Susan Larsen Tuesday, March 21, 2023

Content

Space-time		Line-element	Chap-ter
Example: Four-dimensional space-time	ds^2	$= -dt^2 + L^2(t,r)dr^2 + B^2(t,r)d\phi^2 + M^2(t,r)dz^2$	7
Hyperbolsk plane – Poincaré Halfplane	ds^2	$= \frac{1}{y^2}(dx^2 + dy^2)$	4,7
Lorentz hyperboloid	ds^2	$= d\psi^2 + \sinh^2\psi\ d\theta^2 + \sinh^2\psi \sin^2\theta\, d\phi^2$	7, 13
Poincaré metric	ds^2	$= \frac{4}{1 - x^2 - y^2}(dx^2 + dy^2)$	7
Rindler metric	ds^2	$= \xi^2 d\tau^2 - d\xi^2$	2,3,4,7
Taub-nut space-time	ds^2	$= -\frac{dt^2}{U^2(t)} + (2l)^2 U^2(t)(dr + \cos\theta\, d\phi)^2 + V^2(t)(d\theta^2 + \sin^2\theta\, d\phi^2)$	7
Three-dimensional flat space in polar coordinates	ds^2	$= dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$	2,7, 13
Tolman-Bondi-de Sitter space-time	ds^2	$= dt^2 - e^{-2\psi(t,r)}dr^2 - R^2(t,r)d\theta^2 - R^2(t,r)\sin^2\theta\, d\phi^2$	7
Two-dimensional unit sphere	ds^2	$= d\theta^2 + \sin^2\theta\, d\phi^2$	7

7 Cartan's Structure Equations – a Shortcut to the Einstein Tensor

7.1 [a]One-forms.

For general forms, let α be a p-form and β be a q-form, we have

$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$$

For one-forms this means

$$\alpha \wedge \beta = -\beta \wedge \alpha$$

$$\Rightarrow \quad \alpha \wedge \alpha = -\alpha \wedge \alpha = 0$$

This also holds for $df = \frac{\partial f}{\partial x^a} dx^a$ because df is a one-form as well.

$$df_a \wedge df_b = \frac{\partial f}{\partial x^a} dx^a \wedge \frac{\partial f}{\partial x^b} dx^b = \frac{\partial^2 f}{\partial x^a \partial x^b} dx^a \wedge dx^b = -\frac{\partial^2 f}{\partial x^a \partial x^b} dx^b \wedge dx^a$$

$$\Rightarrow \quad df_a \wedge df_a = \frac{\partial^2 f}{\partial x^a \partial x^b} dx^a \wedge dx^a$$

and $$df_a \wedge df_a = -\frac{\partial^2 f}{\partial x^a \partial x^b} dx^a \wedge dx^a$$

Now, because the partial derivatives commute, this can only be true if:

$$dx^a \wedge dx^a = 0$$

7.1.1 [b]The exterior derivative of a one-form.

The exterior derivative of a one-form $f_a dx^a$:

$$d(f_a dx^a) = df_a \wedge dx^a$$

To use this equation it is important to notice, that the right-hand side includes a summation of the partial derivatives times the differential in the usual way:

$$df(x_i) = \frac{\partial}{\partial x_i} f(x_i) dx_i.$$

7.1.2 Examples

$$\sigma = e^{f(r)} dt$$

$$\Rightarrow \quad d\sigma = d\left(e^{f(r)} dt\right) = d\left(e^{f(r)}\right) \wedge dt = \frac{\partial}{\partial r}\left(e^{f(r)}\right) dr \wedge dt = f'(r) e^{f(r)} dr \wedge dt$$

$$\rho = e^{g(r)} \cos\theta \sin\phi \, dr$$

$$\Rightarrow \quad d\rho = d\left(e^{g(r)} \cos\theta \sin\phi \, dr\right) = d\left(e^{g(r)} \cos\theta \sin\phi\right) \wedge dr$$

$$= {}^{1}\frac{\partial}{\partial\theta}\left(e^{g(r)} \cos\theta \sin\phi\right) d\theta \wedge dr + \frac{\partial}{\partial\phi}\left(e^{g(r)} \cos\theta \sin\phi\right) d\phi \wedge dr$$

$$= -e^{g(r)} \sin\theta \sin\phi \, d\theta \wedge dr + e^{g(r)} \cos\theta \cos\phi \, d\phi \wedge dr$$

7.2 Cartan's first structure equation

$$d\omega^{\hat{a}} = -\Gamma^{\hat{a}}{}_{\hat{b}} \wedge \omega^{\hat{b}}$$

7.3 [c]The curvature two forms and the Riemann tensor

The Riemann tensor and the curvature two forms.

$$\Omega^{\hat{a}}{}_{\hat{b}} = d\Gamma^{\hat{a}}{}_{\hat{b}} + \Gamma^{\hat{a}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{b}} = \frac{1}{2} R^{\hat{a}}{}_{\hat{b}\hat{c}\hat{d}} \omega^{\hat{c}} \wedge \omega^{\hat{d}}$$

7.4 The unit 2-sphere

The line element:

$$ds^2 = d\theta^2 + \sin^2\theta \, d\phi^2$$

7.4.1 [d]The Riemann tensor for the unit 2-sphere

The number of independent elements in the Riemann tensor in a metric of dimension $n = 2$ is $N = \frac{n^2(n^2-1)}{12} = 1$ so we can choose to calculate $R^\theta{}_{\phi\theta\phi}$

The Riemann tensor

$$R^a{}_{bcd} = \partial_c \Gamma^a{}_{bd} - \partial_d \Gamma^a{}_{bc} + \Gamma^e{}_{bd} \Gamma^a{}_{ec} - \Gamma^e{}_{bc} \Gamma^a{}_{ed}$$

We choose $a = c = \theta$, and $b = d = \phi$

$$R^\theta{}_{\phi\theta\phi} = \partial_\theta \Gamma^\theta{}_{\phi\phi} - \partial_\phi \Gamma^\theta{}_{\phi\theta} + \Gamma^e{}_{\phi\phi} \Gamma^\theta{}_{e\theta} - \Gamma^e{}_{\phi\theta} \Gamma^\theta{}_{e\phi} = \partial_\theta \Gamma^\theta{}_{\phi\phi} + \Gamma^e{}_{\phi\phi} \Gamma^\theta{}_{e\theta} - \Gamma^e{}_{\phi\theta} \Gamma^\theta{}_{e\phi}$$

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[1] $= \frac{\partial}{\partial r}\left(e^{g(r)} \cos\theta \sin\phi\right) dr \wedge dr + \frac{\partial}{\partial\theta}\left(e^{g(r)} \cos\theta \sin\phi\right) d\theta \wedge dr + \frac{\partial}{\partial\phi}\left(e^{g(r)} \cos\theta \sin\phi\right) d\phi \wedge dr =$

$= \partial_\theta \Gamma^\theta{}_{\phi\phi} + \Gamma^\theta{}_{\phi\phi}\Gamma^\theta{}_{\theta\theta} - \Gamma^\theta{}_{\phi\theta}\Gamma^\theta{}_{\theta\phi} + \Gamma^\phi{}_{\phi\phi}\Gamma^\theta{}_{\phi\theta} - \Gamma^\phi{}_{\phi\theta}\Gamma^\theta{}_{\phi\phi} =$ [2] $\partial_\theta \Gamma^\theta{}_{\phi\phi} - \Gamma^\phi{}_{\phi\theta}\Gamma^\theta{}_{\phi\phi}$

$= \partial_\theta(-\sin\theta\cos\theta) - \cot\theta\,(-\sin\theta\cos\theta) = -\cos^2\theta + \sin^2\theta + \cos^2\theta = \sin^2\theta$

