Content

13 Cosmology

13.1 [a]Light travelling in the Universe

Light travelling in the universe can be described by the line element $ds^2 = dt^2 - a^2(t)dx^2$, where $dx^2 = dx_1{}^2 + dx_2{}^2 + dx_3{}^2$ and the x_i are commoving coordinates. Light travel along null geodesic i.e. $ds^2 = 0$. We can now write $\int_t^{t_0} \frac{dt}{a(t)}$ for the total commoving distance light emitted at time t can travel by time t_0. If we multiply this by the value of the scale factor $a(t_0)$ at time t_0, then we will have calculated the physical distance that the light has traveled in this time interval. This algorithm can be widely used to calculate how far light can travel in any given time interval, revealing whether to points in space , for example are in causal contact. As you can see, for accelerated expansion, even for arbitrarily large t_0 the integral is bounded, showing that the light will never reach arbitrarily distant commoving locations. Thus, in a universe with accelerated expansion, there are locations with which we can never communicate.

13.2 [b]Spaces of Positive, Negative, and Zero Curvature

The spatial part of a homogenous, isotropic line-element is

$$d\sigma^2 = \frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi$$

rewriting it in a more general form

$$d\sigma^2 = d\chi^2 + r^2(\chi)d\theta^2 + r^2(\chi)\sin^2\theta \, d\phi^2$$

we see that

$$d\chi^2 = \frac{dr^2}{1-kr^2}$$

$$\Rightarrow \quad d\chi = \pm\frac{dr}{\sqrt{1-kr^2}}$$

and in order to identify the line-element for the different k-values we solve the latter differential equation.

$\underline{k = 0:}$

$$\int d\chi = \pm\int dr$$

$$\Rightarrow \quad \chi = \pm r$$

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$\Rightarrow \qquad r = \pm\chi$

$\Rightarrow \qquad r^2(\chi) = \chi^2$

and the line-element[1]

$$d\sigma^2 = d\chi^2 + \chi^2 d\theta^2 + \chi^2 \sin^2\theta \, d\phi^2$$

<u>$k > 0$:</u>

$$\int d\chi = \pm\int \frac{dr}{\sqrt{1-kr^2}}$$

$$\Rightarrow \qquad \chi =^{2} \pm\int \frac{dr}{\sqrt{1-kr^2}} = \pm\frac{1}{\sqrt{k}}\int \frac{dx}{\sqrt{1-x^2}} =^{3} \pm\frac{1}{\sqrt{k}}\sin^{-1}x = \pm\frac{1}{\sqrt{k}}\sin^{-1}\sqrt{k}r$$

$$\Rightarrow \qquad r = \frac{1}{\sqrt{k}}\sin\left(\pm\sqrt{k}\chi\right) = \pm\frac{1}{\sqrt{k}}\sin\left(\sqrt{k}\chi\right)$$

$$\Rightarrow \qquad r^2(\chi) = \frac{1}{k}\sin^2\left(\sqrt{k}\chi\right)$$

if <u>$k = 1$</u> we get

$$r^2(\chi) = \sin^2(\chi)$$

and the line-element[4]

$$d\sigma^2 = d\chi^2 + \sin^2(\chi)\, d\theta^2 + \sin^2(\chi)\sin^2\theta \, d\phi^2$$

<u>$k < 0$:</u>

$$\Rightarrow \qquad \int d\chi = \pm\int \frac{dr}{\sqrt{1+Kr^2}} \qquad K = -k$$

$$\Rightarrow \qquad \chi = \pm\int \frac{dr}{\sqrt{1+\left(\sqrt{K}r\right)^2}}$$

$$= \pm\frac{1}{\sqrt{K}}\int \frac{dx}{\sqrt{1+x^2}} \qquad x = \sqrt{K}r = \sqrt{-k}r$$

$$=^{5} \pm\frac{1}{\sqrt{K}}\ln\left(x + \sqrt{x^2+1}\right)$$

$$=^{6} \pm\frac{1}{\sqrt{K}}\sinh^{-1}x$$

$$= \pm\frac{1}{\sqrt{-k}}\sinh^{-1}\left(\sqrt{-k}r\right)$$

$$\Rightarrow \qquad r = \frac{1}{\sqrt{-k}}\sinh\left(\pm\sqrt{-k}\chi\right) = \pm\frac{1}{\sqrt{-k}}\sinh\left(\sqrt{-k}\chi\right)$$

$$\Rightarrow \qquad r^2(\chi) = \frac{1}{-k}\sinh^2\left(\sqrt{-k}\chi\right)$$

if <u>$k = -1$</u> we get

$$\Rightarrow \qquad r^2(\chi) = \sinh^2(\chi)$$

and the line-element[7]

$$d\sigma^2 = d\chi^2 + \sinh^2(\chi)\, d\theta^2 + \sinh^2(\chi)\sin^2\theta \, d\phi^2$$

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[1] Which is equivalent to the three-dimensional flat space in polar coordinates: $ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta \, d\phi^2$

[2] $x = \sqrt{k}r$

[3] $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$ (14.237) (Spiegel, 1990)

[4] Which is equivalent to the spatial part of the Einstein cylinder: $ds^2 = -dt^2 + (a_0)^2\left(d\theta^2 + \sin^2\theta \left(d\phi^2 + \sin^2\phi \, d\psi^2\right)\right)$

[5] $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left(x + \sqrt{x^2-a^2}\right)$ (14.210) (Spiegel, 1990)

[6] $\sinh^{-1}x = \ln\left(x + \sqrt{x^2+1}\right)$ (8.55) (Spiegel, 1990)

[7] Which is equivalent to the Lorentz hyperboloid: $ds^2 = d\psi^2 + \sinh^2\psi \; d\theta^2 + \sinh^2\psi\sin^2\theta \, d\phi^2$

Note: We have omitted the constants of integration because of symmetry reasons. The metrics have to fulfill the requirement of homogeneity and isotropy

13.3 [c]The critical density

The critical density

$$\rho_c = \frac{3}{8\pi}\frac{H_0^2}{G}$$

can be found from some simple Newtonian considerations[d].

Consider a galaxy with mass m on the surface of a sphere with radius R, density ρ and an expansion rate corresponding to the Hubble expansion H_0. The velocity of the galaxy is given by the Hubble law $v = H_0 R$.

