Content

Space-time		Line-element	Chapter
Example – three dimensional space	ds^2	$= (u^2 + v^2)du^2 + (u^2 + v^2)dv^2 + u^2v^2d\theta^2$	6
Example – three dimensional space	ds^2	$= dx^2 + 2xdy^2 + 2ydz^2$	6
Example – two dimensional space	ds^2	$= y^2 \sin x \, dx^2 + x^2 \tan y \, dy^2$	6

6 The Riemann and Ricci tensor and the Ricci scalar

6.1 [a]The Rieman tensor

6.1.1 Definitions

$$R^a{}_{bcd} \equiv {}^1\partial_c\Gamma^a{}_{bd} - \partial_d\Gamma^a{}_{bc} + \Gamma^e{}_{bd}\Gamma^a{}_{ec} - \Gamma^e{}_{bc}\Gamma^a{}_{ed}$$
$$R_{abcd} \equiv {}^2 g_{ae}R^e{}_{bcd}$$
$$R_{abcd} = {}^3\partial_c\Gamma_{bda} - \partial_d\Gamma_{bca} + \Gamma^e{}_{bc}\Gamma_{ade} - \Gamma^e{}_{bd}\Gamma_{ace}$$
$$= \frac{1}{2}\left(\frac{\partial^2 g_{ad}}{\partial x^c\partial x^b} + \frac{\partial^2 g_{bc}}{\partial x^d\partial x^a} - \frac{\partial^2 g_{bd}}{\partial x^c\partial x^a} - \frac{\partial^2 g_{ac}}{\partial x^d\partial x^b}\right) + \Gamma^e{}_{bc}\Gamma_{ade} - \Gamma^e{}_{bd}\Gamma_{ace}$$

6.1.2 Proofs and Properties

We need

$$\nabla_c g_{ab} = \partial_c g_{ab} - g_{db}\Gamma^d{}_{ac} - g_{ad}\Gamma^d{}_{cb} = \partial_c g_{ab} - \Gamma_{acb} - \Gamma_{cba} = 0$$
$$\Rightarrow R_{abcd} = g_{ae}R^e{}_{bcd} = g_{ae}\left(\partial_c\Gamma^e{}_{bd} - \partial_d\Gamma^e{}_{bc} + \Gamma^f{}_{bd}\Gamma^e{}_{fc} - \Gamma^f{}_{bc}\Gamma^e{}_{fd}\right)$$
$$= g_{ae}\partial_c\Gamma^e{}_{bd} - g_{ae}\partial_d\Gamma^e{}_{bc} + g_{ae}\Gamma^f{}_{bd}\Gamma^e{}_{fc} - g_{ae}\Gamma^f{}_{bc}\Gamma^e{}_{fd}$$
$$= \partial_c(g_{ae}\Gamma^e{}_{bd}) - \Gamma^e{}_{bd}\partial_c g_{ae} - \partial_d(g_{ae}\Gamma^e{}_{bc}) + \Gamma^e{}_{bc}\partial_d g_{ae} + \Gamma^f{}_{bd}\Gamma_{fca} - \Gamma^f{}_{bc}\Gamma_{fda}$$
$$= {}^{4\,5}\partial_c(\Gamma_{bda}) - \partial_d(\Gamma_{bca}) - \Gamma^e{}_{bd}\Gamma_{ace} + \Gamma^e{}_{bc}\Gamma_{ade} \qquad \text{QED}$$
$$\Rightarrow R_{abcd} = \partial_c\Gamma_{bda} - \partial_d\Gamma_{bca} + \Gamma^e{}_{bc}\Gamma_{ade} - \Gamma^e{}_{bd}\Gamma_{ace}$$
$$= {}^{6\,7}\frac{1}{2}\left(\frac{\partial^2 g_{ad}}{\partial x^c\partial x^b} + \frac{\partial^2 g_{bc}}{\partial x^d\partial x^a} - \frac{\partial^2 g_{bd}}{\partial x^c\partial x^a} - \frac{\partial^2 g_{ac}}{\partial x^d\partial x^b}\right) + \Gamma^e{}_{bc}\Gamma_{ade} - \Gamma^e{}_{bd}\Gamma_{ace} \qquad \text{QED}$$
$$\Rightarrow R_{cdab} = \frac{1}{2}\left(\frac{\partial^2 g_{cb}}{\partial x^a\partial x^d} + \frac{\partial^2 g_{da}}{\partial x^b\partial x^c} - \frac{\partial^2 g_{db}}{\partial x^a\partial x^c} - \frac{\partial^2 g_{ca}}{\partial x^b\partial x^d}\right) + \Gamma^e{}_{da}\Gamma_{cbe} - \Gamma^e{}_{db}\Gamma_{cae}$$
$$= \frac{1}{2}\left(\frac{\partial^2 g_{cb}}{\partial x^a\partial x^d} + \frac{\partial^2 g_{da}}{\partial x^b\partial x^c} - \frac{\partial^2 g_{db}}{\partial x^a\partial x^c} - \frac{\partial^2 g_{ca}}{\partial x^b\partial x^d}\right) + g^{ef}\Gamma_{daf}g_{he}\Gamma^h{}_{cb} - \Gamma^e{}_{db}\Gamma_{cae}$$
$$= \frac{1}{2}\left(\frac{\partial^2 g_{cb}}{\partial x^a\partial x^d} + \frac{\partial^2 g_{da}}{\partial x^b\partial x^c} - \frac{\partial^2 g_{db}}{\partial x^a\partial x^c} - \frac{\partial^2 g_{ca}}{\partial x^b\partial x^d}\right) + \delta^f_h\Gamma_{daf}\Gamma^h{}_{cb} - \Gamma^e{}_{db}\Gamma_{cae}$$
$$= \frac{1}{2}\left(\frac{\partial^2 g_{cb}}{\partial x^a\partial x^d} + \frac{\partial^2 g_{da}}{\partial x^b\partial x^c} - \frac{\partial^2 g_{db}}{\partial x^a\partial x^c} - \frac{\partial^2 g_{ca}}{\partial x^b\partial x^d}\right) + \Gamma_{dae}\Gamma^e{}_{cb} - \Gamma^e{}_{db}\Gamma_{cae} = R_{abcd}$$
$$\Rightarrow R_{abdc} = \frac{1}{2}\left(\frac{\partial^2 g_{ac}}{\partial x^d\partial x^b} + \frac{\partial^2 g_{bd}}{\partial x^c\partial x^a} - \frac{\partial^2 g_{bc}}{\partial x^d\partial x^a} - \frac{\partial^2 g_{ad}}{\partial x^c\partial x^b}\right) + \Gamma^e{}_{bd}\Gamma_{ace} - \Gamma^e{}_{bc}\Gamma_{ade} = -R_{abcd}$$
$$\Rightarrow R_{bacd} = \frac{1}{2}\left(\frac{\partial^2 g_{bd}}{\partial x^c\partial x^a} + \frac{\partial^2 g_{ac}}{\partial x^d\partial x^b} - \frac{\partial^2 g_{ad}}{\partial x^c\partial x^b} - \frac{\partial^2 g_{bc}}{\partial x^d\partial x^a}\right) + \Gamma^e{}_{ac}\Gamma_{bde} - \Gamma^e{}_{ad}\Gamma_{bce}$$
$$= {}^8\frac{1}{2}\left(\frac{\partial^2 g_{bd}}{\partial x^c\partial x^a} + \frac{\partial^2 g_{ac}}{\partial x^d\partial x^b} - \frac{\partial^2 g_{ad}}{\partial x^c\partial x^b} - \frac{\partial^2 g_{bc}}{\partial x^d\partial x^a}\right) + \delta^f_h\Gamma_{acf}\Gamma^h{}_{bd} - \delta^i_j\Gamma_{adi}\Gamma^j{}_{bc}$$

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[1] The Riemann tensor of first kind

[2] Raising and lowering of the first index

[3] The Riemann tensor of second kind

[4] $= \partial_c(\Gamma_{bda}) - \partial_d(\Gamma_{bca}) - \Gamma^e{}_{bd}(\Gamma_{ace} + \Gamma_{cea}) + \Gamma^e{}_{bc}(\Gamma_{ade} + \Gamma_{dea}) + \Gamma^f{}_{bd}\Gamma_{fca} - \Gamma^f{}_{bc}\Gamma_{fda} =$

[5] $= \partial_c(\Gamma_{bda}) - \partial_d(\Gamma_{bca}) - \Gamma^e{}_{bd}(\Gamma_{ace} + \Gamma_{cea}) + \Gamma^e{}_{bc}(\Gamma_{ade} + \Gamma_{dea}) + \Gamma^e{}_{bd}\Gamma_{eca} - \Gamma^e{}_{bc}\Gamma_{eda} =$

[6] $= \frac{1}{2}\partial_c\left(\frac{\partial g_{ab}}{\partial x^d} + \frac{\partial g_{ad}}{\partial x^b} - \frac{\partial g_{bd}}{\partial x^a}\right) - \frac{1}{2}\partial_d\left(\frac{\partial g_{ab}}{\partial x^c} + \frac{\partial g_{ac}}{\partial x^b} - \frac{\partial g_{bc}}{\partial x^a}\right) + \Gamma^e{}_{bc}\Gamma_{ade} - \Gamma^e{}_{bd}\Gamma_{ace} =$

[7] $= \frac{1}{2}\left(\frac{\partial^2 g_{ab}}{\partial x^c\partial x^d} + \frac{\partial^2 g_{ad}}{\partial x^c\partial x^b} - \frac{\partial^2 g_{bd}}{\partial x^c\partial x^a}\right) - \frac{1}{2}\left(\frac{\partial^2 g_{ab}}{\partial x^d\partial x^c} + \frac{\partial^2 g_{ac}}{\partial x^d\partial x^b} - \frac{\partial^2 g_{bc}}{\partial x^d\partial x^a}\right)\Gamma_{bca} + \Gamma^e{}_{bc}\Gamma_{ade} - \Gamma^e{}_{bd}\Gamma_{ace} =$