7.4.2 [e]The Ricci scalar

The metric tensor and its inverse:

$$g_{ab} = \begin{Bmatrix} 1 & \\ & \sin^2\theta \end{Bmatrix}$$

$$g^{ab} = \begin{Bmatrix} 1 & \\ & \dfrac{1}{\sin^2\theta} \end{Bmatrix}$$

The Ricci scalar:

$$\begin{aligned} R &= g^{ab}R_{ab} = g^{\theta b}R_{\theta b} + g^{\phi b}R_{\phi b} = g^{\theta\theta}R_{\theta\theta} + g^{\theta\phi}R_{\theta\phi} + g^{\phi\theta}R_{\phi\theta} + g^{\phi\phi}R_{\phi\phi} \\ &= g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi} = g^{\theta\theta}R^c{}_{\theta c\theta} + g^{\phi\phi}R^c{}_{\phi c\phi} \\ &= g^{\theta\theta}R^\theta{}_{\theta\theta\theta} + g^{\theta\theta}R^\phi{}_{\theta\phi\theta} + g^{\phi\phi}R^\theta{}_{\phi\theta\phi} + g^{\phi\phi}R^\phi{}_{\phi\phi\phi} = g^{\theta\theta}R^\phi{}_{\theta\phi\theta} + g^{\phi\phi}R^\theta{}_{\phi\theta\phi} \\ &= g^{\theta\theta}g^{\phi\phi}R_{\phi\theta\phi\theta} + g^{\phi\phi}g^{\theta\theta}R_{\theta\phi\theta\phi} = 2g^{\theta\theta}g^{\phi\phi}R_{\theta\phi\theta\phi} = {}^{3}2g^{\phi\phi}R^\theta{}_{\phi\theta\phi} = 2\frac{1}{\sin^2\theta}\sin^2\theta = 2 \end{aligned}$$

7.4.3 [f]Find the Ricci scalar using Cartan's structure equations of the 2-sphere

The line element:

$$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2$$

The Basis one forms:

$$\omega^{\hat\theta} = d\theta \qquad d\theta = \omega^{\hat\theta}$$

$$\omega^{\hat\phi} = \sin\theta\, d\phi \qquad d\phi = \frac{1}{\sin\theta}\omega^{\hat\phi}$$

$$\eta^{ij} = \begin{Bmatrix} 1 & \\ & 1 \end{Bmatrix}$$

The curvature one forms:

$$\begin{aligned} d\omega^{\hat a} &= -\Gamma^{\hat a}{}_{\hat b} \wedge \omega^{\hat b} \\ d\omega^{\hat\theta} &= 0 \\ d\omega^{\hat\phi} &= d(\sin\theta\, d\phi) = \cos\theta\, d\theta \wedge d\phi = \cot\theta\, \omega^{\hat\theta} \wedge \omega^{\hat\phi} \\ \Rightarrow \quad \Gamma^{\hat\phi}{}_{\hat\theta} &= \cot\theta\, \omega^{\hat\phi} \end{aligned}$$

The curvature two forms:

$$\begin{aligned} \Omega^{\hat a}{}_{\hat b} &= d\Gamma^{\hat a}{}_{\hat b} + \Gamma^{\hat a}{}_{\hat c} \wedge \Gamma^{\hat c}{}_{\hat b} = \frac{1}{2}R^{\hat a}{}_{\hat b\hat c\hat d}\,\omega^{\hat c} \wedge \omega^{\hat d} \\ d\Gamma^{\hat\phi}{}_{\hat\theta} &= d\left(\cot\theta\, \omega^{\hat\phi}\right) = d(\cos\theta\, d\phi) = -\sin\theta\, d\theta \wedge d\phi = -\omega^{\hat\theta} \wedge \omega^{\hat\phi} \\ \Gamma^{\hat\phi}{}_{\hat c} \wedge \Gamma^{\hat c}{}_{\hat\theta} &= \Gamma^{\hat\phi}{}_{\hat\theta} \wedge \Gamma^{\hat\theta}{}_{\hat\theta} + \Gamma^{\hat\phi}{}_{\hat\phi} \wedge \Gamma^{\hat\phi}{}_{\hat\theta} = 0 \\ \Rightarrow \quad \Omega^{\hat\phi}{}_{\hat\theta} &= -\omega^{\hat\theta} \wedge \omega^{\hat\phi} \\ \Rightarrow \quad R^{\hat\theta}{}_{\hat\phi\hat\theta\hat\phi} &= 1 \\ \Rightarrow \quad R &= 2\eta^{\hat\phi\hat\phi}R^{\hat\theta}{}_{\hat\phi\hat\theta\hat\phi} = 2 \end{aligned}$$

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[2] We found the Christoffel symbols earlier in the chapter regarding: Two-dimensional sphere with radius a - $ds^2 = a^2 d\theta^2 + a^2\sin^2\theta\, d\phi^2$

[3] Notice: $R = 2g^{\phi\phi}R^\theta{}_{\phi\theta\phi}$ is a general solution for a 2-dimensional diagonal metric if we write: $R = 2g^{22}R^1{}_{212}$

7.5 The three dimensional flat space in spherical polar coordinates

7.5.1 [g]The Christoffel symbols

The line element:

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

The metric tensor and its inverse:

$$g_{ab} = \begin{Bmatrix} 1 & & \\ & r^2 & \\ & & r^2\sin^2\theta \end{Bmatrix}$$

$$g^{ab} = \begin{Bmatrix} 1 & & \\ & \dfrac{1}{r^2} & \\ & & \dfrac{1}{r^2\sin^2\theta} \end{Bmatrix}$$

<u>The non-zero Christoffel symbols</u>

$$\Gamma_{\theta\theta r} = -\frac{1}{2}\partial_r g_{\theta\theta} = -\frac{1}{2}\partial_r(r^2) = -r \quad \Rightarrow \quad \Gamma^r{}_{\theta\theta} = g^{rr}\Gamma_{\theta\theta r} = -r$$

$$\Gamma_{\phi\phi r} = -\frac{1}{2}\partial_r g_{\phi\phi} = -\frac{1}{2}\partial_r(r^2\sin^2\theta) = -r\sin^2\theta \quad \Rightarrow \quad \Gamma^r{}_{\phi\phi} = g^{rr}\Gamma_{\phi\phi r} = -r\sin^2\theta$$

$$\Gamma_{\phi\phi\theta} = -\frac{1}{2}\partial_\theta g_{\phi\phi} = {}^4 -r^2\cos\theta\sin\theta \quad \Rightarrow \quad \Gamma^\theta{}_{\phi\phi} = g^{\theta\theta}\Gamma_{\phi\phi\theta} = {}^5 -\cos\theta\sin\theta$$

$$\Gamma_{r\theta\theta} = \Gamma_{\theta r\theta} = \frac{1}{2}\partial_r g_{\theta\theta} = \frac{1}{2}\partial_r(r^2) = r \quad \Rightarrow \quad \Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = g^{\theta\theta}\Gamma_{r\theta\theta} = \frac{1}{r^2}r = \frac{1}{r}$$

$$\Gamma_{r\phi\phi} = \Gamma_{\phi r\phi} = \frac{1}{2}\partial_r g_{\phi\phi} = {}^6 r\sin^2\theta \quad \Rightarrow \quad \Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} = g^{\phi\phi}\Gamma_{r\phi\phi} = {}^7 \frac{1}{r}$$

$$\Gamma_{\theta\phi\phi} = \Gamma_{\phi\theta\phi} = \frac{1}{2}\partial_\theta g_{\phi\phi} = {}^8 r^2\cos\theta\sin\theta \quad \Rightarrow \quad \Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta} = g^{\phi\phi}\Gamma_{\theta\phi\phi} = {}^9 \cot\theta$$

7.5.2 [h]The Riemann tensor of the three dimensional flat space in spherical polar coordinates

The number of independent elements in the Riemann tensor in a metric of dimension $n = 3$ is $N = \frac{n^2(n^2-1)}{12} = 6$ and we have to calculate: $R^r{}_{\theta r\theta}$; $R^r{}_{\phi r\phi}$; $R^\theta{}_{\phi\theta\phi}$; $R^r{}_{\theta r\phi}$; $R^\theta{}_{r\theta\phi}$; $R^\phi{}_{r\phi\theta}$