The potential energy of the galaxy is

$$E_{pot} = -\frac{4\pi}{3}R^3\rho * \frac{mG}{R} = -\frac{4\pi R^2 \rho m G}{3}$$

The kinetic energy of the galaxy is

$$E_{kin} = \frac{1}{2}mv^2 = \frac{1}{2}mH_0^2R^2$$

The total energy

$$E_{tot} = E_{kin} + E_{pot} = \frac{1}{2}mH_0^2R^2 - \frac{4\pi R^2\rho m G}{3} = mR^2\left(\frac{1}{2}H_0^2 - \frac{4\pi\rho G}{3}\right)$$

Setting the total energy to zero defines the critical density

$$0 = mR^2\left(\frac{1}{2}H_0^2 - \frac{4\pi\rho_c G}{3}\right)$$

$$\Rightarrow \qquad \rho_c = \frac{3H_0^2}{8\pi G}$$

13.4 [e]The Robertson-Walker space-time

The Robertson-Walker spacetime describes a homogenous, isotropic expanding universe

The line-element

$$ds^2 = -dt^2 + \frac{a^2(t)}{1-kr^2}dr^2 + a^2(t)r^2 d\theta^2 + a^2(t)r^2\sin^2\theta\, d\phi^2$$

13.4.1 [f]The Riemann tensor

The Basis one forms

$$\omega^{\hat{t}} = dt$$

$$\omega^{\hat{r}} = \frac{a(t)}{\sqrt{1-kr^2}}dr \qquad dr = \frac{\sqrt{1-kr^2}}{a(t)}\omega^{\hat{r}}$$

$$\omega^{\hat{\theta}} = a(t)r\,d\theta \qquad d\theta = \frac{1}{a(t)r}\omega^{\hat{\theta}}$$

$$\omega^{\hat{\phi}} = a(t)r\sin\theta\, d\phi \qquad d\phi = \frac{1}{a(t)r\sin\theta}\omega^{\hat{\phi}}$$

$$\eta^{ij} = \begin{Bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{Bmatrix}$$

The curvature one forms:

$$d\omega^{\hat{a}} = -\Gamma^{\hat{a}}{}_{\hat{b}} \wedge \omega^{\hat{b}}$$

$$d\omega^{\hat{t}} = 0$$

$$d\omega^{\hat{r}} = d\left(\frac{a(t)}{\sqrt{1-kr^2}}dr\right) = \frac{\dot{a}}{\sqrt{1-kr^2}}dt\wedge dr = -\frac{\dot{a}}{a}\omega^{\hat{r}}\wedge\omega^{\hat{t}}$$

$$d\omega^{\hat\theta} = d(a(t)rd\theta) = \dot a r dt\wedge d\theta + a dr\wedge d\theta = \frac{\dot a}{a}\omega^{\hat t}\wedge\omega^{\hat\theta} + \frac{\sqrt{1-kr^2}}{ar}\omega^{\hat r}\wedge\omega^{\hat\theta}$$

$$\begin{aligned} d\omega^{\hat\phi} &= d(a(t)r\sin\theta\, d\phi) \\ &= \dot a r\sin\theta\, dt\wedge d\phi + a\sin\theta\, dr\wedge d\phi + ar\cos\theta\, d\theta\wedge d\phi \\ &= {}^{8}\frac{\dot a}{a}\omega^{\hat t}\wedge\omega^{\hat\phi} + \frac{\sqrt{1-kr^2}}{ar}\omega^{\hat r}\wedge\omega^{\hat\phi} + \frac{\cot\theta}{ar}\omega^{\hat\theta}\wedge\omega^{\hat\phi} \end{aligned}$$

Summerized in a matrix where $\hat a$ refers to the column and $\hat b$ the row.

$$\Gamma^{\hat a}{}_{\hat b} = \begin{Bmatrix} 0 & \frac{\dot a}{a}\omega^{\hat r} & \frac{\dot a}{a}\omega^{\hat\theta} & \frac{\dot a}{a}\omega^{\hat\phi} \\ \frac{\dot a}{a}\omega^{\hat r} & 0 & \frac{\sqrt{1-kr^2}}{ar}\omega^{\hat\theta} & \frac{\sqrt{1-kr^2}}{ar}\omega^{\hat\phi} \\ \frac{\dot a}{a}\omega^{\hat\theta} & -\frac{\sqrt{1-kr^2}}{ar}\omega^{\hat\theta} & 0 & \frac{\cot\theta}{ar}\omega^{\hat\phi} \\ \frac{\dot a}{a}\omega^{\hat\phi} & -\frac{\sqrt{1-kr^2}}{ar}\omega^{\hat\phi} & -\frac{\cot\theta}{ar}\omega^{\hat\phi} & 0 \end{Bmatrix}$$

<u>The curvature two forms:</u>

$$\Omega^{\hat a}{}_{\hat b} = d\Gamma^{\hat a}{}_{\hat b} + \Gamma^{\hat a}{}_{\hat c}\wedge\Gamma^{\hat c}{}_{\hat b} = \frac{1}{2}R^{\hat a}{}_{\hat b\hat c\hat d}\omega^{\hat c}\wedge\omega^{\hat d}$$

<u>$\Omega^{\hat r}{}_{\hat t}$:</u>

$$\begin{aligned} d\Gamma^{\hat r}{}_{\hat t} &= d\left(\frac{\dot a}{a}\omega^{\hat r}\right) \\ &= d\left(\frac{\dot a}{a}\frac{a}{\sqrt{1-kr^2}}dr\right) \\ &= d\left(\frac{\dot a}{\sqrt{1-kr^2}}dr\right) \\ &= \frac{\ddot a}{\sqrt{1-kr^2}}dt\wedge dr \\ &= \frac{\ddot a}{\sqrt{1-kr^2}}\omega^{\hat t}\wedge\frac{\sqrt{1-kr^2}}{a}\omega^{\hat r} \\ &= \frac{\ddot a}{a}\omega^{\hat t}\wedge\omega^{\hat r} \end{aligned}$$

$$\Gamma^{\hat r}{}_{\hat c}\wedge\Gamma^{\hat c}{}_{\hat t} = \Gamma^{\hat r}{}_{\hat t}\wedge\Gamma^{\hat t}{}_{\hat t} + \Gamma^{\hat r}{}_{\hat r}\wedge\Gamma^{\hat r}{}_{\hat t} + \Gamma^{\hat r}{}_{\hat\theta}\wedge\Gamma^{\hat\theta}{}_{\hat t} + \Gamma^{\hat r}{}_{\hat\phi}\wedge\Gamma^{\hat\phi}{}_{\hat t} = 0$$

$$\Rightarrow \quad \Omega^{\hat r}{}_{\hat t} = \frac{\ddot a}{a}\omega^{\hat t}\wedge\omega^{\hat r}$$