[8] $= \frac{1}{2}\left(\frac{\partial^2 g_{bd}}{\partial x^c\partial x^a} + \frac{\partial^2 g_{ac}}{\partial x^d\partial x^b} - \frac{\partial^2 g_{ad}}{\partial x^c\partial x^b} - \frac{\partial^2 g_{bc}}{\partial x^d\partial x^a}\right) + g^{ef}\Gamma_{acf}g_{eh}\Gamma^h{}_{bd} - g^{ei}\Gamma_{adi}g_{ej}\Gamma^j{}_{bc} =$

$$= \frac{1}{2}\left(\frac{\partial^2 g_{bd}}{\partial x^c \partial x^a} + \frac{\partial^2 g_{ac}}{\partial x^d \partial x^b} - \frac{\partial^2 g_{ad}}{\partial x^c \partial x^b} - \frac{\partial^2 g_{bc}}{\partial x^d \partial x^a}\right) + \Gamma_{ace}\Gamma^e{}_{bd} - \Gamma_{ade}\Gamma^e{}_{bc} = -R_{abcd}$$

$$R_{aaaa} = \partial_a\Gamma_{aaa} - \partial_a\Gamma_{aaa} + \Gamma^e{}_{aa}\Gamma_{aae} - \Gamma^e{}_{aa}\Gamma_{aae} = 0$$

$$R_{abbb} = \partial_b\Gamma_{bba} - \partial_b\Gamma_{bba} + \Gamma^e{}_{bb}\Gamma_{abe} - \Gamma^e{}_{bb}\Gamma_{abe} = 0$$

$$R_{aacc} = \partial_c\Gamma_{aca} - \partial_c\Gamma_{aca} + \Gamma^e{}_{ac}\Gamma_{ace} - \Gamma^e{}_{ac}\Gamma_{ace} = 0$$

$$R_{bbcd} = \frac{1}{2}\left(\frac{\partial^2 g_{bd}}{\partial x^c \partial x^b} + \frac{\partial^2 g_{bc}}{\partial x^d \partial x^b} - \frac{\partial^2 g_{bd}}{\partial x^c \partial x^b} - \frac{\partial^2 g_{bc}}{\partial x^d \partial x^b}\right) + \Gamma^e{}_{bc}\Gamma_{bde} - \Gamma^e{}_{bd}\Gamma_{bce}$$

$$= g^{ef}\Gamma_{bcf}g_{eh}\Gamma^h{}_{bd} - \Gamma^e{}_{bd}\Gamma_{bce} = \delta^f_h\Gamma_{bcf}\Gamma^h{}_{bd} - \Gamma^e{}_{bd}\Gamma_{bce} = \Gamma_{bce}\Gamma^e{}_{bd} - \Gamma^e{}_{bd}\Gamma_{bce}$$

$$= 0$$

$$R_{abcd} = \partial_c\Gamma_{bda} - \partial_d\Gamma_{bca} + \Gamma^e{}_{bc}\Gamma_{ade} - \Gamma^e{}_{bd}\Gamma_{ace}$$

$$R_{adbc} = \partial_b\Gamma_{dca} - \partial_c\Gamma_{dba} + \Gamma^e{}_{db}\Gamma_{ace} - \Gamma^e{}_{dc}\Gamma_{abe}$$

$$R_{acdb} = \partial_d\Gamma_{cba} - \partial_b\Gamma_{cda} + \Gamma^e{}_{cd}\Gamma_{abe} - \Gamma^e{}_{cb}\Gamma_{ade}$$

$\Rightarrow$ $R_{abcd} + R_{adbc} + R_{acdb} =$ [9] 0

$$\Rightarrow \quad R^a{}_{bcd} + R^a{}_{dbc} + R^a{}_{cdb} = g^{ae}(R_{ebcd} + R_{edbc} + R_{ecdb}) = 0$$

$$R^a{}_{bcd} = g^{ad}R_{dbcd} = -g^{ad}R_{dbdc} = -R^a{}_{bdc}$$

$$R^a{}_{abc} = g^{ad}R_{dabc} = -g^{ad}R_{adbc} = -g^{ad}g_{af}R^f{}_{dbc} = -\delta^d_f R^f{}_{dbc} = -R^d{}_{dbc}$$

$\Rightarrow$ $R^a{}_{abc} =$ [b] 0

<u>Summarizing</u>

$$R_{abcd} = \partial_c\Gamma_{abd} - \partial_d\Gamma_{abc} + \Gamma_{ead}\Gamma^e{}_{bc} - \Gamma_{eac}\Gamma^e{}_{bd}$$

$$R_{abcd} = \frac{1}{2}\left(\frac{\partial^2 g_{ad}}{\partial x^c \partial x^b} + \frac{\partial^2 g_{bc}}{\partial x^d \partial x^a} - \frac{\partial^2 g_{ac}}{\partial x^d \partial x^b} - \frac{\partial^2 g_{bd}}{\partial x^c \partial x^a}\right) - \Gamma_{eac}\Gamma^e{}_{bd} + \Gamma_{ead}\Gamma^e{}_{bc}$$

$$R_{abcd} = R_{cdab} = -R_{abdc} = -R_{bacd}$$

$$R_{aaaa} = R_{baaa} = R_{aaba} = R_{aabb} = R_{aabc} = R_{abcc} = 0$$

$$R_{abcd} + R_{adbc} + R_{acdb} = 0$$

$$R^a{}_{bcd} + R^a{}_{dbc} + R^a{}_{cdb} = 0$$

$$R^a{}_{bcd} = -R^a{}_{bdc}$$

$$R^a{}_{abc} = 0$$

6.1.3 Independent elements in the Riemann, Ricci and Weyl tensor

In n dimensions, there are $N_{Riemann} = n^2(n^2-1)/12$ independent elements in the Riemann tensor. In the Ricci tensor there are $N_{Ricci} = n(n+1)/2$ independent elements, and in the Weyl tensor there are $N_{Weyl} = 10$ independent elements if $n = 4$, if $n < 4$ there are none. Summarized:

	$N_{Riemann} = \frac{n^2(n^2-1)}{12}$		$N_{Ricci} = \frac{n(n+1)}{2}$		N_{Weyl}
$n = 2$	1	R_{abab}	3	R_{aa} R_{ab} R_{bb}	0
$n = 3$	6	R_{abab} R_{abac} R_{abbc} R_{acac} R_{acbc} R_{bcbc}	6	R_{aa} R_{bb} R_{bc} R_{ac} R_{ab} R_{cc}	0

[9] $= \partial_c\Gamma_{bda} - \partial_d\Gamma_{bca} + \Gamma^e{}_{bc}\Gamma_{ade} - \Gamma^e{}_{bd}\Gamma_{ace} + \partial_b\Gamma_{dca} - \partial_c\Gamma_{dba} + \Gamma^e{}_{db}\Gamma_{ace} - \Gamma^e{}_{dc}\Gamma_{abe} + \partial_d\Gamma_{cba} - \partial_b\Gamma_{cda} + \Gamma^e{}_{cd}\Gamma_{abe} - \Gamma^e{}_{cb}\Gamma_{ade} =$

$n=4$	20	${}^{10}R_{abcd}$	10	$R_{ab} = R^{c}{}_{acb}$	10	C_{abcd}
		R_{abab}		R_{aa}		C_{abbc}
		R_{abac}		R_{bb}		C_{abbd}
		R_{abad}		R_{bc}		C_{abcd}
		R_{abbc}		R_{bd}		C_{acbc}
		R_{abbd}		R_{ac}		C_{acbd}
		R_{abcd}		R_{ad}		C_{accd}
		R_{acac}		R_{cc}		~~C_{adbc}~~[12]
		R_{acad}		R_{ab}		C_{adbd}
		R_{acbc}		R_{cd}		C_{adcd}
		R_{acbd}		R_{dd}		C_{bccd}
		R_{accd}				C_{bdcd}
		R_{adad}				
		~~R_{adbc}~~[11]				
		R_{adbd}				
		R_{adcd}				
		R_{bcbc}				
		R_{bcbd}				
		R_{bccd}				
		R_{bdbd}				
		R_{bdcd}				
		R_{cdcd}				

6.2 [c]The Ricci Tensor

6.2.1 Definition

$$
\begin{aligned}
& R_{ab} &&= R^{c}{}_{acb} \\
\Rightarrow\ & R_{ba} &&= R^{c}{}_{bca} = g^{cd} R_{dbca} = g^{cd} R_{cadb} = R^{d}{}_{adb} = R_{ab}
\end{aligned}
$$

6.2.2 The Ricci Tensor of diagonal space in Three Dimensions

The line element

$$ds^2 = g_{xx}dx^2 + g_{yy}dy^2 + g_{zz}dz^2$$

The metric tensor and its inverse

$$g_{ab} = \begin{Bmatrix} g_{xx} & & \\ & g_{yy} & \\ & & g_{zz} \end{Bmatrix}$$

$$g^{ab} = \begin{Bmatrix} \dfrac{1}{g_{xx}} & & \\ & \dfrac{1}{g_{yy}} & \\ & & \dfrac{1}{g_{zz}} \end{Bmatrix}$$

In 3 dimensions the Riemann tensor has six independent elements:
$R_{xyxy}; R_{xyxz}; R_{xyyz}; R_{xzxz}; R_{xzyz}; R_{yzyz}$

[10] Notice: In the case of a diagonal metric, the Riemann tensor elements with raised index has the same properties. E.g. $R^{a}{}_{bab}, R^{a}{}_{bac}, R^{a}{}_{bad}$ etc. are independent, $R^{a}{}_{abb} = 0$, $R^{a}{}_{bcc} = 0$ etc.