The Riemann tensor

$$R^a{}_{bcd} = \partial_c\Gamma^a{}_{bd} - \partial_d\Gamma^a{}_{bc} + \Gamma^e{}_{bd}\Gamma^a{}_{ec} - \Gamma^e{}_{bc}\Gamma^a{}_{ed}$$

$$R^r{}_{\theta r\theta} = \partial_r\Gamma^r{}_{\theta\theta} - \partial_\theta\Gamma^r{}_{\theta r} + \Gamma^e{}_{\theta\theta}\Gamma^r{}_{er} - \Gamma^e{}_{\theta r}\Gamma^r{}_{e\theta} = \partial_r\Gamma^r{}_{\theta\theta} - \Gamma^\theta{}_{\theta r}\Gamma^r{}_{\theta\theta} = -1 - \left(\frac{1}{r}\right)(-r)$$
$$= 0$$

$$R^r{}_{\phi r\phi} = \partial_r\Gamma^r{}_{\phi\phi} - \partial_\phi\Gamma^r{}_{\phi r} + \Gamma^e{}_{\phi\phi}\Gamma^r{}_{er} - \Gamma^e{}_{\phi r}\Gamma^r{}_{e\phi} = \partial_r\Gamma^r{}_{\phi\phi} - \Gamma^\phi{}_{\phi r}\Gamma^r{}_{\phi\phi}$$
$$= -\sin^2\theta - \left(\frac{1}{r}\right)(-r\sin^2\theta) = 0$$

$$R^\theta{}_{\phi\theta\phi} = \partial_\theta\Gamma^\theta{}_{\phi\phi} - \partial_\phi\Gamma^\theta{}_{\phi\theta} + \Gamma^e{}_{\phi\phi}\Gamma^\theta{}_{e\theta} - \Gamma^e{}_{\phi\theta}\Gamma^\theta{}_{e\phi} = \partial_\theta\Gamma^\theta{}_{\phi\phi} + \Gamma^r{}_{\phi\phi}\Gamma^\theta{}_{r\theta} - \Gamma^\phi{}_{\phi\theta}\Gamma^\theta{}_{\phi\phi}$$
$$= -\cos^2\theta + \sin^2\theta + (-r\sin^2\theta)\left(\frac{1}{r}\right) - (\cot\theta)(-\sin\theta\cos\theta) = 0$$

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[4] $= -\frac{1}{2}\partial_\theta(r^2\sin^2\theta) =$

[5] $= \frac{1}{r^2}(-r^2\cos\theta\sin\theta) =$

[6] $= \frac{1}{2}\partial_r(r^2\sin^2\theta) =$

[7] $= \frac{1}{r^2\sin^2\theta} r\sin^2\theta =$

[8] $= \frac{1}{2}\partial_\theta(r^2\sin^2\theta) =$

[9] $= \frac{1}{r^2\sin^2\theta} r^2\cos\theta\sin\theta =$

$$R^{r}{}_{\theta r\phi} = \partial_r \Gamma^{r}{}_{\theta\phi} - \partial_\phi \Gamma^{r}{}_{\theta r} + \Gamma^{e}{}_{\theta\phi}\Gamma^{r}{}_{er} - \Gamma^{e}{}_{\theta r}\Gamma^{r}{}_{e\phi} = 0$$
$$R^{\theta}{}_{r\theta\phi} = \partial_\theta \Gamma^{\theta}{}_{r\phi} - \partial_\phi \Gamma^{\theta}{}_{r\theta} + \Gamma^{e}{}_{r\phi}\Gamma^{\theta}{}_{e\theta} - \Gamma^{e}{}_{r\theta}\Gamma^{\theta}{}_{e\phi} = 0$$
$$R^{\phi}{}_{r\phi\theta} = \partial_\phi \Gamma^{\phi}{}_{r\theta} - \partial_\theta \Gamma^{\phi}{}_{r\phi} + \Gamma^{e}{}_{r\theta}\Gamma^{\phi}{}_{e\phi} - \Gamma^{e}{}_{r\phi}\Gamma^{\phi}{}_{e\theta} = \left(\frac{1}{r}\right)\cot\theta - \left(\frac{1}{r}\right)\cot\theta = 0$$

Not surprisingly all the elements of the Riemann tensor in flat three-dimensional space are zero.

7.5.3 The Ricci rotation coefficients

The line element:

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

The Basis one forms and the transformation matrices

$$\omega^{\hat{r}} = dr \qquad dr = \omega^{\hat{r}}$$
$$\omega^{\hat{\theta}} = r d\theta \qquad d\theta = \frac{1}{r}\omega^{\hat{\theta}}$$
$$\omega^{\hat{\phi}} = r\sin\theta \, d\phi \qquad d\phi = \frac{1}{r\sin\theta}\omega^{\hat{\phi}}$$

$$\eta^{ij} = \begin{Bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{Bmatrix}$$

$$\Lambda^{\hat{a}}{}_{b} = \begin{Bmatrix} 1 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r\sin\theta \end{Bmatrix}$$

$$\Lambda^{a}{}_{\hat{b}} = \begin{Bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r} & 0 \\ 0 & 0 & \frac{1}{r\sin\theta} \end{Bmatrix}$$

The Ricci rotation coefficients $\Gamma^{\hat{a}}{}_{\hat{b}\hat{c}}$

$$d\omega^{\hat{a}} = -\Gamma^{\hat{a}}{}_{\hat{b}} \wedge \omega^{\hat{b}}$$
$$\Gamma^{\hat{a}}{}_{\hat{b}} = \Gamma^{\hat{a}}{}_{\hat{b}\hat{c}}\omega^{\hat{c}}$$
$$d\omega^{\hat{r}} = 0$$
$$d\omega^{\hat{\theta}} = d(rd\theta) = dr \wedge d\theta = \frac{1}{r}\omega^{\hat{r}} \wedge \omega^{\hat{\theta}} = -\frac{1}{r}\omega^{\hat{\theta}} \wedge \omega^{\hat{r}} = -\Gamma^{\hat{\theta}}{}_{\hat{r}} \wedge \omega^{\hat{r}}$$
$$d\omega^{\hat{\phi}} = d(r\sin\theta\, d\phi) = \sin\theta\, dr \wedge d\phi + r\cos\theta\, d\theta \wedge d\phi = \frac{1}{r}\omega^{\hat{r}} \wedge \omega^{\hat{\phi}} + \frac{\cot\theta}{r}\omega^{\hat{\theta}} \wedge \omega^{\hat{\phi}}$$
$$= -\frac{1}{r}\omega^{\hat{\phi}} \wedge \omega^{\hat{r}} - \frac{\cot\theta}{r}\omega^{\hat{\phi}} \wedge \omega^{\hat{\theta}} = -\Gamma^{\hat{\phi}}{}_{\hat{r}} \wedge \omega^{\hat{r}} - \Gamma^{\hat{\phi}}{}_{\hat{\theta}} \wedge \omega^{\hat{\theta}}$$

Summarizing the curvature one forms in a matrix (where $\hat{a}$ refers to column and $\hat{b}$ to row):

$$\Gamma^{\hat{a}}{}_{\hat{b}} = \begin{Bmatrix} 0 & \frac{1}{r}\omega^{\hat{\theta}} & \frac{1}{r}\omega^{\hat{\phi}} \\ -\frac{1}{r}\omega^{\hat{\theta}} & 0 & \frac{\cot\theta}{r}\omega^{\hat{\phi}} \\ -\frac{1}{r}\omega^{\hat{\phi}} & -\frac{\cot\theta}{r}\omega^{\hat{\phi}} & 0 \end{Bmatrix}$$

Now we can read off the Ricci rotation coefficients

$$\Gamma^{\hat{r}}{}_{\hat{\theta}\hat{\theta}} = -\frac{1}{r} \qquad \Gamma^{\hat{\theta}}{}_{\hat{r}\hat{\theta}} = \frac{1}{r} \qquad \Gamma^{\hat{\phi}}{}_{\hat{r}\hat{\phi}} = \frac{1}{r}$$
$$\Gamma^{\hat{r}}{}_{\hat{\phi}\hat{\phi}} = -\frac{1}{r} \qquad \Gamma^{\hat{\theta}}{}_{\hat{\phi}\hat{\phi}} = -\frac{\cot\theta}{r} \qquad \Gamma^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}} = \frac{\cot\theta}{r}$$