<u>$\Omega^{\hat\theta}{}_{\hat t}$:</u>

$$\begin{aligned} d\Gamma^{\hat\theta}{}_{\hat t} &= d\left(\frac{\dot a}{a}\omega^{\hat\theta}\right) \\ &= d\left(\frac{\dot a}{a}ard\theta\right) \\ &= d(\dot a r d\theta) \\ &= \ddot a r dt\wedge d\theta + \dot a dr\wedge d\theta \\ &= \ddot a r\omega^{\hat t}\wedge\frac{1}{ar}\omega^{\hat\theta} + \dot a\frac{\sqrt{1-kr^2}}{a}\omega^{\hat r}\wedge\frac{1}{ar}\omega^{\hat\theta} \\ &= \frac{\ddot a}{a}\omega^{\hat t}\wedge\omega^{\hat\theta} + \dot a\frac{\sqrt{1-kr^2}}{a^2 r}\omega^{\hat r}\wedge\omega^{\hat\theta} \end{aligned}$$

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[8] $= \dot a r\sin\theta\,\omega^{\hat t}\wedge\frac{1}{ar\sin\theta}\omega^{\hat\phi} + a\sin\theta\frac{\sqrt{1-kr^2}}{a}\omega^{\hat r}\wedge\frac{1}{ar\sin\theta}\omega^{\hat\phi} + ar\cos\theta\frac{1}{ar}\omega^{\hat\theta}\wedge\frac{1}{ar\sin\theta}\omega^{\hat\phi} =$

$$\Gamma^{\hat\theta}{}_{\hat c}\wedge\Gamma^{\hat c}{}_{\hat t} = \Gamma^{\hat\theta}{}_{\hat t}\wedge\Gamma^{\hat t}{}_{\hat t} + \Gamma^{\hat\theta}{}_{\hat r}\wedge\Gamma^{\hat r}{}_{\hat t} + \Gamma^{\hat\theta}{}_{\hat\theta}\wedge\Gamma^{\hat\theta}{}_{\hat t} + +\Gamma^{\hat\theta}{}_{\hat\phi}\wedge\Gamma^{\hat\phi}{}_{\hat t}$$
$$= \Gamma^{\hat\theta}{}_{\hat r}\wedge\Gamma^{\hat r}{}_{\hat t}$$
$$= \frac{\dot a\sqrt{1-kr^2}}{a^2 r}\omega^{\hat\theta}\wedge\omega^{\hat r}$$
$$\Rightarrow \quad \Omega^{\hat\theta}{}_{\hat t} = \frac{\ddot a}{a}\omega^{\hat t}\wedge\omega^{\hat\theta}$$

$\Omega^{\hat\phi}{}_{\hat t}$:

$$d\Gamma^{\hat\phi}{}_{\hat t} = d\left(\frac{\dot a}{a}\omega^{\hat\phi}\right)$$
$$= d\left(\frac{\dot a}{a} ar\sin\theta\, d\phi\right)$$
$$= d(\dot a r\sin\theta\, d\phi)$$
$$= \ddot a r\sin\theta\, dt\wedge d\phi + \dot a\sin\theta\, dr\wedge d\phi + \dot a r\cos\theta\, d\theta\wedge d\phi$$
$$= \frac{\ddot a}{a}\omega^{\hat t}\wedge\omega^{\hat\phi} + \dot a\frac{\sqrt{1-kr^2}}{a^2 r}\omega^{\hat r}\wedge\omega^{\hat\phi} + \dot a\frac{\cot\theta}{a^2 r}\omega^{\hat\theta}\wedge\omega^{\hat\phi}$$
$$\Gamma^{\hat\phi}{}_{\hat c}\wedge\Gamma^{\hat c}{}_{\hat t} = \Gamma^{\hat\phi}{}_{\hat t}\wedge\Gamma^{\hat t}{}_{\hat t} + \Gamma^{\hat\phi}{}_{\hat r}\wedge\Gamma^{\hat r}{}_{\hat t} + \Gamma^{\hat\phi}{}_{\hat\theta}\wedge\Gamma^{\hat\theta}{}_{\hat t} + \Gamma^{\hat\phi}{}_{\phi}\wedge\Gamma^{\hat\phi}{}_{\hat t}$$
$$= \Gamma^{\hat\phi}{}_{\hat r}\wedge\Gamma^{\hat r}{}_{\hat t} + \Gamma^{\hat\phi}{}_{\hat\theta}\wedge\Gamma^{\hat\theta}{}_{\hat t}$$
$$= \frac{\dot a\sqrt{1-kr^2}}{a^2 r}\omega^{\hat\phi}\wedge\omega^{\hat r} + \frac{\dot a\cot\theta}{a^2 r}\omega^{\hat\phi}\wedge\omega^{\hat\theta}$$
$$\Rightarrow \quad \Omega^{\hat\phi}{}_{\hat t} = \frac{\ddot a}{a}\omega^{\hat t}\wedge\omega^{\hat\phi}$$

$\Omega^{\hat\theta}{}_{\hat r}$:

$$d\Gamma^{\hat\theta}{}_{\hat r} = d\left(\frac{\sqrt{1-kr^2}}{ar}\omega^{\hat\theta}\right)$$
$$= d\left(\frac{\sqrt{1-kr^2}}{ar} ar d\theta\right)$$
$$= d\left(\sqrt{1-kr^2}d\theta\right)$$
$$= \frac{-kr}{\sqrt{1-kr^2}}dr\wedge d\theta$$
$$= \frac{-k}{a^2}\omega^{\hat r}\wedge\omega^{\hat\theta}$$
$$\Gamma^{\hat\theta}{}_{\hat c}\wedge\Gamma^{\hat c}{}_{\hat r} = \Gamma^{\hat\theta}{}_{\hat r}\wedge\Gamma^{\hat r}{}_{\hat r} + \Gamma^{\hat\theta}{}_{\hat t}\wedge\Gamma^{\hat t}{}_{\hat r} + \Gamma^{\hat\theta}{}_{\hat\theta}\wedge\Gamma^{\hat\theta}{}_{\hat r} + \Gamma^{\hat\theta}{}_{\hat\phi}\wedge\Gamma^{\hat\phi}{}_{\hat r} = \Gamma^{\hat\theta}{}_{\hat t}\wedge\Gamma^{\hat t}{}_{\hat r} = \left(\frac{\dot a}{a}\right)^2\omega^{\hat\theta}\wedge\omega^{\hat r}$$
$$\Rightarrow \quad \Omega^{\hat\theta}{}_{\hat r} = \frac{-(\dot a^2+k)}{a^2}\omega^{\hat r}\wedge\omega^{\hat\theta}$$