[11] Because: $R_{abcd} + R_{acdb} + R_{adbc} = 0$

[12] Because: $C_{abcd} + C_{acdb} + C_{adbc} = 0$

The Ricci Tensor

$$R_{ab} = R^c{}_{acb}$$

$$\Rightarrow R_{xx} = R^c{}_{xcx} = R^x{}_{xxx} + R^y{}_{xyx} + R^z{}_{xzx} = g^{xx}R_{xxxx} + g^{yy}R_{yxyx} + g^{zz}R_{zxzx}$$
$$= g^{yy}R_{xyxy} + g^{zz}R_{xzxz}$$
$$R_{xy} = R^c{}_{xcy} = R^x{}_{xxy} + R^y{}_{xyy} + R^z{}_{xzy} = g^{xx}R_{xxxy} + g^{yy}R_{yxyy} + g^{zz}R_{zxzy} = g^{zz}R_{xzyz}$$
$$R_{xz} = R^c{}_{xcz} = R^x{}_{xxz} + R^y{}_{xyz} + R^z{}_{xzz} = g^{xx}R_{xxxz} + g^{yy}R_{yxyz} + g^{zz}R_{zxzz} = -g^{yy}R_{xyyz}$$
$$R_{yy} = R^c{}_{ycy} = R^x{}_{yxy} + R^y{}_{yyy} + R^z{}_{yzy} = g^{xx}R_{xyxy} + g^{yy}R_{yyy} + g^{zz}R_{zyzy}$$
$$= g^{xx}R_{xyxy} + g^{zz}R_{yzyz}$$
$$R_{yz} = R^c{}_{ycz} = R^x{}_{yxz} + R^y{}_{yyz} + R^z{}_{yzz} = g^{xx}R_{xyxz} + g^{yy}R_{yyyz} + g^{zz}R_{zyzz} = g^{xx}R_{xyxz}$$
$$R_{zz} = R^c{}_{zcz} = R^x{}_{zxz} + R^y{}_{zyz} + R^z{}_{zzz} = g^{xx}R_{xzxz} + g^{yy}R_{yzyz} + g^{zz}R_{zzzz}$$
$$= g^{xx}R_{xzxz} + g^{yy}R_{yzyz}$$

Summerized in a matrix, where a refers to column and b to row.

$$R_{ab} = \begin{Bmatrix} g^{yy}R_{xyxy} + g^{zz}R_{xzxz} & g^{zz}R_{xzyz} & -g^{yy}R_{xyyz} \\ S & g^{xx}R_{xyxy} + g^{zz}R_{yzyz} & g^{xx}R_{xyxz} \\ S & S & g^{xx}R_{xzxz} + g^{yy}R_{yzyz} \end{Bmatrix}$$

6.3 [d]The Ricci scalar

6.3.1 The Ricci scalar of two-dimensional space

The line-element

$$ds^2 = g_{xx}dx^2 + g_{yy}dy$$

A 2-dimensional diagonal metric has only one independent element in the Riemann tensor

$R_{abab} = R_{baba}$ (no summation)

The Ricci scalar

$$R = R^a{}_a = g^{ab}R_{ab} = g^{11}R_{11} + g^{22}R_{22} = g^{11}R^c{}_{1c1} + g^{22}R^c{}_{2c2}$$
$$= g^{11}R^1{}_{111} + g^{22}R^1{}_{212} + g^{11}R^2{}_{121} + g^{22}R^2{}_{222}$$
$$= g^{11}g^{11}R_{1111} + g^{22}g^{11}R_{1212} + g^{11}g^{22}R_{2121} + g^{22}g^{22}R_{2222}$$
$$= g^{22}g^{11}R_{1212} + g^{11}g^{22}R_{2121} = 2g^{11}R^2{}_{121} = 2g^{22}R^1{}_{212}$$

6.3.2 The Ricci Scalar of diagonal space in Three Dimensions

The line-element

$$ds^2 = g_{xx}dx^2 + g_{yy}dy^2 + g_{zz}dz^2$$

The Ricci scalar:

$$R = R^a{}_a = {}^{13}g^{aa}R_{aa} = g^{xx}R_{xx} + g^{yy}R_{yy} + g^{zz}R_{zz}$$
$$= g^{xx}(g^{yy}R_{xyxy} + g^{zz}R_{xzxz}) + g^{yy}(g^{xx}R_{xyxy} + g^{zz}R_{yzyz}) + g^{zz}(g^{xx}R_{xzxz} + g^{yy}R_{yzyz})$$
$$= 2g^{xx}g^{yy}R_{xyxy} + 2g^{xx}g^{zz}R_{xzxz} + 2g^{yy}g^{zz}R_{yzyz}$$
$$= 2g^{xx}g^{yy}g^{zz}(g_{zz}R_{xyxy} + g_{yy}R_{xzxz} + g_{xx}R_{yzyz})$$

6.4 [e]The Weyl tensor

Definition:

$$C_{abcd} = R_{abcd} + \frac{1}{2}(g_{ad}R_{cb} + g_{bc}R_{da} - g_{ac}R_{db} - g_{bd}R_{ca}) + \frac{1}{6}(g_{ac}g_{db} - g_{ad}g_{cb})R$$

Properties:

The Weyl tensor possesses the same symmetries as the Riemann tensor: $C_{abcd} = -C_{abdc} = -C_{bacd} = C_{cdab}$ and $C_{abcd} + C_{adbc} + C_{acdb} = 0$. It possesses an additional symmetry: $C^c{}_{acb} = 0$. It follows that the

[13] Notice: The general formula is $R = g^{ab}R_{ab}$, which in the diagonal case reduces to $R = g^{aa}R_{aa}$

Weyl tensor is trace-free, in other words, it vanishes for any pair of contracted indices. One can think of the Weyl tensor as that part of the curvature tensor for which all contractions vanish. The Weyl tensor is the same for a given metric g_{ab} and a metric that is conformally related to it $g'_{ab} = f(x) g_{ab}$. To put it a bit more informally we can say that the Weyl tensor is the same as the Riemann tensor but with the Ricci tensor parts removed. This means that, as the Ricci tensor represents the source[14], the Weyl tensor represents the pure gravitational part.

6.5 [f]Contractions: The Metric tensor, the Riemann tensor, the Ricci tensor and the Ricci scalar.

Contraction of the metric tensor with itself:

$$\begin{aligned} g_{ac} g^{cb} &= \delta_a^b \\ g_{ab} g^{ab} &= 4 \end{aligned}$$

Contracting the metric tensor with the Riemann tensor to get the Ricci tensor

$$\begin{aligned} g^{ab} R_{abcd} &= g^{ab} R_{cdab} = R^{b}{}_{bcd} = 0 \text{ if the metric is diagonal, otherwise not.} \\ g^{ab} R_{acbd} &= g^{ab} R_{bdac} = R^{a}{}_{cad} = R_{cd} \\ g^{ab} R_{adbc} &= g^{ab} R_{bcad} = R^{b}{}_{dbc} = R_{cd} \end{aligned}$$

Contracting the metric tensor twice with the Riemann tensor to get the Ricci scalar

$$\begin{aligned} g^{cd} g^{ab} R_{abcd} &= 0 \text{ if the metric is diagonal, otherwise not.} \\ g^{cd} g^{ab} R_{acbd} &= g^{cd} R_{cd} = R^{d}{}_{d} \\ g^{cd} g^{ab} R_{adbc} &= g^{cd} R_{cd} = R^{d}{}_{d} \end{aligned}$$

6.6 [g]A metric example 1: $ds^2 = y^2 \sin x \, dx^2 + x^2 \tan y \, dy^2$

The line element:

$$ds^2 = y^2 \sin x \, dx^2 + x^2 \tan y \, dy^2$$

The metric tensor and its inverse:

$$g_{ab} = \begin{Bmatrix} y^2 \sin x & \\ & x^2 \tan y \end{Bmatrix}$$

$$g^{ab} = \begin{Bmatrix} \dfrac{1}{y^2 \sin x} & \\ & \dfrac{1}{x^2 \tan y} \end{Bmatrix}$$

6.6.1 The Christoffel symbols

<u>The Christoffel symbols of first kind</u> / <u>Christoffel symbols of the second kind</u>

$$\Gamma_{abc} = \frac{1}{2}(\partial_a g_{bc} + \partial_b g_{ca} - \partial_c g_{ab}) \qquad \Gamma^{a}{}_{bc} = g^{ad}\Gamma_{bcd}$$

$$\Gamma_{xxx} = \frac{1}{2}\partial_x g_{xx} = \frac{1}{2}\partial_x(y^2 \sin x) = \frac{1}{2} y^2 \cos x \quad \Rightarrow \quad \Gamma^{x}{}_{xx} = g^{xx}\Gamma_{xxx} =^{15} \frac{1}{2}\cot x$$

$$\Gamma_{yyy} = \frac{1}{2}\partial_y g_{yy} = \frac{1}{2}\partial_y(x^2 \tan y) = \frac{1}{2} x^2(1 + \tan^2 y) \quad \Rightarrow \quad \Gamma^{y}{}_{yy} = g^{yy}\Gamma_{yyy} =^{16} \frac{1 + \tan^2 y}{2 \tan y}$$

$$\Gamma_{xxy} = -\frac{1}{2}\partial_y g_{xx} = -\frac{1}{2}\partial_y(y^2 \sin x) = -y \sin x \quad \Rightarrow \quad \Gamma^{y}{}_{xx} = g^{yy}\Gamma_{xxy} =^{17} -\frac{y \sin x}{x^2 \tan y}$$

[14] See later chapters on the Einstein Equations with source.