7.5.4 [j]Transformation of the Ricci rotation coefficients $\Gamma^{\hat{a}}{}_{\hat{b}\hat{c}}$ into the Christoffel symbols $\Gamma^{a}{}_{bc}$ of the three dimensional flat space in spherical polar coordinates

The transformation

$$\Gamma^a{}_{bc} = \Lambda^a{}_{\hat{a}}\Gamma^{\hat{a}}{}_{\hat{e}\hat{f}}\Lambda^{\hat{e}}{}_b\Lambda^{\hat{f}}{}_c$$

$$\Rightarrow \quad \Gamma^\theta{}_{r\theta} = \Lambda^\theta{}_{\hat{a}}\Gamma^{\hat{a}}{}_{\hat{e}\hat{f}}\Lambda^{\hat{e}}{}_r\Lambda^{\hat{f}}{}_\theta = \Gamma^{\hat{\theta}}{}_{\hat{r}\hat{\theta}}\Lambda^{\hat{r}}{}_r = \frac{1}{r}\cdot 1 = \frac{1}{r}$$

$$\Gamma^r{}_{\theta\theta} = \Lambda^r{}_{\hat{a}}\Gamma^{\hat{a}}{}_{\hat{e}\hat{f}}\Lambda^{\hat{e}}{}_\theta\Lambda^{\hat{f}}{}_\theta = \Lambda^r{}_{\hat{r}}\Gamma^{\hat{r}}{}_{\hat{\theta}\hat{\theta}}\left(\Lambda^{\hat{\theta}}{}_\theta\right)^2 = 1\left(-\frac{1}{r}\right)r^2 = -r$$

$$\Gamma^\phi{}_{r\phi} = \Lambda^\phi{}_{\hat{a}}\Gamma^{\hat{a}}{}_{\hat{e}\hat{f}}\Lambda^{\hat{e}}{}_r\Lambda^{\hat{f}}{}_\phi = \Gamma^{\hat{\phi}}{}_{\hat{r}\hat{\phi}}\Lambda^{\hat{r}}{}_r = \frac{1}{r}\cdot 1 = \frac{1}{r}$$

$$\Gamma^r{}_{\phi\phi} = \Lambda^r{}_{\hat{a}}\Gamma^{\hat{a}}{}_{\hat{e}\hat{f}}\Lambda^{\hat{e}}{}_\phi\Lambda^{\hat{f}}{}_\phi = \Lambda^r{}_{\hat{r}}\Gamma^{\hat{r}}{}_{\hat{\phi}\hat{\phi}}\left(\Lambda^{\hat{\phi}}{}_\phi\right)^2 = 1\left(-\frac{1}{r}\right)r^2\sin^2\theta = -r\sin^2\theta$$

$$\Gamma^\phi{}_{\theta\phi} = \Lambda^\phi{}_{\hat{a}}\Gamma^{\hat{a}}{}_{\hat{e}\hat{f}}\Lambda^{\hat{e}}{}_\theta\Lambda^{\hat{f}}{}_\phi = \Gamma^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}}\Lambda^{\hat{\theta}}{}_\theta = \frac{\cot\theta}{r}\cdot r = \cot\theta$$

$$\Gamma^\theta{}_{\phi\phi} = \Lambda^\theta{}_{\hat{a}}\Gamma^{\hat{a}}{}_{\hat{e}\hat{f}}\Lambda^{\hat{e}}{}_\phi\Lambda^{\hat{f}}{}_\phi = \Lambda^\theta{}_{\hat{\theta}}\Gamma^{\hat{\theta}}{}_{\hat{\phi}\hat{\phi}}\left(\Lambda^{\hat{\phi}}{}_\phi\right)^2 = \frac{1}{r}\left(-\frac{\cot\theta}{r}\right)r^2\sin^2\theta = -\sin\theta\cos\theta$$

Collecting the results

$$\Gamma^r{}_{\theta\theta} = -r \quad \Gamma^r{}_{\phi\phi} = -r\sin^2\theta \quad \Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^\theta{}_{r\theta} = \frac{1}{r} \quad \Gamma^\phi{}_{r\phi} = \frac{1}{r} \quad \Gamma^\phi{}_{\theta\phi} = \cot\theta$$

Which is consistent with the ordinary method used above.

7.6 The Rindler metric

[10]The Rindler coordinate system or frame describes a uniformly accelerating frame of reference in Minkowski space.

7.6.1 [k]Use the geodesic equations to find the Christoffel symbols for the Rindler metric.

The line element:

$$ds^2 = \xi^2 d\tau^2 - d\xi^2$$

The metric tensor:

$$g_{ab} = \begin{Bmatrix} \xi^2 & \\ & -1 \end{Bmatrix}$$

The geodesic equation

$$\frac{\partial K}{\partial x^a} = \frac{d}{ds}\left(\frac{\partial K}{\partial \dot{x}^a}\right)$$

$$K = \frac{1}{2}g_{ab}\dot{x}^a\dot{x}^b = \frac{1}{2}\xi^2(\dot{\tau})^2 - \frac{1}{2}\dot{\xi}^2$$

$\underline{x^a = \xi:}$

$$\frac{\partial K}{\partial \xi} = \frac{d}{ds}\left(\frac{\partial K}{\partial \dot{\xi}}\right)$$

$$\Rightarrow \quad \xi\dot{\tau}^2 = \frac{d}{ds}(-\dot{\xi}) = -\ddot{\xi}$$

$$\Rightarrow \quad 0 = \ddot{\xi} + \xi\dot{\tau}^2$$

$\underline{x^a = \tau:}$

$$\frac{\partial K}{\partial \tau} = \frac{d}{ds}\left(\frac{\partial K}{\partial \dot{\tau}}\right)$$

$$0 = \frac{d}{ds}(\xi^2\dot{\tau}) = 2\xi\dot{\xi}\dot{\tau} + \xi^2\ddot{\tau}$$

$$\Rightarrow \quad 0 = \ddot{\tau} + \frac{1}{\xi}\dot{\xi}\dot{\tau} + \frac{1}{\xi}\dot{\tau}\dot{\xi}$$

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[10] We have looked at this space-time earlier: $ds^2 = -X^2dT^2 + dX^2$

Collecting the results

$$\begin{aligned} 0 &= \ddot{\xi} + \xi\dot{\tau}^2 \\ 0 &= 2\xi\dot{\xi}\dot{\tau} + \xi^2\ddot{\tau} \\ \Rightarrow \quad \Gamma^{\xi}{}_{\tau\tau} &= \xi \\ \Gamma^{\tau}{}_{\xi\tau} &= \frac{1}{\xi} \\ \Gamma^{\tau}{}_{\tau\xi} &= \frac{1}{\xi} \end{aligned}$$

7.6.2 [l]Ricci rotation coefficients of the Rindler metric

The line element:

$$ds^2 = u^2 dv^2 - du^2$$

The Basis one forms

$$\omega^{\hat{u}} = du \qquad du = \omega^{\hat{u}}$$

$$\omega^{\hat{v}} = u dv \qquad dv = \frac{1}{u}\omega^{\hat{v}}$$

$$\eta^{ij} = \begin{Bmatrix} 1 & \\ & -1 \end{Bmatrix}$$

Cartan's First Structure equation and the calculation of the Ricci rotation coefficients $\Gamma^{\hat{a}}{}_{\hat{b}\hat{c}}$:

$$\begin{aligned} d\omega^{\hat{a}} &= -\Gamma^{\hat{a}}{}_{\hat{b}} \wedge \omega^{\hat{b}} \\ \Gamma^{\hat{a}}{}_{\hat{b}} &= \Gamma^{\hat{a}}{}_{\hat{b}\hat{c}}\omega^{\hat{c}} \\ \Rightarrow \quad d\omega^{\hat{u}} &= 0 \\ d\omega^{\hat{v}} &= d(udv) = du \wedge dv = \frac{1}{u}\omega^{\hat{u}} \wedge \omega^{\hat{v}} \\ \Rightarrow \quad \Gamma^{\hat{v}}{}_{\hat{u}} &= \frac{1}{u}\omega^{\hat{v}} \\ \Gamma^{\hat{v}}{}_{\hat{u}\hat{v}} &= \frac{1}{u} \\ \Gamma^{\hat{u}}{}_{\hat{v}\hat{v}} &= \eta^{\hat{u}\hat{u}}\Gamma_{\hat{u}\hat{v}\hat{v}} = -\eta^{\hat{u}\hat{u}}\Gamma_{\hat{v}\hat{u}\hat{v}} = -\eta^{\hat{u}\hat{u}}\eta_{\hat{v}\hat{v}}\Gamma^{\hat{v}}{}_{\hat{u}\hat{v}} = \frac{1}{u} \end{aligned}$$

7.7 [m]The Lorentz hyperboloid

7.7.1 [n]The Christoffel symbols.