$\Omega^{\hat\phi}{}_{\hat r}$:

$$d\Gamma^{\hat\phi}{}_{\hat r} = d\left(\frac{\sqrt{1-kr^2}}{ar}\omega^{\hat\phi}\right)$$
$$= d\left(\sqrt{1-kr^2}\sin\theta\, d\phi\right)$$
$$= \frac{-kr}{\sqrt{1-kr^2}}\sin\theta\, dr\wedge d\phi + \sqrt{1-kr^2}\cos\theta\, d\theta\wedge d\phi$$
$$= \frac{-kr}{a^2 r}\omega^{\hat r}\wedge\omega^{\hat\phi} + \frac{\sqrt{1-kr^2}\cot\theta}{a^2 r^2}\omega^{\hat\theta}\wedge\omega^{\hat\phi}$$
$$\Gamma^{\hat\phi}{}_{\hat c}\wedge\Gamma^{\hat c}{}_{\hat r} = \Gamma^{\hat\phi}{}_{\hat r}\wedge\Gamma^{\hat r}{}_{\hat r} + \Gamma^{\hat\phi}{}_{\hat t}\wedge\Gamma^{\hat t}{}_{\hat r} + \Gamma^{\hat\phi}{}_{\hat\theta}\wedge\Gamma^{\hat\theta}{}_{\hat r} + \Gamma^{\hat\phi}{}_{\hat\phi}\wedge\Gamma^{\hat\phi}{}_{\hat r}$$

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$$= \Gamma^{\hat\phi}{}_{\hat t} \wedge \Gamma^{\hat t}{}_{\hat r} + \Gamma^{\hat\phi}{}_{\hat\theta} \wedge \Gamma^{\hat\theta}{}_{\hat r}$$
$$= -\frac{\dot a^2}{a^2}\omega^{\hat r} \wedge \omega^{\hat\phi} - \frac{\sqrt{1-kr^2}\cot\theta}{a^2r^2}\omega^{\hat\theta} \wedge \omega^{\hat\phi}$$
$$\Rightarrow \quad \Omega^{\hat\phi}{}_{\hat r} = \frac{-(\dot a^2 + k)}{a^2}\omega^{\hat r} \wedge \omega^{\hat\phi}$$

$\Omega^{\hat\phi}{}_{\hat\theta}$:

$$d\Gamma^{\hat\phi}{}_{\hat\theta} = d\left(\frac{\cot\theta}{ar}\omega^{\hat\phi}\right)$$
$$= d\left(\frac{\cot\theta}{ar} ar\sin\theta\, d\phi\right)$$
$$= d(\cos\theta\, d\phi)$$
$$= -\sin\theta\, d\theta \wedge d\phi$$
$$= -\frac{1}{a^2r^2}\omega^{\hat\theta} \wedge \omega^{\hat\phi}$$
$$\Gamma^{\hat\phi}{}_{\hat c} \wedge \Gamma^{\hat c}{}_{\hat\theta} = \Gamma^{\hat\phi}{}_{\hat t} \wedge \Gamma^{\hat t}{}_{\hat\theta} + \Gamma^{\hat\phi}{}_{\hat r} \wedge \Gamma^{\hat r}{}_{\hat\theta} + \Gamma^{\hat\phi}{}_{\hat\theta} \wedge \Gamma^{\hat\theta}{}_{\hat\theta} + \Gamma^{\hat\phi}{}_{\hat\phi} \wedge \Gamma^{\hat\phi}{}_{\hat\theta}$$
$$= \Gamma^{\hat\phi}{}_{\hat t} \wedge \Gamma^{\hat t}{}_{\hat\theta} + \Gamma^{\hat\phi}{}_{\hat r} \wedge \Gamma^{\hat r}{}_{\hat\theta}$$
$$= \left(\frac{\dot a}{a}\right)^2 \omega^{\hat\phi} \wedge \omega^{\hat\theta} - \frac{1-kr^2}{(ar)^2}\omega^{\hat\phi} \wedge \omega^{\hat\theta}$$
$$\Rightarrow \quad \Omega^{\hat\phi}{}_{\hat\theta} = \frac{(\dot a^2 + k)}{a^2}\omega^{\hat\phi} \wedge \omega^{\hat\theta}$$

Summerized in a matrix where $\hat a$ refers to the column and $\hat b$ the row.

$$\Omega^{\hat a}{}_{\hat b} = \begin{Bmatrix} 0 & \frac{\ddot a}{a}\omega^{\hat t} \wedge \omega^{\hat r} & \frac{\ddot a}{a}\omega^{\hat t} \wedge \omega^{\hat\theta} & \frac{\ddot a}{a}\omega^{\hat t} \wedge \omega^{\hat\phi} \\ S & 0 & \frac{-(\dot a^2 + k)}{a^2}\omega^{\hat r} \wedge \omega^{\hat\theta} & \frac{-(\dot a^2 + k)}{a^2}\omega^{\hat r} \wedge \omega^{\hat\phi} \\ S & AS & 0 & \frac{(\dot a^2 + k)}{a^2}\omega^{\hat\phi} \wedge \omega^{\hat\theta} \\ S & AS & AS & 0 \end{Bmatrix}$$

Now we can find the independent elements of the Riemann tensor in the non-coordinate basis:

$$R^{\hat r}{}_{\hat t\hat r\hat t} = -\frac{\ddot a}{a}$$
$$R^{\hat\theta}{}_{\hat t\hat\theta\hat t} = -\frac{\ddot a}{a}$$
$$R^{\hat\theta}{}_{\hat r\hat\theta\hat r} = \frac{(\dot a^2 + k)}{a^2}$$
$$R^{\hat\phi}{}_{\hat t\hat\phi\hat t} = -\frac{\ddot a}{a}$$
$$R^{\hat\phi}{}_{\hat r\hat\phi\hat r} = \frac{(\dot a^2 + k)}{a^2}$$
$$\Gamma^{\hat\phi}{}_{\hat\theta\hat\phi\hat\theta} = \frac{\dot a^2 + k}{a^2}$$