[15] $= \frac{1}{y^2 \sin x}\frac{1}{2} y^2 \cos x =$

[16] $= \frac{1}{x^2 \tan y}\frac{1}{2} x^2 (1 + \tan^2 y) =$

[17] $= \frac{1}{x^2 \tan y}(-y \sin x) =$

$$\begin{aligned}
\Gamma_{yyx} &= -\frac{1}{2}\partial_x g_{yy} = -\frac{1}{2}\partial_x(x^2\tan y) = -x\tan y &&\Rightarrow\quad \Gamma^x{}_{yy} = g^{xx}\Gamma_{yyx} =^{18} -\frac{x\tan y}{y^2\sin x}\\
\Gamma_{xyy} &= \Gamma_{yxy} = \frac{1}{2}\partial_x g_{yy} = \frac{1}{2}\partial_x(x^2\tan y) = x\tan y &&\Rightarrow\quad \Gamma^y{}_{xy} = \Gamma^y{}_{yx} = g^{yy}\Gamma_{xyy} =^{19} \frac{1}{x}\\
\Gamma_{yxx} &= \Gamma_{xyx} = \frac{1}{2}\partial_y g_{xx} = \frac{1}{2}\partial_y(y^2\sin x) = y\sin x &&\Rightarrow\quad \Gamma^x{}_{yx} = \Gamma^x{}_{xy} = g^{xx}\Gamma_{yxx} =^{20} \frac{1}{y}
\end{aligned}$$

6.6.2 The Ricci scalar

A 2-dimensional diagonal metric has only one independent element in the Riemann tensor

$R_{xyxy} = R_{yxyx}$

The Ricci scalar

$$\begin{aligned}
\Rightarrow\qquad R &= 2g^{yy}R^x{}_{yxy}\\
R^x{}_{yxy} &= \partial_x\Gamma^x{}_{yy} - \partial_y\Gamma^x{}_{yx} + \Gamma^e{}_{yy}\Gamma^x{}_{ex} - \Gamma^e{}_{yx}\Gamma^x{}_{ey}\\
&= \partial_x\Gamma^x{}_{yy} - \partial_y\Gamma^x{}_{yx} + \Gamma^x{}_{yy}\Gamma^x{}_{xx} - \Gamma^x{}_{yx}\Gamma^x{}_{xy} + \Gamma^y{}_{yy}\Gamma^x{}_{yx} - \Gamma^y{}_{yx}\Gamma^x{}_{yy}\\
&=^{21\ 22} \left(\frac{x\tan y\cos x}{y^2\sin^2 x}\right) - \left(\frac{1}{2}\frac{x}{y^2}\frac{\tan y\cos x}{\sin^2 x}\right) + \left(\frac{1+\tan^2 y}{2y\tan y}\right)\\
&= \left(\frac{x\cos x\tan^2 y + y\sin^2 x + y\sin^2 x\tan^2 y}{2y^2\sin^2 x\tan y}\right)\\
\Rightarrow\qquad R &= 2g^{yy}R^x{}_{yxy} = 2\left(\frac{1}{x^2\tan y}\right)\left(\frac{x\cos x\tan^2 y + y\sin^2 x + y\sin^2 x\tan^2 y}{2y^2\sin^2 x\tan y}\right)\\
&= \left(\frac{x\cos x\tan^2 y + y\sin^2 x + y\sin^2 x\tan^2 y}{x^2y^2\sin^2 x\tan^2 y}\right)
\end{aligned}$$

6.7 [h]A metric example 3: $ds^2 = (u^2+v^2)du^2 + (u^2+v^2)dv^2 + u^2v^2d\theta^2$

The line element:

$$ds^2 = (u^2+v^2)du^2 + (u^2+v^2)dv^2 + u^2v^2d\theta^2$$

The metric tensor and its inverse:

$$g_{ab} = \begin{Bmatrix} u^2+v^2 & 0 & 0\\ 0 & u^2+v^2 & 0\\ 0 & 0 & u^2v^2 \end{Bmatrix}$$

$$g^{ab} = \begin{Bmatrix} \frac{1}{u^2+v^2} & & \\ & \frac{1}{u^2+v^2} & \\ & & \frac{1}{u^2v^2} \end{Bmatrix}$$

6.7.1 The Christoffel symbols

<u>The Christoffel symbols of first kind</u>

$$\Gamma_{abc} = \frac{1}{2}(\partial_a g_{bc} + \partial_b g_{ca} - \partial_c g_{ab})$$

<u>Christoffel symbols of the second kind</u>

$$\Gamma^a{}_{bc} = g^{ad}\Gamma_{bcd}$$

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[18] $= \frac{1}{y^2\sin x}(-x\tan y) =$

[19] $= \frac{1}{x^2\tan y}x\tan y =$

[20] $= \frac{1}{y^2\sin x}y\sin x =$

[21] $= \partial_x\left(-\frac{x\tan y}{y^2\sin x}\right) - \partial_y\left(\frac{1}{y}\right) + \left(-\frac{x\tan y}{y^2\sin x}\right)\left(\frac{1}{2}\cot x\right) - \left(\frac{1}{y}\right)^2 + \left(\frac{1+\tan^2 y}{2\tan y}\right)\left(\frac{1}{y}\right) - \left(\frac{1}{x}\right)\left(-\frac{x\tan y}{y^2\sin x}\right) =$

[22] $= -\frac{\tan y}{y^2}\left(\frac{1}{\sin x} - \frac{x\cos x}{\sin^2 x}\right) + \left(\frac{1}{y}\right)^2 - \left(\frac{1}{2}\frac{x}{y^2}\frac{\tan y\cos x}{\sin^2 x}\right) - \left(\frac{1}{y}\right)^2 + \left(\frac{1+\tan^2 y}{2y\tan y}\right) + \left(\frac{\tan y}{y^2\sin x}\right) =$

$$\begin{aligned}
\Gamma_{uuu} &= \frac{1}{2}\partial_u g_{uu} = \frac{1}{2}\partial_u(u^2+v^2) = u && \Rightarrow \Gamma^u{}_{uu} &= g^{uu}\Gamma_{uuu} = \frac{u}{(u^2+v^2)} \\
\Gamma_{vvv} &= \frac{1}{2}\partial_v g_{vv} = \frac{1}{2}\partial_v(u^2+v^2) = v && \Rightarrow \Gamma^v{}_{vv} &= g^{vv}\Gamma_{vvv} = \frac{v}{(u^2+v^2)} \\
\Gamma_{uuv} &= -\frac{1}{2}\partial_v g_{uu} = -\frac{1}{2}\partial_v(u^2+v^2) = -v && \Rightarrow \Gamma^v{}_{uu} &= g^{vv}\Gamma_{uuv} = -\frac{v}{(u^2+v^2)} \\
\Gamma_{vvu} &= -\frac{1}{2}\partial_u g_{vv} = -\frac{1}{2}\partial_u(u^2+v^2) = -u && \Rightarrow \Gamma^u{}_{vv} &= g^{uu}\Gamma_{vvu} = -\frac{u}{(u^2+v^2)} \\
\Gamma_{\theta\theta u} &= -\frac{1}{2}\partial_u g_{\theta\theta} = -\frac{1}{2}\partial_u(u^2v^2) = -uv^2 && \Rightarrow \Gamma^u{}_{\theta\theta} &= g^{uu}\Gamma_{\theta\theta u} = -\frac{uv^2}{(u^2+v^2)} \\
\Gamma_{\theta\theta v} &= -\frac{1}{2}\partial_v g_{\theta\theta} = -\frac{1}{2}\partial_v(u^2v^2) = -u^2v && \Rightarrow \Gamma^v{}_{\theta\theta} &= g^{vv}\Gamma_{\theta\theta v} = -\frac{u^2v}{(u^2+v^2)} \\
\Gamma_{uvv} &= \Gamma_{vuv} = \frac{1}{2}\partial_u g_{vv} = u && \Rightarrow \Gamma^v{}_{uv} &= \Gamma^v{}_{vu} = g^{vv}\Gamma_{uvv} = \frac{u}{(u^2+v^2)} \\
\Gamma_{u\theta\theta} &= \Gamma_{\theta u\theta} = \frac{1}{2}\partial_u g_{\theta\theta} = uv^2 && \Rightarrow \Gamma^\theta{}_{u\theta} &= \Gamma^\theta{}_{\theta u} = g^{\theta\theta}\Gamma_{u\theta\theta} = \frac{uv^2}{u^2v^2} = \frac{1}{u} \\
\Gamma_{vuu} &= \Gamma_{uvu} = \frac{1}{2}\partial_v g_{uu} = v && \Rightarrow \Gamma^u{}_{vu} &= \Gamma^u{}_{uv} = g^{uu}\Gamma_{vuu} = \frac{v}{(u^2+v^2)} \\
\Gamma_{v\theta\theta} &= \Gamma_{\theta v\theta} = \frac{1}{2}\partial_v g_{\theta\theta} = u^2v && \Rightarrow \Gamma^\theta{}_{v\theta} &= \Gamma^\theta{}_{\theta v} = g^{\theta\theta}\Gamma_{v\theta\theta} = \frac{u^2v}{u^2v^2} = \frac{1}{v}
\end{aligned}$$