The line element:

$$ds^2 = d\psi^2 + \sinh^2\psi \; d\theta^2 + \sinh^2\psi \sin^2\theta \, d\phi^2$$

To find the Christoffel symbols we calculate the geodesic from the Euler-Lagrange equation

$$\begin{aligned} 0 &= \frac{d}{ds}\left(\frac{\partial F}{\partial \dot{x}^a}\right) - \frac{\partial F}{\partial x^a} \\ \Rightarrow \quad F &= \dot{\psi}^2 + \sinh^2\psi\,\dot{\theta}^2 + \sinh^2\psi \sin^2\theta\,\dot{\phi}^2 \end{aligned}$$

$\underline{x^a = \psi:}$

$$\begin{aligned} \frac{\partial F}{\partial \psi} &= 2\cosh\psi\sinh\psi\,\dot{\theta}^2 + 2\cosh\psi\sinh\psi\sin^2\theta\,\dot{\phi}^2 \\ \frac{\partial F}{\partial \dot{\psi}} &= 2\dot{\psi} \\ \frac{d}{ds}\left(\frac{\partial F}{\partial \dot{\psi}}\right) &= 2\ddot{\psi} \\ \Rightarrow \quad 0 &= 2\ddot{\psi} - 2\cosh\psi\sinh\psi\,\dot{\theta}^2 - 2\cosh\psi\sinh\psi\sin^2\theta\,\dot{\phi}^2 \\ \Rightarrow \quad 0 &= \ddot{\psi} - \cosh\psi\sinh\psi\,\dot{\theta}^2 - \cosh\psi\sinh\psi\sin^2\theta\,\dot{\phi}^2 \end{aligned}$$

$\underline{x^a = \theta:}$

$$\frac{\partial F}{\partial \theta} = 2\cos\theta\sin\theta\sinh^2\psi\,\dot{\phi}^2$$

$$\frac{\partial F}{\partial \dot\theta} = 2\sinh^2\psi\,\dot\theta$$
$$\frac{d}{ds}\left(\frac{\partial F}{\partial \dot\theta}\right) = 4\cosh\psi\sinh\psi\,\dot\psi\dot\theta + 2\sinh^2\psi\,\ddot\theta$$
$$\Rightarrow \quad 0 = 2\sinh^2\psi\,\ddot\theta + 4\cosh\psi\sinh\psi\,\dot\psi\dot\theta - 2\cos\theta\sin\theta\sinh^2\psi\,\dot\phi^2$$
$$\Rightarrow \quad 0 = \ddot\theta + 2\coth\psi\,\dot\psi\dot\theta - \cos\theta\sin\theta\,\dot\phi^2$$

$\underline{x^a = \phi:}$

$$\frac{\partial F}{\partial \phi} = 0$$
$$\frac{\partial F}{\partial \dot\phi} = 2\sinh^2\psi\sin^2\theta\,\dot\phi$$
$$\frac{d}{ds}\left(\frac{\partial F}{\partial \dot\phi}\right) = 4\cosh\psi\sinh\psi\,\dot\psi\dot\phi + 4\cos\theta\sin\theta\,\dot\theta\dot\phi + 2\sinh^2\psi\sin^2\theta\,\ddot\phi$$
$$\Rightarrow \quad 0 = 4\cosh\psi\sinh\psi\,\dot\psi\dot\phi + 4\cos\theta\sin\theta\,\dot\theta\dot\phi + 2\sinh^2\psi\sin^2\theta\,\ddot\phi$$
$$\Rightarrow \quad 0 = \ddot\phi + 2\frac{\coth\psi}{\sin^2\theta}\dot\psi\dot\phi + 2\frac{\cot\theta}{\sinh^2\psi}\dot\theta\dot\phi$$

Collecting the results

$$0 = \ddot\psi - \cosh\psi\sinh\psi\,\dot\theta^2 - \cosh\psi\sinh\psi\sin^2\theta\,\dot\phi^2$$
$$0 = \ddot\theta + 2\cosh\psi\sinh\psi\,\dot\psi\dot\theta - \cos\theta\sin\theta\,\dot\phi^2$$
$$0 = \ddot\phi + 2\frac{\coth\psi}{\sin^2\theta}\dot\psi\dot\phi + 2\frac{\cot\theta}{\sinh^2\psi}\dot\theta\dot\phi$$

We can now find the non-zero Christoffel symbols:

$$\Gamma^{\psi}{}_{\theta\theta} = -\cosh\psi\sinh\psi$$
$$\Gamma^{\psi}{}_{\phi\phi} = -\cosh\psi\sinh\psi\sin^2\theta$$
$$\Gamma^{\theta}{}_{\psi\theta} = \Gamma^{\theta}{}_{\theta\psi} = \cosh\psi\sinh\psi$$
$$\Gamma^{\theta}{}_{\phi\phi} = -\cos\theta\sin\theta$$
$$\Gamma^{\phi}{}_{\psi\phi} = \Gamma^{\phi}{}_{\phi\psi} = \frac{\coth\psi}{\sin^2\theta}$$
$$\Gamma^{\phi}{}_{\theta\phi} = \Gamma^{\phi}{}_{\phi\theta} = \frac{\cot\theta}{\sinh^2\psi}$$

7.7.2 °The Ricci rotation coefficients

The line element:

$$ds^2 = d\psi^2 + \sinh^2\psi\;d\theta^2 + \sinh^2\psi\sin^2\theta\,d\phi^2$$

<u>The Basis one forms</u>

$$\omega^{\hat\psi} = d\psi \qquad d\psi = \omega^{\hat\psi}$$
$$\omega^{\hat\theta} = \sinh\psi\,d\theta \qquad d\theta = \frac{1}{\sinh\psi}\omega^{\hat\theta}$$
$$\omega^{\hat\phi} = \sinh\psi\sin\theta\,d\phi \qquad d\phi = \frac{1}{\sinh\psi\sin\theta}\omega^{\hat\phi}$$
$$\eta^{ij} = \begin{Bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{Bmatrix}$$

<u>Cartan's First Structure equation and the calculation of the Ricci rotation coefficients $\Gamma^{\hat a}{}_{\hat b\hat c}$:</u>

$$d\omega^{\hat a} = -\Gamma^{\hat a}{}_{\hat b} \wedge \omega^{\hat b}$$
$$\Gamma^{\hat a}{}_{\hat b} = \Gamma^{\hat a}{}_{\hat b\hat c}\omega^{\hat c}$$
$$\Rightarrow \quad d\omega^{\hat\psi} = 0$$
$$d\omega^{\hat\theta} = d(\sinh\psi\,d\theta) = \cosh\psi\,d\psi\wedge d\theta = \cosh\psi\,\omega^{\hat\psi}\wedge\frac{1}{\sinh\psi}\omega^{\hat\theta} = \coth\psi\,\omega^{\hat\psi}\wedge\omega^{\hat\theta}$$

$$\begin{aligned} d\omega^{\hat\phi} &= d(\sinh\psi \sin\theta\, d\phi) = \cosh\psi\, d\psi \wedge d\phi + \cos\theta\, d\theta \wedge d\phi \\ &= \cosh\psi\, \omega^{\hat\psi} \wedge \frac{1}{\sinh\psi \sin\theta}\omega^{\hat\phi} + \cos\theta \frac{1}{\sinh\psi}\omega^{\hat\theta} \wedge \frac{1}{\sinh\psi\sin\theta}\omega^{\hat\phi} \\ &= \frac{\coth\psi}{\sin\theta}\omega^{\hat\psi} \wedge \omega^{\hat\phi} + \frac{\cot\theta}{\sinh^2\psi}\omega^{\hat\theta} \wedge \omega^{\hat\phi} \end{aligned}$$

Summarizing the curvature one forms in a matrix:

$$\Gamma^{\hat a}{}_{\hat b} = \begin{Bmatrix} 0 & \coth\psi\, \omega^{\hat\theta} & \frac{\coth\psi}{\sin\theta}\omega^{\hat\phi} \\ -\coth\psi\,\omega^{\hat\theta} & 0 & \frac{\cot\theta}{\sinh^2\psi}\omega^{\hat\phi} \\ -\frac{\coth\psi}{\sin\theta}\omega^{\hat\phi} & -\frac{\cot\theta}{\sinh^2\psi}\omega^{\hat\phi} & 0 \end{Bmatrix}$$

Where $\hat a$ refers to column and $\hat b$ to row.