13.4.2 The Ricci tensor

$$R_{\hat a\hat b} = R^{\hat c}{}_{\hat a\hat c\hat b}$$
$$R_{\hat a\hat b} = 0 \qquad \text{If } \hat a \neq \hat b$$
$$R_{\hat t\hat t} = R^{\hat c}{}_{\hat t\hat c\hat t} = R^{\hat t}{}_{\hat t\hat t\hat t} + R^{\hat r}{}_{\hat t\hat r\hat t} + R^{\hat\theta}{}_{\hat t\hat\theta\hat t} + R^{\hat\phi}{}_{\hat t\hat\phi\hat t} = -\frac{\ddot a}{a} - \frac{\ddot a}{a} - \frac{\ddot a}{a} = -3\frac{\ddot a}{a}$$
$$R_{\hat r\hat r} = R^{\hat c}{}_{\hat r\hat c\hat r}$$

$$
\begin{aligned}
&= R^{\hat{t}}{}_{\hat{r}\hat{t}\hat{r}} + R^{\hat{r}}{}_{\hat{r}\hat{r}\hat{r}} + R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} + R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{r}} \\
&= -R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}} + R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} + R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{r}} \\
&= \frac{\ddot{a}}{a} + \frac{(\dot{a}^2+k)}{a^2} + \frac{(\dot{a}^2+k)}{a^2} \\
&= \frac{\ddot{a}}{a} + 2\frac{(\dot{a}^2+k)}{a^2} \\
R_{\hat{\theta}\hat{\theta}} &= R^{\hat{c}}{}_{\hat{\theta}\hat{c}\hat{\theta}} \\
&= R^{\hat{t}}{}_{\hat{\theta}\hat{t}\hat{\theta}} + R^{\hat{r}}{}_{\hat{\theta}\hat{r}\hat{\theta}} + R^{\hat{\theta}}{}_{\hat{\theta}\hat{\theta}\hat{\theta}} + R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{\theta}} \\
&= -R^{\hat{\theta}}{}_{\hat{t}\hat{\theta}\hat{t}} + R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} + R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{\theta}} \\
&= \frac{\ddot{a}}{a} + 2\frac{(\dot{a}^2+k)}{a^2} \\
R_{\hat{\phi}\hat{\phi}} &= \frac{\ddot{a}}{a} + 2\frac{(\dot{a}^2+k)}{a^2}
\end{aligned}
$$

Summarized in a matrix where $\hat{a}$ refers to column and $\hat{b}$ to row

$$
R_{\hat{a}\hat{b}} = \begin{Bmatrix} -3\frac{\ddot{a}}{a} & 0 & 0 & 0 \\ 0 & \frac{\ddot{a}}{a} + 2\frac{(\dot{a}^2+k)}{a^2} & 0 & 0 \\ 0 & 0 & \frac{\ddot{a}}{a} + 2\frac{(\dot{a}^2+k)}{a^2} & 0 \\ 0 & 0 & 0 & \frac{\ddot{a}}{a} + 2\frac{(\dot{a}^2+k)}{a^2} \end{Bmatrix}
$$

13.4.3 The Einstein tensor and Friedmann-equations

<u>The Ricci scalar:</u>

$$
\begin{aligned}
R &= \eta^{\hat{a}\hat{b}} R_{\hat{a}\hat{b}} \\
&= -R_{\hat{t}\hat{t}} + R_{\hat{r}\hat{r}} + R_{\hat{\theta}\hat{\theta}} + R_{\hat{\phi}\hat{\phi}} \\
&= -R^{\hat{c}}{}_{\hat{t}\hat{c}\hat{t}} + R^{\hat{c}}{}_{\hat{r}\hat{c}\hat{r}} + R^{\hat{c}}{}_{\hat{\theta}\hat{c}\hat{\theta}} + R^{\hat{c}}{}_{\hat{\phi}\hat{c}\hat{\phi}} \\
&= {}^{9} - 2R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}} - 2R^{\hat{\theta}}{}_{\hat{t}\hat{\theta}\hat{t}} - 2R^{\hat{\phi}}{}_{\hat{t}\hat{\phi}\hat{t}} + 2R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} + 2R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{r}} + 2R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{\theta}} \\
&= 2\frac{\ddot{a}}{a} + 2\frac{\ddot{a}}{a} + 2\frac{\ddot{a}}{a} + 2\frac{(\dot{a}^2+k)}{a^2} + 2\frac{(\dot{a}^2+k)}{a^2} + 2\frac{(\dot{a}^2+k)}{a^2} \\
&= \left(\frac{\ddot{a}}{a} + \frac{(\dot{a}^2+k)}{a^2}\right)
\end{aligned}
$$

<u>The Einstein tensor:</u>

$$
\begin{aligned}
G_{\hat{a}\hat{b}} &= R_{\hat{a}\hat{b}} - \frac{1}{2}\eta_{\hat{a}\hat{b}} R \\
G_{\hat{t}\hat{t}} &= R_{\hat{t}\hat{t}} - \frac{1}{2}\eta_{\hat{t}\hat{t}} R \\
&= {}^{10} R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} + R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{r}} + R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{\theta}}
\end{aligned}
$$

---

[9] $= -R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}} - R^{\hat{\theta}}{}_{\hat{t}\hat{\theta}\hat{t}} - R^{\hat{\phi}}{}_{\hat{t}\hat{\phi}\hat{t}} + R^{\hat{t}}{}_{\hat{r}\hat{t}\hat{r}} + R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} + R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{r}} + R^{\hat{t}}{}_{\hat{\theta}\hat{t}\hat{\theta}} + R^{\hat{r}}{}_{\hat{\theta}\hat{r}\hat{\theta}} + R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{\theta}} + R^{\hat{t}}{}_{\hat{\phi}\hat{t}\hat{\phi}} + R^{\hat{r}}{}_{\hat{\phi}\hat{r}\hat{\phi}} + R^{\hat{\theta}}{}_{\hat{\phi}\hat{\theta}\hat{\phi}} =$

[10] $= R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}} + R^{\hat{\theta}}{}_{\hat{t}\hat{\theta}\hat{t}} + R^{\hat{\phi}}{}_{\hat{t}\hat{\phi}\hat{t}} + \frac{1}{2}\left(-2R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}} - 2R^{\hat{\theta}}{}_{\hat{t}\hat{\theta}\hat{t}} - 2R^{\hat{\phi}}{}_{\hat{t}\hat{\phi}\hat{t}} + 2R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} + 2R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{r}} + 2R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{\theta}}\right) =$

$$= 3\left(\frac{(\dot{a}^2 + \mathrm{k})}{\mathrm{a}^2}\right)$$

$$G_{\hat{r}\hat{t}} = R_{\hat{r}\hat{t}} - \frac{1}{2}\eta_{\hat{r}\hat{t}}R = R^{\hat{c}}{}_{\hat{r}\hat{c}\hat{t}} = 0$$

$$G_{\hat{\theta}\hat{t}} = G_{\hat{\phi}\hat{t}} = G_{\hat{t}\hat{r}} = G_{\hat{\theta}\hat{r}} = G_{\hat{\phi}\hat{r}} = G_{\hat{t}\hat{\theta}} = G_{\hat{r}\hat{\theta}} = G_{\hat{\phi}\hat{\theta}} = G_{\hat{t}\hat{\phi}} = G_{\hat{r}\hat{\phi}} = G_{\hat{\theta}\hat{\phi}} = 0$$