6.7.2 The Riemann tensor

The Riemann tensor has six independent elements: $R_{uvuv}, R_{uvu\theta}, R_{uvv\theta}, R_{u\theta u\theta}, R_{u\theta v\theta}, R_{v\theta v\theta}$

$$\begin{aligned}
R_{abcd} &= \partial_c\Gamma_{bda} - \partial_d\Gamma_{bca} + \Gamma^e{}_{bc}\Gamma_{ade} - \Gamma^e{}_{bd}\Gamma_{ace} \\
\Rightarrow R_{uvuv} &= \partial_u\Gamma_{vvu} - \partial_v\Gamma_{vuu} + \Gamma^e{}_{vu}\Gamma_{uve} - \Gamma^e{}_{vv}\Gamma_{uue} \\
&= -2 + \left(\frac{v}{(u^2+v^2)}\right)\cdot v - \left(-\frac{u}{(u^2+v^2)}\right)u + \left(\frac{u}{(u^2+v^2)}\right)u - \left(\frac{v}{(u^2+v^2)}\right)(-v) = {}^{23}0 \\
R_{uvu\theta} &= \partial_u\Gamma_{v\theta u} - \partial_\theta\Gamma_{vuu} + \Gamma^e{}_{vu}\Gamma_{u\theta e} - \Gamma^e{}_{v\theta}\Gamma_{uue} \\
&= \Gamma^u{}_{vu}\Gamma_{u\theta u} - \Gamma^u{}_{v\theta}\Gamma_{uuu} + \Gamma^v{}_{vu}\Gamma_{u\theta v} - \Gamma^v{}_{v\theta}\Gamma_{uuv} + \Gamma^\theta{}_{vu}\Gamma_{u\theta\theta} - \Gamma^\theta{}_{v\theta}\Gamma_{uu\theta} = 0 \\
R_{uvv\theta} &= \partial_v\Gamma_{v\theta u} - \partial_\theta\Gamma_{vvu} + \Gamma^e{}_{vv}\Gamma_{u\theta e} - \Gamma^e{}_{v\theta}\Gamma_{uve} \\
&= \Gamma^u{}_{vv}\Gamma_{u\theta u} - \Gamma^u{}_{v\theta}\Gamma_{uvu} + \Gamma^v{}_{vv}\Gamma_{u\theta v} - \Gamma^v{}_{v\theta}\Gamma_{uvv} + \Gamma^\theta{}_{vv}\Gamma_{u\theta\theta} - \Gamma^\theta{}_{v\theta}\Gamma_{uv\theta} = 0 \\
R_{u\theta u\theta} &= \partial_u\Gamma_{\theta\theta u} - \partial_\theta\Gamma_{\theta uu} + \Gamma^e{}_{\theta u}\Gamma_{u\theta e} - \Gamma^e{}_{\theta\theta}\Gamma_{uue} \\
&= \partial_u(-uv^2) + \Gamma^u{}_{\theta u}\Gamma_{u\theta u} - \Gamma^u{}_{\theta\theta}\Gamma_{uuu} + \Gamma^v{}_{\theta u}\Gamma_{u\theta v} - \Gamma^v{}_{\theta\theta}\Gamma_{uuv} + \Gamma^\theta{}_{\theta u}\Gamma_{u\theta\theta} - \Gamma^\theta{}_{\theta\theta}\Gamma_{uu\theta} \\
&= -v^2 - \Gamma^u{}_{\theta\theta}\Gamma_{uuu} - \Gamma^v{}_{\theta\theta}\Gamma_{uuv} + \Gamma^\theta{}_{\theta u}\Gamma_{u\theta\theta} \\
&= -v^2 - \left(-\frac{uv^2}{(u^2+v^2)}\right)u - \left(-\frac{u^2v}{(u^2+v^2)}\right)(-v) + \frac{1}{u}\cdot uv^2 = 0 \\
R_{u\theta v\theta} &= \partial_v\Gamma_{\theta\theta u} - \partial_\theta\Gamma_{\theta vu} + \Gamma^e{}_{\theta v}\Gamma_{u\theta e} - \Gamma^e{}_{\theta\theta}\Gamma_{uve} \\
&= \partial_v(-uv^2) + \Gamma^u{}_{\theta v}\Gamma_{u\theta u} - \Gamma^u{}_{\theta\theta}\Gamma_{uvu} + \Gamma^v{}_{\theta v}\Gamma_{u\theta v} - \Gamma^v{}_{\theta\theta}\Gamma_{uvv} + \Gamma^\theta{}_{\theta v}\Gamma_{u\theta\theta} - \Gamma^\theta{}_{\theta\theta}\Gamma_{uv\theta} \\
&= -2uv - \Gamma^u{}_{\theta\theta}\Gamma_{uvu} - \Gamma^v{}_{\theta\theta}\Gamma_{uvv} + \Gamma^\theta{}_{\theta v}\Gamma_{u\theta\theta} \\
&= -2uv - \left(-\frac{uv^2}{(u^2+v^2)}\right)v - \left(-\frac{u^2v}{(u^2+v^2)}\right)u + \frac{1}{v}uv^2 \\
&= uv\left(-2 + \frac{v^2}{(u^2+v^2)} + \frac{u^2}{(u^2+v^2)} + 1\right) = 0 \\
R_{v\theta v\theta} &= \partial_v\Gamma_{\theta\theta v} - \partial_\theta\Gamma_{\theta vv} + \Gamma^e{}_{\theta v}\Gamma_{v\theta e} - \Gamma^e{}_{\theta\theta}\Gamma_{vve} \\
&= \partial_v(-u^2v) + \Gamma^u{}_{\theta v}\Gamma_{v\theta u} - \Gamma^u{}_{\theta\theta}\Gamma_{vvu} + \Gamma^v{}_{\theta v}\Gamma_{v\theta v} - \Gamma^v{}_{\theta\theta}\Gamma_{vvv} + \Gamma^\theta{}_{\theta v}\Gamma_{v\theta\theta} - \Gamma^\theta{}_{\theta\theta}\Gamma_{vv\theta} \\
&= -u^2 - \Gamma^u{}_{\theta\theta}\Gamma_{vvu} - \Gamma^v{}_{\theta\theta}\Gamma_{vvv} + \Gamma^\theta{}_{\theta v}\Gamma_{v\theta\theta}
\end{aligned}$$

[23] $= \partial_u(-u) - \partial_v(v) + \Gamma^u{}_{vu}\Gamma_{uvu} - \Gamma^u{}_{vv}\Gamma_{uuu} + \Gamma^v{}_{vu}\Gamma_{uvv} - \Gamma^v{}_{vv}\Gamma_{uuv} + \Gamma^\theta{}_{vu}\Gamma_{uv\theta} - \Gamma^\theta{}_{vv}\Gamma_{uu\theta} =$

$$= -u^2 - \left(-\frac{uv^2}{(u^2+v^2)}\right)(-u) - \left(-\frac{u^2v}{(u^2+v^2)}\right)v + \frac{1}{v}u^2v = 0$$

6.7.2.1 The Riemann tensor – [i]Alternative version

$R_{abcd} = 0$ for the following reason: Consider the global Minkowski spacetime $ds^2 = du'^2 + dv'^2 + d\theta'^2$, in some coordinates (u', v', θ') [the corresponding Riemannian curvature tensor identically vanishing, of course], and consider the following coordinate transformation:

$$\begin{aligned} u' &= uv\cos\theta \\ v' &= uv\sin\theta \\ \theta' &= \frac{1}{2}(u^2 - v^2) \end{aligned}$$

The differentials are related as

$$\begin{aligned} du' &= (udv + vdu)\cos\theta - uv\sin\theta\, d\theta \\ dv' &= (udv + vdu)\sin\theta + uv\cos\theta\, d\theta \\ d\theta' &= udu - vdv \end{aligned}$$

from which it readily follows that

$$\begin{aligned} du'^2 + dv'^2 + d\theta'^2 &= (udv + vdu)^2 + (uv)^2 d\theta^2 + (udu - vdv)^2 \\ &= (u^2 + v^2)(du^2 + dv^2) + u^2v^2 d\theta^2 \end{aligned}$$

Thus the line element $ds^2 = (u^2 + v^2)(du^2 + dv^2) + u^2v^2 d\theta^2$ must correspond to an identically vanishing Riemannian curvature tensor.

6.8 [j]Metric example 4: A three-dimensional spacetime $ds^2 = dx^2 + 2xdy^2 + 2ydz^2$

The line element

$$ds^2 = dx^2 + 2xdy^2 + 2ydz^2$$

The metric tensor and its inverse

$$g_{ab} = \begin{Bmatrix} 1 & & \\ & 2x & \\ & & 2y \end{Bmatrix}$$

$$g^{ab} = \begin{Bmatrix} 1 & & \\ & \frac{1}{2x} & \\ & & \frac{1}{2y} \end{Bmatrix}$$

6.8.1 The Christoffel symbols, the Riemann tensor, Ricci tensor and scalar

<u>The Christoffel symbols</u>

$$\begin{aligned}
\Gamma_{abc} &= \frac{1}{2}(\partial_a g_{bc} + \partial_b g_{ca} - \partial_c g_{ab}) & \Gamma^a{}_{bc} &= \frac{1}{2}g^{ad}(\partial_c g_{db} + \partial_b g_{dc} - \partial_d g_{bc}) \\
\Gamma_{xyy} &= \Gamma_{yxy} = \frac{1}{2}\partial_x g_{yy} = \frac{1}{2}\partial_x(2x) = 1 \quad \Rightarrow & \Gamma^y{}_{xy} &= \Gamma^y{}_{yx} = g^{yy}\Gamma_{xyy} = \frac{1}{2x} \\
\Gamma_{yzz} &= \Gamma_{zyz} = \frac{1}{2}\partial_y g_{zz} = \frac{1}{2}\partial_y(2y) = 1 \quad \Rightarrow & \Gamma^z{}_{yz} &= \Gamma^z{}_{zy} = g^{zz}\Gamma_{yzz} = \frac{1}{2y} \\
\Gamma_{yyx} &= -\frac{1}{2}\partial_x g_{yy} = -\frac{1}{2}\partial_x(2x) = -1 \quad \Rightarrow & \Gamma^x{}_{yy} &= g^{xx}\Gamma_{yyx} = -1 \\
\Gamma_{zzy} &= -\frac{1}{2}\partial_y g_{zz} = -\frac{1}{2}\partial_y(2y) = -1 \quad \Rightarrow & \Gamma^y{}_{zz} &= g^{yy}\Gamma_{zzy} = -\frac{1}{2x}
\end{aligned}$$

<u>The Riemann tensor</u>

In 3 dimensions the Riemann tensor has six independent elements:

