\ \Lambda\omega^{\hat{v}} \wedge \omega^{\hat{u}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \Lambda\omega^{\hat{y}} \wedge \omega^{\hat{x}} \\ 0 & 0 & \Lambda\omega^{\hat{x}} \wedge \omega^{\hat{y}} & 0 \end{Bmatrix}$$

Where $\hat{a}$ refers to column and $\hat{b}$ to row

Now we can write down the independent elements of the Riemann tensor in the non-coordinate basis:

$$R^{\hat{v}}{}_{\hat{u}\hat{v}\hat{u}} = \Lambda$$
$$R^{\hat{x}}{}_{\hat{y}\hat{x}\hat{y}} = \Lambda$$

14.13.5 The Ricci tensor

$$R_{\hat{a}\hat{b}} = R^{\hat{c}}{}_{\hat{a}\hat{c}\hat{b}}$$
$$R_{\hat{v}\hat{v}} = R^{\hat{c}}{}_{\hat{v}\hat{c}\hat{v}} = R^{\hat{u}}{}_{\hat{v}\hat{u}\hat{v}} + R^{\hat{v}}{}_{\hat{v}\hat{v}\hat{v}} + R^{\hat{x}}{}_{\hat{v}\hat{x}\hat{v}} + R^{\hat{y}}{}_{\hat{v}\hat{y}\hat{v}} = R^{\hat{u}}{}_{\hat{v}\hat{u}\hat{v}} = -\Lambda$$
$$R_{\hat{v}\hat{u}} = R^{\hat{c}}{}_{\hat{v}\hat{c}\hat{u}} = R^{\hat{u}}{}_{\hat{v}\hat{u}\hat{u}} + R^{\hat{v}}{}_{\hat{v}\hat{v}\hat{u}} + R^{\hat{x}}{}_{\hat{v}\hat{x}\hat{u}} + R^{\hat{y}}{}_{\hat{v}\hat{y}\hat{u}} = 0$$
$$R_{\hat{v}\hat{x}} = R_{\hat{v}\hat{y}} = 0$$
$$R_{\hat{u}\hat{u}} = R^{\hat{c}}{}_{\hat{u}\hat{c}\hat{u}} = R^{\hat{u}}{}_{\hat{u}\hat{u}\hat{u}} + R^{\hat{v}}{}_{\hat{u}\hat{v}\hat{u}} + R^{\hat{x}}{}_{\hat{u}\hat{x}\hat{u}} + R^{\hat{y}}{}_{\hat{u}\hat{y}\hat{u}} = R^{\hat{v}}{}_{\hat{u}\hat{v}\hat{u}} = \Lambda$$
$$R_{\hat{u}\hat{x}} = R_{\hat{u}\hat{y}} = 0$$
$$R_{\hat{x}\hat{x}} = R^{\hat{c}}{}_{\hat{x}\hat{c}\hat{x}} = R^{\hat{u}}{}_{\hat{x}\hat{u}\hat{x}} + R^{\hat{v}}{}_{\hat{x}\hat{v}\hat{x}} + R^{\hat{x}}{}_{\hat{x}\hat{x}\hat{x}} + R^{\hat{y}}{}_{\hat{x}\hat{y}\hat{x}} = R^{\hat{y}}{}_{\hat{x}\hat{y}\hat{x}} = \Lambda$$
$$R_{\hat{y}\hat{x}} = R^{\hat{c}}{}_{\hat{y}\hat{c}\hat{x}} = R^{\hat{u}}{}_{\hat{y}\hat{u}\hat{x}} + R^{\hat{v}}{}_{\hat{y}\hat{v}\hat{x}} + R^{\hat{x}}{}_{\hat{y}\hat{x}\hat{x}} + R^{\hat{y}}{}_{\hat{y}\hat{y}\hat{x}} = 0$$
$$R_{\hat{y}\hat{y}} = R^{\hat{c}}{}_{\hat{y}\hat{c}\hat{y}} = R^{\hat{u}}{}_{\hat{y}\hat{u}\hat{y}} + R^{\hat{v}}{}_{\hat{y}\hat{v}\hat{y}} + R^{\hat{x}}{}_{\hat{y}\hat{x}\hat{y}} + R^{\hat{y}}{}_{\hat{y}\hat{y}\hat{y}} = R^{\hat{x}}{}_{\hat{y}\hat{x}\hat{y}} = \Lambda$$

Summarized in a matrix:

$$R_{\hat{a}\hat{b}} = \begin{Bmatrix} -\Lambda & 0 & 0 & 0 \\ 0 & \Lambda & 0 & 0 \\ 0 & 0 & \Lambda & 0 \\ 0 & 0 & 0 & \Lambda \end{Bmatrix} = -\eta_{\hat{a}\hat{b}}\Lambda$$

Where $\hat{a}$ refers to column and $\hat{b}$ to row.