$$G_{\hat{r}\hat{r}} = R_{\hat{r}\hat{r}} - \frac{1}{2}\eta_{\hat{r}\hat{r}}R$$

$$= {}^{11}R^{\hat{t}}{}_{\hat{r}\hat{t}\hat{r}} + R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} + R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{r}} + R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}} + R^{\hat{\theta}}{}_{\hat{t}\hat{\theta}\hat{t}} + R^{\hat{\phi}}{}_{\hat{t}\hat{\phi}\hat{t}} - R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} - R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{r}} - R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{\theta}}$$

$$= R^{\hat{\theta}}{}_{\hat{t}\hat{\theta}\hat{t}} + R^{\hat{\phi}}{}_{\hat{t}\hat{\phi}\hat{t}} - R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{\theta}}$$

$$= -\frac{\ddot{a}}{a} - \frac{\ddot{a}}{a} - \frac{\dot{a}^2 + k}{a^2}$$

$$= -\left(\frac{2\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2}\right)$$

$$G_{\hat{\theta}\hat{\theta}} = R_{\hat{\theta}\hat{\theta}} - \frac{1}{2}\eta_{\hat{\theta}\hat{\theta}}R$$

$$= {}^{12}R^{\hat{t}}{}_{\hat{\theta}\hat{t}\hat{\theta}} + R^{\hat{r}}{}_{\hat{\theta}\hat{r}\hat{\theta}} + R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{\theta}} + R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}} + R^{\hat{\theta}}{}_{\hat{t}\hat{\theta}\hat{t}} + R^{\hat{\phi}}{}_{\hat{t}\hat{\phi}\hat{t}} - R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} - R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{r}} - R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{\theta}}$$

$$= R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}} + R^{\hat{\phi}}{}_{\hat{t}\hat{\phi}\hat{t}} - R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{r}}$$

$$= -\frac{\ddot{a}}{a} - \frac{\ddot{a}}{a} - \frac{\dot{a}^2 + k}{a^2}$$

$$= -\left(\frac{2\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2}\right)$$

$$G_{\hat{\phi}\hat{\phi}} = R_{\hat{\phi}\hat{\phi}} - \frac{1}{2}\eta_{\hat{\phi}\hat{\phi}}R$$

$$= {}^{13}R^{\hat{t}}{}_{\hat{\phi}\hat{t}\hat{\phi}} + R^{\hat{r}}{}_{\hat{\phi}\hat{r}\hat{\phi}} + R^{\hat{\theta}}{}_{\hat{\phi}\hat{\theta}\hat{\phi}} + R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}} + R^{\hat{\theta}}{}_{\hat{t}\hat{\theta}\hat{t}} + R^{\hat{\phi}}{}_{\hat{t}\hat{\phi}\hat{t}} - R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} - R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{r}} - R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{\theta}}$$

$$= R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}} + R^{\hat{\theta}}{}_{\hat{t}\hat{\theta}\hat{t}} - R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}}$$

$$= -\frac{\ddot{a}}{a} - \frac{\ddot{a}}{a} - \frac{\dot{a}^2 + k}{a^2}$$

$$= -\left(\frac{2\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2}\right)$$

Summarized in a matrix where $\hat{a}$ refers to column and $\hat{b}$ to row

$$G_{\hat{a}\hat{b}} = \begin{Bmatrix} 3\left(\frac{(\dot{a}^2 + \mathrm{k})}{\mathrm{a}^2}\right) & 0 & 0 & 0 \\ 0 & -\left(\frac{2\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2}\right) & 0 & 0 \\ 0 & 0 & -\left(\frac{2\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2}\right) & 0 \\ 0 & 0 & 0 & -\left(\frac{2\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2}\right) \end{Bmatrix}$$

[11] $= R^{\hat{t}}{}_{\hat{r}\hat{t}\hat{r}} + R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} + R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{r}} - \frac{1}{2}\left(-2R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}} - 2R^{\hat{\theta}}{}_{\hat{t}\hat{\theta}\hat{t}} - 2R^{\hat{\phi}}{}_{\hat{t}\hat{\phi}\hat{t}} + 2R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} + 2R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{r}} + 2R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{\theta}}\right) =$

[12] $= R^{\hat{t}}{}_{\hat{\theta}\hat{t}\hat{\theta}} + R^{\hat{r}}{}_{\hat{\theta}\hat{r}\hat{\theta}} + R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{\theta}} - \frac{1}{2}\left(-2R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}} - 2R^{\hat{\theta}}{}_{\hat{t}\hat{\theta}\hat{t}} - 2R^{\hat{\phi}}{}_{\hat{t}\hat{\phi}\hat{t}} + 2R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} + 2R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{r}} + 2R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{\theta}}\right) =$

[13] $= R^{\hat{t}}{}_{\hat{\phi}\hat{t}\hat{\phi}} + R^{\hat{r}}{}_{\hat{\phi}\hat{r}\hat{\phi}} + R^{\hat{\theta}}{}_{\hat{\phi}\hat{\theta}\hat{\phi}} - \frac{1}{2}\left(-2R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}} - 2R^{\hat{\theta}}{}_{\hat{t}\hat{\theta}\hat{t}} - 2R^{\hat{\phi}}{}_{\hat{t}\hat{\phi}\hat{t}} + 2R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} + 2R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{r}} + 2R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{\theta}}\right) =$

13.4.3.1 Calculationg the Einstein tensor – Alternative version.

The line element:

$$ds^2 = -dt^2 + \frac{a^2(t)}{1-kr^2}dr^2 + a^2(t)r^2 d\theta^2 + a^2(t)r^2 \sin^2\theta \, d\phi^2$$

Now we can compare with the Tolman-Bondi – de Sitter line element, where the primes should not be mistaken for the derivative d/dr.

$$ds^2 = {}^{14}dt'^2 - e^{-2\psi(t',r')}dr'^2 - R^2(t',r')d\theta'^2 - R^2(t',r')\sin^2\theta' \, d\phi'^2$$

And chose:

$$\begin{aligned}
dt' &= dt \\
e^{-\psi(t',r')}dr' &= \frac{a(t)}{\sqrt{1-kr^2}}dr \\
R(t',r')d\theta' &= a(t)rd\theta \\
R(t',r')\sin\theta' \, d\phi' &= a(t)r\sin\theta \, d\phi
\end{aligned}$$

Comparing the two metrics we see: $d\phi' = d\phi, d\theta' = d\theta, \theta' = \theta, R(t',r') = a(t)r, dt' = dt$

Next we can use the former calculations of the Tolman-Bondi – de Sitter metric to find the Einstein tensor for the Robertson-Walker metric.