$R_{xyxy}; R_{xyxz}; R_{xyyz}; R_{xzxz}; R_{xzyz}; R_{yzyz}$

$$R_{abcd} = \partial_c\Gamma_{bda} - \partial_d\Gamma_{bca} + \Gamma^e{}_{bc}\Gamma_{ade} - \Gamma^e{}_{bd}\Gamma_{ace}$$

$$R_{xyxy} = \partial_x\Gamma_{yyx} - \partial_y\Gamma_{yxx} + \Gamma^e{}_{yx}\Gamma_{xye} - \Gamma^e{}_{yy}\Gamma_{xxe} = \Gamma^y{}_{yx}\Gamma_{xyy} = \frac{1}{2x}$$
$$R_{xyxz} = \partial_x\Gamma_{yzx} - \partial_z\Gamma_{yxx} + \Gamma^e{}_{yx}\Gamma_{xze} - \Gamma^e{}_{yz}\Gamma_{xxe} = 0$$
$$R_{xyyz} = \partial_y\Gamma_{yzx} - \partial_z\Gamma_{yyx} + \Gamma^e{}_{yy}\Gamma_{xze} - \Gamma^e{}_{yz}\Gamma_{xye} = 0$$
$$R_{xzxz} = \partial_x\Gamma_{zzx} - \partial_z\Gamma_{zxz} + \Gamma^e{}_{zx}\Gamma_{xze} - \Gamma^e{}_{zz}\Gamma_{xxe} = 0$$
$$R_{xzyz} = \partial_y\Gamma_{zzx} - \partial_z\Gamma_{zyx} + \Gamma^e{}_{zy}\Gamma_{xze} - \Gamma^e{}_{zz}\Gamma_{xye} = -\Gamma^y{}_{zz}\Gamma_{xyy} = \frac{1}{2x}$$
$$R_{yzyz} = \partial_y\Gamma_{zzy} - \partial_d\Gamma_{zyy} + \Gamma^e{}_{zy}\Gamma_{yze} - \Gamma^e{}_{zz}\Gamma_{yy} = \Gamma^z{}_{zy}\Gamma_{yzz} = \frac{1}{2y}$$

<u>The Ricci tensor</u>

$$R_{ab} = \begin{Bmatrix} g^{yy}R_{xyxy} + g^{zz}R_{xzxz} & g^{zz}R_{xzyz} & -g^{yy}R_{xyyz} \\ S & g^{xx}R_{xyxy} + g^{zz}R_{yzyz} & g^{xx}R_{xyxz} \\ S & S & g^{xx}R_{xzxz} + g^{yy}R_{yzyz} \end{Bmatrix}$$
$$= \begin{Bmatrix} \frac{1}{2x}\frac{1}{2x} + \frac{1}{2y}0 & \frac{1}{2y}\frac{1}{2x} & -\frac{1}{2x}0 \\ S & \frac{1}{2x} + \frac{1}{2y}\frac{1}{2y} & 0 \\ S & S & 0 + \frac{1}{2x}\frac{1}{2y} \end{Bmatrix} = \begin{Bmatrix} \left(\frac{1}{2x}\right)^2 & \frac{1}{4xy} & 0 \\ S & \frac{1}{2x} + \left(\frac{1}{2y}\right)^2 & 0 \\ S & S & \frac{1}{4xy} \end{Bmatrix}$$

<u>The Ricci scalar</u>

$$R = g^{xx}R_{xx} + g^{yy}R_{yy} + g^{zz}R_{zz} = \left(\frac{1}{2x}\right)^2 + \frac{1}{2x}\left(\frac{1}{2x} + \left(\frac{1}{2y}\right)^2\right) + \frac{1}{2y}\frac{1}{4xy} = 2\left(\frac{1}{2x}\right)^2 + \frac{1}{4xy^2}$$

6.9 [k]Metric example 5: A three-dimensional spacetime $ds^2 = f(x)dx^2 + g(x)dy^2 + h(x)dz^2$

The line element

$$ds^2 = f(x)dx^2 + g(x)dy^2 + h(x)dz^2$$

The metric tensor and its inverse

$$g_{ab} = \begin{Bmatrix} f(x) & & \\ & g(x) & \\ & & h(x) \end{Bmatrix}$$
$$g^{ab} = \begin{Bmatrix} \frac{1}{f(x)} & & \\ & \frac{1}{g(x)} & \\ & & \frac{1}{h(x)} \end{Bmatrix}$$

6.9.1 The Christoffel Symbols, Riemann tensor, Ricci tensor and Ricci scalar

<u>The non-zero Christoffel symbols</u>

$$\Gamma_{abc} = \frac{1}{2}(\partial_a g_{bc} + \partial_b g_{ca} - \partial_c g_{ab}) \qquad \Gamma^a{}_{bc} = g^{ad}\Gamma_{bcd}$$
$$\Gamma_{xxx} = \frac{1}{2}\partial_x g_{xx} = \frac{1}{2}\frac{\partial f(x)}{\partial x} \quad \Rightarrow \quad \Gamma^x{}_{xx} = g^{xx}\Gamma_{xxx} = \frac{1}{2}\frac{1}{f(x)}\frac{\partial f(x)}{\partial x}$$
$$\Gamma_{xyy} = \Gamma_{yxy} = \frac{1}{2}\partial_x g_{yy} = \frac{1}{2}\frac{\partial g(x)}{\partial x} \quad \Rightarrow \quad \Gamma^y{}_{xy} = \Gamma^y{}_{yx} = g^{yy}\Gamma_{xyy} = \frac{1}{2}\frac{1}{g(x)}\frac{\partial g(x)}{\partial x}$$

$$\Gamma_{xzz} = \Gamma_{zxz} = \frac{1}{2}\partial_x g_{zz} = \frac{1}{2}\frac{\partial h(x)}{\partial x} \quad\Rightarrow\quad \Gamma^z{}_{xz} = \Gamma^z{}_{zx} = g^{zz}\Gamma_{xzz} = \frac{1}{2}\frac{1}{h(x)}\frac{\partial h(x)}{\partial x}$$

$$\Gamma_{yyx} = -\frac{1}{2}\partial_x g_{yy} = -\frac{1}{2}\frac{\partial g(x)}{\partial x} \quad\Rightarrow\quad \Gamma^x{}_{yy} = g^{xx}\Gamma_{yyx} = -\frac{1}{2}\frac{1}{f(x)}\frac{\partial g(x)}{\partial x}$$

$$\Gamma_{zzx} = -\frac{1}{2}\partial_x g_{zz} = -\frac{1}{2}\frac{\partial h(x)}{\partial x} \quad\Rightarrow\quad \Gamma^x{}_{zz} = g^{xx}\Gamma_{zzx} = -\frac{1}{2}\frac{1}{f(x)}\frac{\partial h(x)}{\partial x}$$

<u>The Riemann tensor</u>

In 3 dimensions the Riemann tensor has six independent elements:

$R_{xyxy}; R_{xyxz}; R_{xyyz}; R_{xzxz}; R_{xzyz}; R_{yzyz}$

$$R_{abcd} = \partial_c\Gamma_{bda} - \partial_d\Gamma_{bca} + \Gamma^e{}_{bc}\Gamma_{ade} - \Gamma^e{}_{bd}\Gamma_{ace}$$

$$R_{xyxy} = \partial_x\Gamma_{yyx} - \partial_y\Gamma_{yxx} + \Gamma^e{}_{yx}\Gamma_{xye} - \Gamma^e{}_{yy}\Gamma_{xxe} = -\frac{1}{2}\frac{\partial^2 g(x)}{\partial^2 x} - \Gamma^x{}_{yy}\Gamma_{xxx} + \Gamma^y{}_{yx}\Gamma_{xyy}$$

$$= -\frac{1}{2}\frac{\partial^2 g(x)}{\partial^2 x} - \left(-\frac{1}{2}\frac{1}{f(x)}\frac{\partial g(x)}{\partial x}\right)\left(\frac{1}{2}\frac{\partial f(x)}{\partial x}\right) + \left(\frac{1}{2}\frac{1}{g(x)}\frac{\partial g(x)}{\partial x}\right)\left(\frac{1}{2}\frac{\partial g(x)}{\partial x}\right)$$

$$= -\frac{1}{2}\frac{\partial^2 g(x)}{\partial^2 x} + \frac{1}{4}\left(\frac{\partial g(x)}{\partial x}\right)\left(\frac{1}{f(x)}\frac{\partial f(x)}{\partial x} + \frac{1}{g(x)}\frac{\partial g(x)}{\partial x}\right)$$

$$R_{xyxz} = \partial_x\Gamma_{yzx} - \partial_z\Gamma_{yxx} + \Gamma^e{}_{yx}\Gamma_{xze} - \Gamma^e{}_{yz}\Gamma_{xxe} = 0$$

$$R_{xyyz} = \partial_y\Gamma_{yzx} - \partial_z\Gamma_{yyx} + \Gamma^e{}_{yy}\Gamma_{xze} - \Gamma^e{}_{yz}\Gamma_{xye} = 0$$

$$R_{xzxz} = \partial_x\Gamma_{zzx} - \partial_z\Gamma_{zxx} + \Gamma^e{}_{zx}\Gamma_{xze} - \Gamma^e{}_{zz}\Gamma_{xxe}$$

$$= -\frac{1}{2}\frac{\partial^2 h(x)}{\partial^2 x} - \left(-\frac{1}{2}\frac{1}{f(x)}\frac{\partial h(x)}{\partial x}\right)\left(\frac{1}{2}\frac{\partial f(x)}{\partial x}\right) + \left(\frac{1}{2}\frac{1}{h(x)}\frac{\partial h(x)}{\partial x}\right)\left(\frac{1}{2}\frac{\partial h(x)}{\partial x}\right)$$

$$= -\frac{1}{2}\frac{\partial^2 h(x)}{\partial^2 x} + \frac{1}{4}\left(\frac{\partial h(x)}{\partial x}\right)\left(\frac{1}{f(x)}\frac{\partial f(x)}{\partial x} + \frac{1}{h(x)}\frac{\partial h(x)}{\partial x}\right)$$

$$R_{xzyz} = \partial_y\Gamma_{zzx} - \partial_z\Gamma_{zyx} + \Gamma^e{}_{zy}\Gamma_{xze} - \Gamma^e{}_{zz}\Gamma_{xye} = 0$$

$$R_{yzyz} = \partial_y\Gamma_{zzy} - \partial_z\Gamma_{zyy} + \Gamma^e{}_{zy}\Gamma_{yze} - \Gamma^e{}_{zz}\Gamma_{yye} = -\Gamma^x{}_{zz}\Gamma_{yyx}$$

$$= -\left(-\frac{1}{2}\frac{1}{f(x)}\frac{\partial h(x)}{\partial x}\right)\left(-\frac{1}{2}\frac{\partial g(x)}{\partial x}\right) = -\frac{1}{4}\frac{1}{f(x)}\frac{\partial g(x)}{\partial x}\frac{\partial h(x)}{\partial x}$$