Which implies that $R_{ab} = -g_{ab}\Lambda$

14.13.6 The Ricci scalar

$$R = \eta^{\hat{a}\hat{b}}R_{\hat{a}\hat{b}} = \eta^{\hat{v}\hat{v}}R_{\hat{v}\hat{v}} + \eta^{\hat{u}\hat{u}}R_{\hat{u}\hat{u}} + \eta^{\hat{x}\hat{x}}R_{\hat{x}\hat{x}} + \eta^{\hat{y}\hat{y}}R_{\hat{y}\hat{y}} = \eta^{\hat{v}\hat{v}}(-\Lambda) + \eta^{\hat{u}\hat{u}}\Lambda + \eta^{\hat{x}\hat{x}}\Lambda + \eta^{\hat{y}\hat{y}}\Lambda = -4\Lambda$$

14.13.7 Change of signature

14.13.7.1 The Line element

The line element:

$$ds^2 = \Lambda v^2du^2 - 2dudv + \frac{1}{\Omega^2}(dx^2+dy^2)$$
$$= -\frac{1}{\Lambda v^2}dv^2 + \left(\sqrt{\Lambda}vdu - \frac{1}{\sqrt{\Lambda}v}dv\right)^2 + \frac{1}{\Omega^2}(dx^2+dy^2)$$
$$\Omega = 1 + \frac{\Lambda}{4}(x^2+y^2)$$

The metric tensor

$$g_{ab} = \begin{Bmatrix} & \frac{-1}{\Lambda v^2} & & \\ -1 & & & \\ & & \frac{1}{\Omega^2} & \\ & & & \frac{1}{\Omega^2} \end{Bmatrix}$$

14.13.7.2 The non-zero Christoffel symbols

$$\Gamma^{v}{}_{uv} = -\Gamma^{u}{}_{uu} = -\Lambda v$$
$$\Gamma^{v}{}_{uu} = \Lambda^2 v^3$$
$$\Gamma^{x}{}_{xx} = \Gamma^{y}{}_{yx} = -\Gamma^{x}{}_{yy} = -\frac{\Lambda x}{2\Omega}$$
$$\Gamma^{x}{}_{xy} = -\Gamma^{y}{}_{xx} = \Gamma^{y}{}_{yy} = -\frac{\Lambda y}{2\Omega}$$

14.13.7.3 The Riemann tensor

$$R^{\hat{v}}{}_{\hat{u}\hat{v}\hat{u}} = \Lambda$$
$$R^{\hat{x}}{}_{\hat{y}\hat{x}\hat{y}} = \Lambda$$

14.13.7.4 The Ricci tensor

$$R_{\hat{a}\hat{b}} = \begin{Bmatrix} -\Lambda & 0 & 0 & 0 \\ 0 & \Lambda & 0 & 0 \\ 0 & 0 & \Lambda & 0 \\ 0 & 0 & 0 & \Lambda \end{Bmatrix} = {}^{71}\eta_{\hat{a}\hat{b}}\Lambda$$

Where $\hat{a}$ refers to column and $\hat{b}$ to row.

Which implies that $R_{ab} = g_{ab}\Lambda$

14.13.7.5 The Ricci scalar

$$R = \eta^{\hat{a}\hat{b}}R_{\hat{a}\hat{b}} = \eta^{\hat{v}\hat{v}}R_{\hat{v}\hat{v}} + \eta^{\hat{u}\hat{u}}R_{\hat{u}\hat{u}} + \eta^{\hat{x}\hat{x}}R_{\hat{x}\hat{x}} + \eta^{\hat{y}\hat{y}}R_{\hat{y}\hat{y}} = \eta^{\hat{v}\hat{v}}(-\Lambda) + \eta^{\hat{u}\hat{u}}\Lambda + \eta^{\hat{x}\hat{x}}\Lambda + \eta^{\hat{y}\hat{y}}\Lambda = 4\Lambda$$

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[a] (McMahon, 2006, s. 304), (Penrose, 2004, s. 105), (d'Inverno, 1992, p. 282)

[b] (Choquet-Bruhat, 2015, s. 79), (McMahon, 2006, s. 280), (Carroll, 2004, p. 274), (Hartle, 2003, p. 549), (d'Inverno, 1992, p. 271)

[c] (McMahon, 2006, s. 286)

[71] $\eta^{ij} = \begin{Bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{Bmatrix}$

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[d] (McMahon, 2006, s. 288)
[e] (McMahon, 2006, s. 290)
[f] (Choquet-Bruhat, 2015, s. 80) I.33: $h_{23}^{(1)} = c_{23}\sin(\omega t - \omega x^1)$, $h_{22}^{(1)} = -h_{33}^{(1)} = c_{22}\sin(\omega t - \omega x^1)$
[g] (McMahon, 2006, s. 291)
[h] (McMahon, 2006, s. 298), (Carroll, 2004, p. 321)
[i] (d'Inverno, 1992, p. 280)
[j] (d'Inverno, 1992, p. 280)
[k] (d'Inverno, 1992, p. 288)
[l] (McMahon, 2006, s. 75, 92)
[m] (McMahon, 2006, s. 92)
[n] (McMahon, 2006, s. 195)
[o] (McMahon, 2006, s. 322)
[p] (d'Inverno, 1992, p. 288)
[q] (McMahon, 2006, s. 304), (d'Inverno, 1992, p. 282)
[r] http://www-staff.lboro.ac.uk/~majbg/jbg/book/chap3.pdf
[s] (McMahon, 2006, s. 305), (d'Inverno, 1992, p. 282)
[t] http://www-staff.lboro.ac.uk/~majbg/jbg/book/chap3.pdf
[u] (McMahon, 2006, s. 313)
[v] (McMahon, 2006, s. 322)
[w] (McMahon, 2006, s. 318)

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