But first we need to find

$$\begin{aligned}
\dot{\psi} &= \frac{d\psi(t',r')}{dt'} \\
&= e^{-\psi(t',r')}\frac{d}{dt'}\left(e^{\psi(t',r')}\right) \\
&= \frac{a(t)}{\sqrt{1-kr^2}}\frac{dr}{dr'}\frac{d}{dt}\left(\frac{\sqrt{1-kr^2}}{a(t)}\frac{dr'}{dr}\right) \\
&= -\frac{\dot{a}(t)}{a(t)} \\
\ddot{\psi} &= \frac{d}{dt'}\left(-\frac{\dot{a}(t)}{a(t)}\right) \\
&= \frac{d}{dt}\left(-\frac{\dot{a}(t)}{a(t)}\right) \\
&= -\frac{\left(\ddot{a}(t)a(t) - \dot{a}(t)\dot{a}(t)\right)}{a(t)^2} \\
&= \left(\frac{\dot{a}(t)}{a(t)}\right)^2 - \frac{\ddot{a}(t)}{a(t)} \\
\psi' &= \frac{d\psi(t',r')}{dr'} \\
&= e^{-\psi(t',r')}\frac{d}{dr'}\left(e^{\psi(t',r')}\right) \\
&= e^{-\psi(t',r')}\frac{dr}{dr'}\frac{d}{dr}\left(\frac{\sqrt{1-kr^2}}{a(t)}\frac{dr'}{dr}\right) \\
&= -\frac{kr}{a(t)\sqrt{1-kr^2}}e^{-\psi(t',r')} \\
\dot{R} &= \frac{dR(t',r')}{dt'} = \frac{da(t)r}{dt} = \dot{a}(t)r \\
\ddot{R} &= \frac{d\dot{a}(t)r}{dt'} = \frac{d\dot{a}(t)r}{dt} = \ddot{a}(t)r
\end{aligned}$$

[14] Notice: the calculations were made for a metric with negative signature. This has no influence on the calculation of the Einstein tensor.

$$
\begin{aligned}
R' &= \frac{dR(t',r')}{dr'} \\
&= \frac{dr}{dr'}\frac{d}{dr}(a(t)r) \\
&= \frac{\sqrt{1-kr^2}}{a(t)}e^{-\psi(t',r')}a(t) \\
&= \sqrt{1-kr^2}e^{-\psi(t',r')} \\
R'' &= \frac{d}{dr'}\left(\sqrt{1-kr^2}e^{-\psi(t',r')}\right) = \frac{dr}{dr'}\frac{d}{dr}\left(\sqrt{1-kr^2}\frac{a(t)}{\sqrt{1-kr^2}}\frac{dr}{dr'}\right) = 0 \\
\dot{R}' &= \frac{d}{dr'}(\dot{a}(t)r) = \frac{dr}{dr'}\frac{d}{dr}(\dot{a}(t)r) = \frac{\dot{a}(t)\sqrt{1-kr^2}}{a(t)}e^{-\psi(t',r')}
\end{aligned}
$$

The Einstein tensor:

Tolman –Bondi – de Sitter Robertson-Walker

$$
\begin{aligned}
G_{\hat{t}\hat{t}} &= \frac{1}{R^2}\left[1 - 2R\dot{R}\dot{\psi} + (\dot{R})^2 - (2RR'' + 2RR'\psi' + (R')^2)e^{2\psi(t,r)}\right] &\Rightarrow\quad G_{\hat{t}\hat{t}} &= 3\frac{\dot{a}(t)^2 + k}{a(t)^2} \\
G_{\hat{r}\hat{t}} &= -2\left[(\dot{R})' + R'\dot{\psi}\right]\frac{e^{\psi(t,r)}}{R} &\Rightarrow\quad G_{\hat{r}\hat{t}} &= 0 \\
G_{\hat{r}\hat{r}} &= \frac{1}{R^2}\left[(R')^2e^{2\psi(t,r)} - 2R\ddot{R} - 1 - (\dot{R})^2\right] &\Rightarrow\quad G_{\hat{r}\hat{r}} &= -\left(2\frac{\ddot{a}(t)}{a(t)} + \frac{\dot{a}(t)^2 + k}{a(t)^2}\right) \\
G_{\hat{\theta}\hat{\theta}} &= \left[\ddot{\psi} - (\dot{\psi})^2\right] + \frac{1}{R}\left[(R'' + R'\psi')e^{2\psi(t,r)} + \dot{R}\dot{\psi} - \ddot{R}\right] &\Rightarrow\quad G_{\hat{\theta}\hat{\theta}} &= -\left(2\frac{\ddot{a}(t)}{a(t)} + \frac{\dot{a}(t)^2 + k}{a(t)^2}\right) \\
G_{\hat{\phi}\hat{\phi}} &= \left[\ddot{\psi} - (\dot{\psi})^2\right] + \frac{1}{R}\left[(R'' + R'\psi')e^{2\psi(t,r)} + \dot{R}\dot{\psi} - \ddot{R}\right] &\Rightarrow\quad G_{\hat{\phi}\hat{\phi}} &= -\left(2\frac{\ddot{a}(t)}{a(t)} + \frac{\dot{a}(t)^2 + k}{a(t)^2}\right)
\end{aligned}
$$

13.4.4 The Friedmann equations:

Given the Einstein equation (if $c = G = 1$):

$$G_{\hat{a}\hat{b}} + \Lambda\eta_{\hat{a}\hat{b}} = 8\pi T_{\hat{a}\hat{b}}$$

The stress-energy tensor:

$$T_{\hat{a}\hat{b}} = 8\pi\begin{Bmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{Bmatrix}$$

You can find the Friedmann- equations

$$\begin{Bmatrix} 3\left(\frac{(\dot{a}^2+\mathrm{k})}{\mathrm{a}^2}\right) & 0 & 0 & 0 \\ 0 & -\left(\frac{2\ddot{a}}{a} + \frac{\dot{a}^2+k}{a^2}\right) & 0 & 0 \\ 0 & 0 & -\left(\frac{2\ddot{a}}{a} + \frac{\dot{a}^2+k}{a^2}\right) & 0 \\ 0 & 0 & 0 & -\left(\frac{2\ddot{a}}{a} + \frac{\dot{a}^2+k}{a^2}\right) \end{Bmatrix} + \Lambda\begin{Bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{Bmatrix} = 8\pi\begin{Bmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{Bmatrix}$$

$$
\begin{aligned}
\frac{3}{a^2}(k + \dot{a}^2) - \Lambda &= 8\pi\rho \\
2\frac{\ddot{a}}{a} + \frac{1}{a^2}(k + \dot{a}^2) - \Lambda &= -8\pi P
\end{aligned}
$$

13.4.5 [g]Manipulating the Friedmann equations.