Collecting the results

$$R_{xyxy} = -\frac{1}{2}\frac{\partial^2 g(x)}{\partial^2 x} + \frac{1}{4}\left(\frac{\partial g(x)}{\partial x}\right)\left(\frac{1}{f(x)}\left(\frac{\partial f(x)}{\partial x}\right) + \frac{1}{g(x)}\frac{\partial g(x)}{\partial x}\right)$$

$$R_{xzxz} = -\frac{1}{2}\frac{\partial^2 h(x)}{\partial^2 x} + \frac{1}{4}\left(\frac{\partial h(x)}{\partial x}\right)\left(\frac{1}{f(x)}\left(\frac{\partial f(x)}{\partial x}\right) + \frac{1}{h(x)}\frac{\partial h(x)}{\partial x}\right)$$

$$R_{yzyz} = -\frac{1}{4}\frac{1}{f(x)}\left(\frac{\partial g(x)}{\partial x}\right)\left(\frac{\partial h(x)}{\partial x}\right)$$

$$R_{xyxz} = R_{xyyz} = R_{xzyz} = 0$$

<u>The Ricci tensor</u>[24]

$$R_{xx} = g^{yy}R_{xyxy} + g^{zz}R_{xzxz}$$

[24] The non-diagonal elements are zero

$$
\begin{aligned}
&= [25] -\frac{1}{2}\left(\frac{1}{g(x)}\frac{\partial^2 g(x)}{\partial^2 x}+\frac{1}{h(x)}\frac{\partial^2 h(x)}{\partial^2 x}\right)\\
&\quad +\frac{1}{4}\frac{1}{f(x)}\left(\frac{\partial f(x)}{\partial x}\right)\left(\frac{1}{g(x)}\left(\frac{\partial g(x)}{\partial x}\right)+\frac{1}{h(x)}\left(\frac{\partial h(x)}{\partial x}\right)\right)\\
&\quad +\frac{1}{4}\left(\frac{1}{g^2(x)}\left(\frac{\partial g(x)}{\partial x}\right)^2+\frac{1}{h^2(x)}\left(\frac{\partial h(x)}{\partial x}\right)^2\right)
\end{aligned}
$$

$$
\begin{aligned}
R_{yy} &= g^{xx}R_{xyxy}+g^{zz}R_{yzyz}\\
&= [26] -\frac{1}{2}\frac{1}{f(x)}\frac{\partial^2 g(x)}{\partial^2 x}\\
&\quad +\frac{1}{4}\frac{1}{f(x)}\left(\frac{\partial g(x)}{\partial x}\right)\left(\frac{1}{f(x)}\left(\frac{\partial f(x)}{\partial x}\right)+\frac{1}{g(x)}\left(\frac{\partial g(x)}{\partial x}\right)-\frac{1}{h(x)}\left(\frac{\partial h(x)}{\partial x}\right)\right)\\
R_{zz} &= g^{xx}R_{xzxz}+g^{yy}R_{yzyz}\\
&= [27] -\frac{1}{2}\frac{1}{f(x)}\frac{\partial^2 h(x)}{\partial^2 x}+\frac{1}{4}\frac{1}{f(x)}\left(\frac{\partial h(x)}{\partial x}\right)\left(\frac{1}{f(x)}\left(\frac{\partial f(x)}{\partial x}\right)+\frac{1}{h(x)}\frac{\partial h(x)}{\partial x}-\frac{1}{g(x)}\left(\frac{\partial g(x)}{\partial x}\right)\right)
\end{aligned}
$$

<u>The Ricci scalar</u>

$$
\begin{aligned}
R &= 2g^{xx}g^{yy}g^{zz}\left(g_{zz}R_{xyxy}+g_{yy}R_{xzxz}+g_{xx}R_{yzyz}\right)\\
&= \frac{2}{f(x)g(x)h(x)}\left(h(x)R_{xyxy}+g(x)R_{xzxz}+f(x)R_{yzyz}\right)\\
&= \frac{2}{f(x)g(x)h(x)}\left(h(x)\left[-\frac{1}{2}\frac{\partial^2 g(x)}{\partial^2 x}+\frac{1}{4}\left(\frac{\partial g(x)}{\partial x}\right)\left(\frac{1}{f(x)}\left(\frac{\partial f(x)}{\partial x}\right)+\frac{1}{g(x)}\frac{\partial g(x)}{\partial x}\right)\right]\right.\\
&\quad +g(x)\left[-\frac{1}{2}\frac{\partial^2 h(x)}{\partial^2 x}+\frac{1}{4}\left(\frac{\partial h(x)}{\partial x}\right)\left(\frac{1}{f(x)}\left(\frac{\partial f(x)}{\partial x}\right)+\frac{1}{h(x)}\frac{\partial h(x)}{\partial x}\right)\right]\\
&\quad \left.+f(x)\left[-\frac{1}{4}\frac{1}{f(x)}\left(\frac{\partial g(x)}{\partial x}\right)\left(\frac{\partial h(x)}{\partial x}\right)\right]\right)
\end{aligned}
$$

6.10 Metric example 6: A three-dimensional spacetime $\boldsymbol{ds^2 = f(x)dx^2 + f(x)dy^2 + f(x)dz^2}$

The line element

$$ds^2 = f(x)dx^2 + f(x)dy^2 + f(x)dz^2$$

The metric tensor and its inverse

$$
g_{ab} = \begin{Bmatrix} f(x) & & \\ & f(x) & \\ & & f(x) \end{Bmatrix}
$$

---

[25] $= \frac{1}{g(x)}\left(-\frac{1}{2}\frac{\partial^2 g(x)}{\partial^2 x}+\frac{1}{4}\left(\frac{\partial g(x)}{\partial x}\right)\left(\frac{1}{f(x)}\left(\frac{\partial f(x)}{\partial x}\right)+\frac{1}{g(x)}\frac{\partial g(x)}{\partial x}\right)\right)+\frac{1}{h(x)}\left(-\frac{1}{2}\frac{\partial^2 h(x)}{\partial^2 x}+\frac{1}{4}\left(\frac{\partial h(x)}{\partial x}\right)\left(\frac{1}{f(x)}\left(\frac{\partial f(x)}{\partial x}\right)+\frac{1}{h(x)}\frac{\partial h(x)}{\partial x}\right)\right) =$

[26] $= \frac{1}{f(x)}\left(-\frac{1}{2}\frac{\partial^2 g(x)}{\partial^2 x}+\frac{1}{4}\left(\frac{\partial g(x)}{\partial x}\right)\left(\frac{1}{f(x)}\left(\frac{\partial f(x)}{\partial x}\right)+\frac{1}{g(x)}\frac{\partial g(x)}{\partial x}\right)\right)+\frac{1}{h(x)}\left(-\frac{1}{4}\frac{1}{f(x)}\left(\frac{\partial g(x)}{\partial x}\right)\left(\frac{\partial h(x)}{\partial x}\right)\right) =$

[27] $= \frac{1}{f(x)}\left(-\frac{1}{2}\frac{\partial^2 h(x)}{\partial^2 x}+\frac{1}{4}\left(\frac{\partial h(x)}{\partial x}\right)\left(\frac{1}{f(x)}\left(\frac{\partial f(x)}{\partial x}\right)+\frac{1}{h(x)}\frac{\partial h(x)}{\partial x}\right)\right)+\frac{1}{g(x)}\left(-\frac{1}{4}\frac{1}{f(x)}\left(\frac{\partial g(x)}{\partial x}\right)\left(\frac{\partial h(x)}{\partial x}\right)\right) =$

$$g^{ab} = \begin{Bmatrix} \frac{1}{f(x)} & & \\ & \frac{1}{f(x)} & \\ & & \frac{1}{f(x)} \end{Bmatrix}$$

6.10.1 The Christoffel Symbols, Riemann tensor, Ricci tensor and Ricci scalar

The Christoffel symbols

$$\Gamma_{abc} = \frac{1}{2}(\partial_a g_{bc} + \partial_b g_{ca} - \partial_c g_{ab}) \qquad \Gamma^a{}_{bc} = g^{ad}\Gamma_{bcd}$$

$$\Gamma_{xxx} = \frac{1}{2}\partial_x g_{xx} = \frac{1}{2}\frac{\partial f(x)}{\partial x} \quad \Rightarrow \quad \Gamma^x{}_{xx} = g^{xx}\Gamma_{xxx} = \frac{1}{2}\frac{1}{f(x)}\frac{\partial f(x)}{\partial x}$$

$$\Gamma_{xyy} = \Gamma_{yxy} = \frac{1}{2}\frac{\partial f(x)}{\partial x} \quad \Rightarrow \quad \Gamma^y{}_{xy} = \Gamma^y{}_{yx} = \frac{1}{2}\frac{1}{f(x)}\frac{\partial f(x)}{\partial x}$$

$$\Gamma_{xzz} = \Gamma_{zxz} = \frac{1}{2}\frac{\partial f(x)}{\partial x} \quad \Rightarrow \quad \Gamma^z{}_{xz} = \Gamma^z{}_{zx} = \frac{1}{2}\frac{1}{f(x)}\frac{\partial f(x)}{\partial x}$$

$$\Gamma_{yyx} = -\frac{1}{2}\frac{\partial f(x)}{\partial x} \quad \Rightarrow \quad \Gamma^x{}_{yy} = -\frac{1}{2}\frac{1}{f(x)}\frac{\partial f(x)}{\partial x}$$

$$\Gamma_{zzx} = -\frac{1}{2}\frac{\partial f(x)}{\partial x} \quad \Rightarrow \quad \Gamma^x{}_{zz} = -\frac{1}{2}\frac{1}{f(x)}\frac{\partial f(x)}{\partial x}$$