Now we can find the non-zero Ricci rotation coefficients:

$$\begin{aligned} \Gamma^{\hat\psi}{}_{\hat\theta\hat\theta} &= -\coth\psi \\ \Gamma^{\hat\psi}{}_{\hat\phi\hat\phi} &= -\frac{\coth\psi}{\sin\theta} \\ \Gamma^{\hat\theta}{}_{\hat\psi\hat\theta} &= \coth\psi \\ \Gamma^{\hat\theta}{}_{\hat\phi\hat\phi} &= -\frac{\cot\theta}{\sinh^2\psi} \\ \Gamma^{\hat\phi}{}_{\hat\psi\hat\phi} &= \frac{\coth\psi}{\sin\theta} \\ \Gamma^{\hat\phi}{}_{\hat\theta\hat\phi} &= \frac{\cot\theta}{\sinh^2\psi} \end{aligned}$$

7.8 [P]Ricci rotation coefficients, Ricci scalar and Einstein equations for a general 4-dimensional metric: $ds^2 = -dt^2 + L^2(t,r)dr^2 + B^2(t,r)d\phi^2 + M^2(t,r)dz^2$

The line element:

$$ds^2 = -dt^2 + L^2(t,r)dr^2 + B^2(t,r)d\phi^2 + M^2(t,r)dz^2$$

7.8.1 The Basis one forms

$$\begin{aligned} \omega^{\hat t} &= dt \\ \omega^{\hat r} &= L(t,r)dr & dr &= \frac{1}{L(t,r)}\omega^{\hat r} \\ \omega^{\hat\phi} &= B(t,r)d\phi & d\phi &= \frac{1}{B(t,r)}\omega^{\hat\phi} \\ \omega^{\hat z} &= M(t,r)dz & dz &= \frac{1}{M(t,r)}\omega^{\hat z} \end{aligned}$$

$$\eta^{ij} = \begin{Bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{Bmatrix}$$

7.8.2 Cartan's First Structure equation and the calculation of the Ricci rotation coefficients $\Gamma^{\hat a}{}_{\hat b\hat c}$

$$\begin{aligned} d\omega^{\hat a} &= -\Gamma^{\hat a}{}_{\hat b} \wedge \omega^{\hat b} \\ \Gamma^{\hat a}{}_{\hat b} &= \Gamma^{\hat a}{}_{\hat b\hat c}\omega^{\hat c} \\ d\omega^{\hat t} &= 0 \end{aligned}$$

$$d\omega^{\hat{r}} = d(L(t,r)dr) = \dot{L}dt \wedge dr = \frac{\dot{L}}{L}\omega^{\hat{t}} \wedge \omega^{\hat{r}}$$

$$d\omega^{\hat{\phi}} = d(B(t,r)d\phi) = \dot{B}dt \wedge d\phi + B'dr \wedge d\phi = \frac{\dot{B}}{B}\omega^{\hat{t}} \wedge \omega^{\hat{\phi}} + \frac{B'}{LB}\omega^{\hat{r}} \wedge \omega^{\hat{\phi}}$$

$$d\omega^{\hat{z}} = d(M(t,r)dz) = \dot{M}dt \wedge dz + M'dr \wedge dz = \frac{\dot{M}}{M}\omega^{\hat{t}} \wedge \omega^{\hat{z}} + \frac{M'}{LM}\omega^{\hat{r}} \wedge \omega^{\hat{z}}$$

Summarizing the curvature one forms in a matrix:

$$\Gamma^{\hat{a}}{}_{\hat{b}} = \begin{Bmatrix} 0 & \frac{\dot{L}}{L}\omega^{\hat{r}} & \frac{\dot{B}}{B}\omega^{\hat{\phi}} & \frac{\dot{M}}{M}\omega^{\hat{z}} \\ \frac{\dot{L}}{L}\omega^{\hat{r}} & 0 & \frac{B'}{LB}\omega^{\hat{\phi}} & \frac{M'}{LM}\omega^{\hat{z}} \\ \frac{\dot{B}}{B}\omega^{\hat{\phi}} & -\frac{B'}{LB}\omega^{\hat{\phi}} & 0 & 0 \\ \frac{\dot{M}}{M}\omega^{\hat{z}} & -\frac{M'}{LM}\omega^{\hat{z}} & 0 & 0 \end{Bmatrix}$$

Where $\hat{a}$ refers to column and $\hat{b}$ to row

Now we can read off the Ricci rotation coefficients

$$\Gamma^{\hat{t}}{}_{\hat{r}\hat{r}} = \frac{\dot{L}}{L} \quad \Gamma^{\hat{r}}{}_{\hat{t}\hat{r}} = \frac{\dot{L}}{L} \quad \Gamma^{\hat{\phi}}{}_{\hat{t}\hat{\phi}} = \frac{\dot{B}}{B} \quad \Gamma^{\hat{z}}{}_{\hat{t}\hat{z}} = \frac{\dot{M}}{M}$$

$$\Gamma^{\hat{t}}{}_{\hat{\phi}\hat{\phi}} = \frac{\dot{B}}{B} \quad \Gamma^{\hat{r}}{}_{\hat{\phi}\hat{\phi}} = -\frac{B'}{LB} \quad \Gamma^{\hat{\phi}}{}_{\hat{r}\hat{\phi}} = \frac{B'}{LB} \quad \Gamma^{\hat{z}}{}_{\hat{r}\hat{z}} = \frac{M'}{LM}$$

$$\Gamma^{\hat{t}}{}_{\hat{z}\hat{z}} = \frac{\dot{M}}{M} \quad \Gamma^{\hat{r}}{}_{\hat{z}\hat{z}} = -\frac{M'}{LM}$$

7.8.3 The curvature two forms

$$\Omega^{\hat{a}}{}_{\hat{b}} = d\Gamma^{\hat{a}}{}_{\hat{b}} + \Gamma^{\hat{a}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{b}} = \frac{1}{2}R^{\hat{a}}{}_{\hat{b}\hat{c}\hat{d}}\omega^{\hat{c}} \wedge \omega^{\hat{d}}$$

$$d\Gamma^{\hat{r}}{}_{\hat{t}} = d\left(\frac{\dot{L}}{L}\omega^{\hat{r}}\right) = d\big(\dot{L}(t,r)dr\big) = \ddot{L}dt \wedge dr + \dot{L}'dr \wedge dr = \frac{\ddot{L}}{L}\omega^{\hat{t}} \wedge \omega^{\hat{r}}$$

$$\Gamma^{\hat{r}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{t}} = \Gamma^{\hat{r}}{}_{\hat{t}} \wedge \Gamma^{\hat{t}}{}_{\hat{t}} + \Gamma^{\hat{r}}{}_{\hat{r}} \wedge \Gamma^{\hat{r}}{}_{\hat{t}} + \Gamma^{\hat{r}}{}_{\hat{\phi}} \wedge \Gamma^{\hat{\phi}}{}_{\hat{t}} + \Gamma^{\hat{r}}{}_{\hat{z}} \wedge \Gamma^{\hat{z}}{}_{\hat{t}} = 0$$

$$\Rightarrow \quad \Omega^{\hat{r}}{}_{\hat{t}} = -\frac{\ddot{L}}{L}\omega^{\hat{r}} \wedge \omega^{\hat{t}}$$