Show that the two Friedman equations

$$\frac{3}{a^2}(k+\dot{a}^2)-\Lambda = 8\pi\rho \quad \text{(i)}$$

$$2\frac{\ddot{a}}{a}+\frac{1}{a^2}(k+\dot{a}^2)-\Lambda = -8\pi P \quad \text{(ii)}$$

can be manipulated into:

$$\frac{d}{dt}(\rho a^3)+P\frac{d}{dt}(a^3) = 0$$

Rewriting (i):

$$8\pi\rho = \frac{3}{a^2}(k+\dot{a}^2)-\Lambda$$

$$\Rightarrow \quad 8\pi\rho a^3 = 3a(k+\dot{a}^2)-\Lambda a^3$$

$$\Rightarrow \quad 8\pi\frac{d}{dt}(\rho a^3) = \frac{d}{dt}(3a(k+\dot{a}^2)-\Lambda a^3) = 3\dot{a}(k+\dot{a}^2)+6a\dot{a}\ddot{a}-3\Lambda a^2\dot{a}$$

Rewriting (ii):

$$-8\pi P = 2\frac{\ddot{a}}{a}+\frac{1}{a^2}(k+\dot{a}^2)-\Lambda$$

$$\Rightarrow \quad -8\pi P\frac{d}{dt}(a^3) = \left(2\frac{\ddot{a}}{a}+\frac{1}{a^2}(k+\dot{a}^2)-\Lambda\right)\frac{d}{dt}(a^3)$$

$$= \left(2\frac{\ddot{a}}{a}+\frac{1}{a^2}(k+\dot{a}^2)-\Lambda\right)3a^2\dot{a}$$

$$= 6a\dot{a}\ddot{a}+3\dot{a}(k+\dot{a}^2)-3\Lambda a^2\dot{a}$$

$$\Leftrightarrow \quad 8\pi P\frac{d}{dt}(a^3) = -(6a\dot{a}\ddot{a}+3\dot{a}(k+\dot{a}^2)-3\Lambda a^2\dot{a})$$

Now adding

$$8\pi\frac{d}{dt}(\rho a^3)+8\pi P\frac{d}{dt}(a^3) = 3\dot{a}(k+\dot{a}^2)+6a\dot{a}\ddot{a}-3\Lambda a^2\dot{a}-(6a\dot{a}\ddot{a}+3\dot{a}(k+\dot{a}^2)-3\Lambda a^2\dot{a}) = 0$$

Q.E.D.

13.4.6 [h]Parameters in an flat universe with positive cosmological constant

Rewrite (i) and use the change of variable: $u=\frac{2\Lambda}{3C}a^3$

$$\dot{a}^2 = \frac{C}{a}+\frac{\Lambda}{3}a^2$$

$$\Rightarrow \quad \left(\frac{\dot{a}}{a}\right)^2 = \frac{C}{a^3}+\frac{\Lambda}{3}$$

and

$$u = \frac{2\Lambda}{3C}a^3$$

$$\Rightarrow \quad \dot{u} = \frac{2\Lambda}{C}\dot{a}a^2 = 3\frac{u}{a}\dot{a}$$

$$\Rightarrow \quad \frac{\dot{u}}{u} = 3\frac{\dot{a}}{a}$$

Rearranging we get

$$\left(\frac{1}{3}\frac{\dot{u}}{u}\right)^2 = \frac{2\Lambda}{3u}+\frac{\Lambda}{3}$$

$$\Rightarrow \quad \dot{u} = 3\sqrt{\frac{2\Lambda}{3}u+\frac{\Lambda}{3}u^2} = \sqrt{3\Lambda}\sqrt{u}\sqrt{u+2}$$

$$\Rightarrow \quad \int dt = \int\frac{du}{\sqrt{3\Lambda}\sqrt{u}\sqrt{u+2}}$$

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$$\Rightarrow \quad t - t_0 = {}^{15}\frac{1}{\sqrt{3\Lambda}}\ln\left(\sqrt{u+2}+\sqrt{u}\right)^2$$
$$= \frac{1}{\sqrt{3\Lambda}}\ln\left(2u+2+2\sqrt{u}\sqrt{u+2}\right)$$
$$= \frac{1}{\sqrt{3\Lambda}}\ln\left(2u+2+2\sqrt{u^2+2u}\right)$$
$$= \frac{1}{\sqrt{3\Lambda}}\ln\left(2u+2+2\sqrt{(u+1)^2-1}\right)$$
$$= \frac{1}{\sqrt{3\Lambda}}\ln 2\left(u+1+\sqrt{(u+1)^2-1}\right)$$
$$= \frac{1}{\sqrt{3\Lambda}}\left(\ln 2 + \ln\left(u+1+\sqrt{(u+1)^2-1}\right)\right)$$
$$= \frac{1}{\sqrt{3\Lambda}}2\ln\left(\sqrt{u+2}+\sqrt{u}\right)$$
$$= {}^{16}\frac{1}{\sqrt{3\Lambda}}\left(\ln 2 + \cosh^{-1}(u+1)\right)$$
$$\Rightarrow \quad u = \cosh\left(\sqrt{3\Lambda}(t-t_0)-\ln 2\right)-1$$
$$\Rightarrow \quad a^3 = \frac{3C}{2\Lambda}\left[\cosh\left(\sqrt{3\Lambda}(t-t_0)-\ln 2\right)-1\right]$$

Leaving out the constants of integration $-\sqrt{3\Lambda}t_0 - \ln 2$ we get

$$a^3 = \frac{3C}{2\Lambda}\left[\cosh\left(\sqrt{3\Lambda}t\right)-1\right]$$

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[a] (Greene, s. 516) note 10
[b] (McMahon, Relativity Demystified, 2006, s. 262)
[c] (McMahon, Relativity Demystified, 2006, s. 267)
[d] (Weinberg, 1979, s. 157)
[e] (McMahon, Relativity Demystified, 2006, s. 161), (Hartle, 2003, s. 547)
[f] (McMahon, Relativity Demystified, 2006, p. 116)
[g] (McMahon, Relativity Demystified, 2006, s. 165)
[h] (McMahon, Relativity Demystified, 2006, s. 278)

[15] $\int \frac{dx}{\sqrt{(ax+b)(px+q)}} = \frac{2}{\sqrt{ap}}\ln\left(\sqrt{a(px+q)}+p(ax+b)\right)$ (14.280) (Spiegel, 1990)
[16] $\cosh^{-1}x = \ln\left(x+\sqrt{x^2-1}\right)$ (8.56) (Spiegel, 1990)