The Riemann tensor

In 3 dimensions the Riemann tensor has six independent elements:

$R_{xyxy}; R_{xyxz}; R_{xyyz}; R_{xzxz}; R_{xzyz}; R_{yzyz}$

$$R_{xyxy} = -\frac{1}{2}\frac{\partial^2 f(x)}{\partial^2 x} + \frac{1}{2}\frac{1}{f(x)}\left(\frac{\partial f(x)}{\partial x}\right)^2$$

$$R_{xzxz} = -\frac{1}{2}\frac{\partial^2 f(x)}{\partial^2 x} + \frac{1}{2}\frac{1}{f(x)}\left(\frac{\partial f(x)}{\partial x}\right)^2 = R_{xyxy}$$

$$R_{yzyz} = -\frac{1}{4}\frac{1}{f(x)}\left(\frac{\partial f(x)}{\partial x}\right)^2$$

$$R_{xyxz} = R_{xyyz} = R_{xzyz} = 0$$

The Ricci tensor

$$R_{ab} = \begin{Bmatrix} g^{yy}R_{xyxy} + g^{zz}R_{xzxz} & g^{zz}R_{xzyz} & -g^{yy}R_{xyyz} \\ S & g^{xx}R_{xyxy} + g^{zz}R_{yzyz} & g^{xx}R_{xyxz} \\ S & S & g^{xx}R_{xzxz} + g^{yy}R_{yzyz} \end{Bmatrix}$$

$$= \begin{Bmatrix} \frac{2}{f(x)}R_{xyxy} & 0 & 0 \\ S & \frac{1}{f(x)}(R_{xyxy} + R_{yzyz}) & 0 \\ S & S & \frac{1}{f(x)}(R_{xzxz} + R_{yzyz}) \end{Bmatrix}$$

$$= \begin{Bmatrix} \begin{bmatrix} -\frac{1}{f(x)}\frac{\partial^2 f(x)}{\partial^2 x} \\ +\left(\frac{1}{f(x)}\frac{\partial f(x)}{\partial x}\right)^2 \end{bmatrix} & 0 & 0 \\ S & \begin{bmatrix} -\frac{1}{2}\frac{1}{f(x)}\frac{\partial^2 f(x)}{\partial^2 x} \\ +\frac{1}{4}\left(\frac{1}{f(x)}\frac{\partial f(x)}{\partial x}\right)^2 \end{bmatrix} & 0 \\ S & S & \begin{bmatrix} -\frac{1}{2}\frac{1}{f(x)}\frac{\partial^2 f(x)}{\partial^2 x} \\ +\frac{1}{4}\left(\frac{1}{f(x)}\frac{\partial f(x)}{\partial x}\right)^2 \end{bmatrix} \end{Bmatrix}$$

<u>The Ricci scalar</u>

$$R = 2g^{xx}g^{yy}g^{zz}\left(g_{zz}R_{xyxy} + g_{yy}R_{xzxz} + g_{xx}R_{yzyz}\right) = \frac{2}{f^2(x)}\left(2R_{xyxy} + R_{yzyz}\right)$$

$$= \frac{2}{f^2(x)}\left(2\left(-\frac{1}{2}\frac{\partial^2 f(x)}{\partial^2 x} + \frac{1}{2}\frac{1}{f(x)}\left(\frac{\partial f(x)}{\partial x}\right)^2\right) - \frac{1}{4}\frac{1}{f(x)}\left(\frac{\partial f(x)}{\partial x}\right)^2\right)$$

$$= \frac{2}{f^2(x)}\left(-\frac{\partial^2 f(x)}{\partial^2 x} + \frac{3}{4}\frac{1}{f(x)}\left(\frac{\partial f(x)}{\partial x}\right)^2\right)$$

6.11 [m]Metric example 7: A three-dimensional spacetime $ds^2 = \frac{1}{x^2}dx^2 + \frac{1}{x^2}dy^2 + \frac{1}{x^2}dz^2$

The line element

$$ds^2 = \frac{1}{x^2}dx^2 + \frac{1}{x^2}dy^2 + \frac{1}{x^2}dz^2$$

The metric tensor and its inverse

$$g_{ab} = \begin{Bmatrix} \frac{1}{x^2} & & \\ & \frac{1}{x^2} & \\ & & \frac{1}{x^2} \end{Bmatrix}$$

$$g^{ab} = \begin{Bmatrix} x^2 & & \\ & x^2 & \\ & & x^2 \end{Bmatrix}$$

We need

$$f(x) = x^{-2}$$

$$\frac{\partial f(x)}{\partial x} = -2x^{-3}$$

$$\frac{\partial^2 f(x)}{\partial^2 x} = 6x^{-4}$$

6.11.1 The Christoffel Symbols, Riemann tensor, Ricci tensor and Ricci scalar

<u>The non-zero Christoffel symbols</u>

$$\Gamma_{abc} = \frac{1}{2}\left(\partial_a g_{bc} + \partial_b g_{ca} - \partial_c g_{ab}\right) \qquad \Gamma^a{}_{bc} = g^{ad}\Gamma_{bcd}$$

$$\Gamma_{xxx} = \frac{1}{2}\frac{\partial f(x)}{\partial x} = -x^{-3} \quad \Rightarrow \quad \Gamma^{x}{}_{xx} = \frac{1}{2}\frac{1}{f(x)}\frac{\partial f(x)}{\partial x} = x^2(-x^{-3}) = -\frac{1}{x}$$

$$\Gamma_{xyy} = \Gamma_{yxy} = \frac{1}{2}\frac{\partial f(x)}{\partial x} = -x^{-3} \quad \Rightarrow \quad \Gamma^{y}{}_{xy} = \Gamma^{y}{}_{yx} = \frac{1}{2}\frac{1}{f(x)}\frac{\partial f(x)}{\partial x} = -\frac{1}{x}$$

$$\Gamma_{xzz} = \Gamma_{zxz} = \frac{1}{2}\frac{\partial f(x)}{\partial x} = -x^{-3} \quad \Rightarrow \quad \Gamma^{z}{}_{xz} = \Gamma^{z}{}_{zx} = \frac{1}{2}\frac{1}{f(x)}\frac{\partial f(x)}{\partial x} = -\frac{1}{x}$$

$$\Gamma_{yyx} = -\frac{1}{2}\frac{\partial f(x)}{\partial x} = x^{-3} \quad \Rightarrow \quad \Gamma^{x}{}_{yy} = -\frac{1}{2}\frac{1}{f(x)}\frac{\partial f(x)}{\partial x} = \frac{1}{x}$$

$$\Gamma_{zzx} = -\frac{1}{2}\frac{\partial f(x)}{\partial x} = -x^{-3} \quad \Rightarrow \quad \Gamma^{x}{}_{zz} = -\frac{1}{2}\frac{1}{f(x)}\frac{\partial f(x)}{\partial x} = \frac{1}{x}$$

The Riemann tensor

In 3 dimensions the Riemann tensor has six independent elements:

$R_{xyxy}; R_{xyxz}; R_{xyyz}; R_{xzxz}; R_{xzyz}; R_{yzyz}$

$$R_{xyxy} = -\frac{1}{2}\frac{\partial^2 f(x)}{\partial^2 x} + \frac{1}{2}\frac{1}{f(x)}\left(\frac{\partial f(x)}{\partial x}\right)^2 = -\frac{1}{2}6x^{-4} + \frac{1}{2}x^2(-2x^{-3})^2 = -x^{-4}$$

$$R_{xzxz} = -\frac{1}{2}\frac{\partial^2 f(x)}{\partial^2 x} + \frac{1}{2}\frac{1}{f(x)}\left(\frac{\partial f(x)}{\partial x}\right)^2 = R_{xyxy} = -x^{-4}$$

$$R_{yzyz} = -\frac{1}{4}\frac{1}{f(x)}\left(\frac{\partial f(x)}{\partial x}\right)^2 = -\frac{1}{4}x^2(-2x^{-3})^2 = -x^{-4}$$

$$R_{xyxz} = R_{xyyz} = R_{xzyz} = 0$$

The Ricci tensor

$$R_{ab} = \begin{Bmatrix} \frac{2}{f(x)}R_{xyxy} & 0 & 0 \\ S & \frac{1}{f(x)}(R_{xyxy} + R_{yzyz}) & 0 \\ S & S & \frac{1}{f(x)}(R_{xzxz} + R_{yzyz}) \end{Bmatrix}$$

$$= \begin{Bmatrix} -2x^{-2} & 0 & 0 \\ S & -2x^{-2} & 0 \\ S & S & -2x^{-2} \end{Bmatrix}$$

The Ricci scalar

$$R = 2g^{xx}g^{yy}g^{zz}(g_{zz}R_{xyxy} + g_{yy}R_{xzxz} + g_{xx}R_{yzyz}) = \frac{6}{f^2(x)}R_{xyxy} = 6x^4(-x^{-4}) = -6$$

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[a] (McMahon, 2006, s. 85), (Hartle, 2003, s. 467), (d'Inverno, 1992, p. 86), (Kay, 1988, s. 101)
[b] (Kay, 1988, s. 126)
[c] (d'Inverno, 1992, p. 87)
[d] (d'Inverno, 1992, p. 87)
[e] (McMahon, 2006, s. 90), (Penrose, 2004, s. 464), (d'Inverno, 1992, p. 88)
[f] (McMahon, 2006, s. 88)
[g] (McMahon, 2006, s. 92)

[h] (McMahon, 2006, s. 324)
[i] Kindly provided by Mr. John Fredsted: http://johnfredsted.dk/science/publications.php
[j] (Kay, 1988, s. 111)
[k] (Kay, 1988, s. 113)
[l] (Kay, 1988, s. 113)
[m] (Kay, 1988, s. 110, 113)