$$d\Gamma^{\hat{\phi}}{}_{\hat{t}} = d\left(\frac{\dot{B}}{B}\omega^{\hat{\phi}}\right) = d\big(\dot{B}(t,r)d\phi\big) = \ddot{B}dt \wedge d\phi + \dot{B}'dr \wedge d\phi$$

$$= \frac{\ddot{B}}{B}\omega^{\hat{t}} \wedge \omega^{\hat{\phi}} + \frac{\dot{B}'}{BL}\omega^{\hat{r}} \wedge \omega^{\hat{\phi}}$$

$$\Gamma^{\hat{\phi}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{t}} = \Gamma^{\hat{\phi}}{}_{\hat{t}} \wedge \Gamma^{\hat{t}}{}_{\hat{t}} + \Gamma^{\hat{\phi}}{}_{\hat{r}} \wedge \Gamma^{\hat{r}}{}_{\hat{t}} + \Gamma^{\hat{\phi}}{}_{\hat{\phi}} \wedge \Gamma^{\hat{\phi}}{}_{\hat{t}} + \Gamma^{\hat{\phi}}{}_{\hat{z}} \wedge \Gamma^{\hat{z}}{}_{\hat{t}} = \Gamma^{\hat{\phi}}{}_{\hat{r}} \wedge \Gamma^{\hat{r}}{}_{\hat{t}} = \frac{B'}{BL}\omega^{\hat{\phi}} \wedge \frac{\dot{L}}{L}\omega^{\hat{r}}$$

$$\Rightarrow \quad \Omega^{\hat{\phi}}{}_{\hat{t}} = \frac{\ddot{B}}{B}\omega^{\hat{t}} \wedge \omega^{\hat{\phi}} + \frac{\dot{B}'}{BL}\omega^{\hat{r}} \wedge \omega^{\hat{\phi}} + \frac{B'}{BL}\omega^{\hat{\phi}} \wedge \frac{\dot{L}}{L}\omega^{\hat{r}}$$

$$= -\frac{\ddot{B}}{B}\omega^{\hat{\phi}} \wedge \omega^{\hat{t}} + \left(\frac{B'\dot{L}}{BL^2} - \frac{\dot{B}'}{BL}\right)\omega^{\hat{\phi}} \wedge \omega^{\hat{r}}$$

$$d\Gamma^{\hat{z}}{}_{\hat{t}} = d\left(\frac{\dot{M}}{M}\omega^{\hat{z}}\right) = d\big(\dot{M}(t,r)dz\big) = \ddot{M}dt \wedge dz + \dot{M}'dr \wedge dz$$

$$= \frac{\ddot{M}}{M}\omega^{\hat{t}} \wedge \omega^{\hat{z}} + \frac{\dot{M}'}{ML}\omega^{\hat{r}} \wedge \omega^{\hat{z}}$$

$$
\begin{aligned}
\Gamma^{\hat z}{}_{\hat c}\wedge\Gamma^{\hat c}{}_{\hat t} &= \Gamma^{\hat z}{}_{\hat t}\wedge\Gamma^{\hat t}{}_{\hat t}+\Gamma^{\hat z}{}_{\hat r}\wedge\Gamma^{\hat r}{}_{\hat t}+\Gamma^{\hat z}{}_{\hat\phi}\wedge\Gamma^{\hat\phi}{}_{\hat t}+\Gamma^{\hat z}{}_{\hat z}\wedge\Gamma^{\hat z}{}_{\hat t}=\Gamma^{\hat z}{}_{\hat r}\wedge\Gamma^{\hat r}{}_{\hat t}=\frac{M'}{ML}\omega^{\hat z}\wedge\frac{\dot L}{L}\omega^{\hat r}\\
\Rightarrow\quad \Omega^{\hat z}{}_{\hat t} &= \frac{\ddot M}{M}\omega^{\hat t}\wedge\omega^{\hat z}+\frac{\dot M'}{ML}\omega^{\hat r}\wedge\omega^{\hat z}+\frac{M'}{ML}\omega^{\hat z}\wedge\frac{\dot L}{L}\omega^{\hat r}\\
&= -\frac{\ddot M}{M}\omega^{\hat z}\wedge\omega^{\hat t}+\left(\frac{M'\dot L}{ML^2}-\frac{\dot M'}{ML}\right)\omega^{\hat z}\wedge\omega^{\hat r}\\
d\Gamma^{\hat\phi}{}_{\hat r} &= d\left(\frac{B'}{BL}\omega^{\hat\phi}\right)=d\left(\frac{B'}{L}d\phi\right)=\frac{\dot B'L-B'\dot L}{L^2}dt\wedge d\phi+\frac{B''L-B'L'}{L^2}dr\wedge d\phi\\
&= \frac{B'\dot L-\dot B'L}{BL^2}\omega^{\hat\phi}\wedge\omega^{\hat t}+\frac{B'L'-B''L}{BL^3}\omega^{\hat\phi}\wedge\omega^{\hat r}\\
\Gamma^{\hat\phi}{}_{\hat c}\wedge\Gamma^{\hat c}{}_{\hat r} &= \Gamma^{\hat\phi}{}_{\hat r}\wedge\Gamma^{\hat r}{}_{\hat r}+\Gamma^{\hat\phi}{}_{\hat t}\wedge\Gamma^{\hat t}{}_{\hat r}+\Gamma^{\hat\phi}{}_{\hat\phi}\wedge\Gamma^{\hat\phi}{}_{\hat r}+\Gamma^{\hat\phi}{}_{\hat z}\wedge\Gamma^{\hat z}{}_{\hat r}=\Gamma^{\hat\phi}{}_{\hat t}\wedge\Gamma^{\hat t}{}_{\hat r}=\frac{\dot B\dot L}{BL}\omega^{\hat\phi}\wedge\omega^{\hat r}\\
\Rightarrow\quad \Omega^{\hat\phi}{}_{\hat r} &= \frac{B'\dot L-\dot B'L}{BL^2}\omega^{\hat\phi}\wedge\omega^{\hat t}+\frac{B'L'-B''L}{BL^3}\omega^{\hat\phi}\wedge\omega^{\hat r}+\frac{\dot B\dot L}{BL}\omega^{\hat\phi}\wedge\omega^{\hat r}\\
&= \frac{B'\dot L-\dot B'L}{BL^2}\omega^{\hat\phi}\wedge\omega^{\hat t}+\left(\frac{B'L'-B''L}{BL^3}+\frac{\dot B\dot L}{BL}\right)\omega^{\hat\phi}\wedge\omega^{\hat r}\\
d\Gamma^{\hat z}{}_{\hat r} &= d\left(\frac{M'}{ML}\omega^{\hat z}\right)=d\left(\frac{M'}{L}dz\right)=\frac{\dot M'L-M'\dot L}{L^2}dt\wedge dz+\frac{M''L-M'L'}{L^2}dr\wedge dz\\
&= \frac{M'\dot L-\dot M'L}{ML^2}\omega^{\hat z}\wedge\omega^{\hat t}+\frac{M'L'-M''L}{ML^3}\omega^{\hat z}\wedge\omega^{\hat r}\\
\Gamma^{\hat z}{}_{\hat c}\wedge\Gamma^{\hat c}{}_{\hat r} &= \Gamma^{\hat z}{}_{\hat r}\wedge\Gamma^{\hat r}{}_{\hat r}+\Gamma^{\hat z}{}_{\hat t}\wedge\Gamma^{\hat t}{}_{\hat r}+\Gamma^{\hat z}{}_{\hat\phi}\wedge\Gamma^{\hat\phi}{}_{\hat r}+\Gamma^{\hat z}{}_{\hat z}\wedge\Gamma^{\hat z}{}_{\hat r}=\Gamma^{\hat z}{}_{\hat t}\wedge\Gamma^{\hat t}{}_{\hat r}=\frac{\dot M\dot L}{ML}\omega^{\hat z}\wedge\omega^{\hat r}\\
\Rightarrow\quad \Omega^{\hat z}{}_{\hat r} &= \frac{M'\dot L-\dot M'L}{ML^2}\omega^{\hat z}\wedge\omega^{\hat t}+\frac{M'L'-M''L}{ML^3}\omega^{\hat z}\wedge\omega^{\hat r}+\frac{\dot M\dot L}{ML}\omega^{\hat z}\wedge\omega^{\hat r}\\
&= \frac{M'\dot L-\dot M'L}{ML^2}\omega^{\hat z}\wedge\omega^{\hat t}+\left(\frac{M'L'-M''L}{ML^3}+\frac{\dot M\dot L}{ML}\right)\omega^{\hat z}\wedge\omega^{\hat r}\\
d\Gamma^{\hat z}{}_{\hat\phi} &= 0\\
\Rightarrow\quad \Omega^{\hat z}{}_{\hat\phi} &= \Gamma^{\hat z}{}_{\hat c}\wedge\Gamma^{\hat c}{}_{\hat\phi}=\Gamma^{\hat z}{}_{\hat t}\wedge\Gamma^{\hat t}{}_{\hat\phi}+\Gamma^{\hat z}{}_{\hat r}\wedge\Gamma^{\hat r}{}_{\hat\phi}+\Gamma^{\hat z}{}_{\hat\phi}\wedge\Gamma^{\hat\phi}{}_{\hat\phi}+\Gamma^{\hat z}{}_{\hat z}\wedge\Gamma^{\hat z}{}_{\hat\phi}\\
&= \Gamma^{\hat z}{}_{\hat t}\wedge\Gamma^{\hat t}{}_{\hat\phi}+\Gamma^{\hat z}{}_{\hat r}\wedge\Gamma^{\hat r}{}_{\hat\phi}=\frac{\dot M}{M}\omega^{\hat z}\wedge\frac{\dot B}{B}\omega^{\hat\phi}+\frac{M'}{LM}\omega^{\hat z}\wedge\left(-\frac{B'}{LB}\omega^{\hat\phi}\right)\\
&= \left(\frac{\dot B\dot M}{BM}-\frac{B'M'}{L^2BM}\right)\omega^{\hat z}\wedge\omega^{\hat\phi}
\end{aligned}
$$

Summarized in a matrix:

$$
\Omega^{\hat a}{}_{\hat b}=\left\{\begin{array}{cccc}
0 & -\frac{\ddot L}{L}\omega^{\hat r}\wedge\omega^{\hat t} & \begin{bmatrix}-\frac{\ddot B}{B}\omega^{\hat\phi}\wedge\omega^{\hat t}\\ +\left(\frac{B'\dot L}{BL^2}-\frac{\dot B'}{BL}\right)\omega^{\hat\phi}\wedge\omega^{\hat r}\end{bmatrix} & \begin{bmatrix}-\frac{\ddot M}{M}\omega^{\hat z}\wedge\omega^{\hat t}\\ +\left(\frac{M'\dot L}{ML^2}-\frac{\dot M'}{ML}\right)\omega^{\hat z}\wedge\omega^{\hat r}\end{bmatrix}\\
S & 0 & \begin{bmatrix}\left(\frac{B'\dot L}{BL^2}-\frac{\dot B'}{BL}\right)\omega^{\hat\phi}\wedge\omega^{\hat t}\\ +\left(\frac{B'L'-B''L}{BL^3}+\frac{\dot B\dot L}{BL}\right)\omega^{\hat\phi}\wedge\omega^{\hat r}\end{bmatrix} & \begin{bmatrix}\left(\frac{M'\dot L}{ML^2}-\frac{\dot M'}{ML}\right)\omega^{\hat z}\wedge\omega^{\hat t}\\ +\left(\frac{M'L'-M''L}{ML^3}+\frac{\dot M\dot L}{ML}\right)\omega^{\hat z}\wedge\omega^{\hat r}\end{bmatrix}\\
S & AS & 0 & \left(\frac{\dot B\dot M}{BM}-\frac{B'M'}{L^2BM}\right)\omega^{\hat z}\wedge\omega^{\hat\phi}\\
S & AS & AS & 0
\end{array}\right\}
$$

Where $\hat{a}$ refers to column and $\hat{b}$ to row.
Now we can find the independent elements of the Riemann tensor in the non-coordinate basis:

$$R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}}(A) = -\frac{\ddot{L}}{L} \quad R^{\hat{\Phi}}{}_{\hat{t}\hat{\Phi}\hat{t}}(B) = -\frac{\ddot{B}}{B} \quad R^{\hat{z}}{}_{\hat{t}z\hat{t}}(C) = -\frac{\ddot{M}}{M}$$

$$R^{\hat{\Phi}}{}_{\hat{t}\hat{\Phi}\hat{r}}(D) = \frac{B'\dot{L}}{BL^2} - \frac{\dot{B}'}{BL} \quad R^{\hat{z}}{}_{\hat{t}\hat{z}\hat{r}}(E) = \frac{M'\dot{L}}{ML^2} - \frac{\dot{M}'}{ML}$$

$$R^{\hat{\Phi}}{}_{\hat{r}\hat{\Phi}\hat{r}}(F) = \frac{B'L' - B''L}{BL^3} + \frac{\dot{B}\dot{L}}{BL} \quad R^{\hat{z}}{}_{\hat{r}\hat{z}\hat{r}}(G) = \frac{M'L' - M''L}{ML^3} + \frac{\dot{M}\dot{L}}{ML}$$

$$R^{\hat{z}}{}_{\hat{\Phi}\hat{z}\hat{\Phi}}(H) = \frac{\dot{B}\dot{M}}{BM} - \frac{B'M'}{L^2BM}$$

Where A,B,C,D,E,F,G,H will be used later, to make the calculations easier

7.8.4 The Ricci tensor

$$R_{\hat{a}\hat{b}} = R^{\hat{c}}{}_{\hat{a}\hat{c}\hat{b}}$$

$$R_{\hat{t}\hat{t}} = R^{\hat{c}}{}_{\hat{t}\hat{c}\hat{t}} = R^{\hat{t}}{}_{\hat{t}\hat{t}\hat{t}} + R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}} + R^{\hat{\Phi}}{}_{\hat{t}\hat{\Phi}\hat{t}} + R^{\hat{z}}{}_{\hat{t}\hat{z}\hat{t}} = R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}} + R^{\hat{\Phi}}{}_{\hat{t}\hat{\Phi}\hat{t}} + R^{\hat{z}}{}_{\hat{t}\hat{z}\hat{t}}$$
$$= -\frac{\ddot{L}}{L} - \frac{\ddot{B}}{B} - \frac{\ddot{M}}{M} = A + B + C$$

$$R_{\hat{r}\hat{t}} = R^{\hat{c}}{}_{\hat{r}\hat{c}\hat{t}} = R^{\hat{t}}{}_{\hat{r}\hat{t}\hat{t}} + R^{\hat{r}}{}_{\hat{r}\hat{r}\hat{t}} + R^{\hat{\Phi}}{}_{\hat{r}\hat{\Phi}\hat{t}} + R^{\hat{z}}{}_{\hat{r}\hat{z}\hat{t}} = R^{\hat{\Phi}}{}_{\hat{r}\hat{\Phi}\hat{t}} + R^{\hat{z}}{}_{\hat{r}\hat{z}\hat{t}}$$
$$= \frac{B'\dot{L}}{BL^2} - \frac{\dot{B}'}{BL} + \frac{M'\dot{L}}{ML^2} - \frac{\dot{M}'}{ML} = D + E$$

$$R_{\hat{\Phi}\hat{t}} = R^{\hat{c}}{}_{\hat{\Phi}\hat{c}\hat{t}} = R^{\hat{t}}{}_{\hat{\Phi}\hat{t}\hat{t}} + R^{\hat{r}}{}_{\hat{\Phi}\hat{r}\hat{t}} + R^{\hat{\Phi}}{}_{\hat{\Phi}\hat{\Phi}\hat{t}} + R^{\hat{z}}{}_{\hat{\Phi}\hat{z}\hat{t}} = 0$$

$$R_{\hat{z}\hat{t}} = R^{\hat{c}}{}_{\hat{z}\hat{c}\hat{t}} = R^{\hat{t}}{}_{\hat{z}\hat{t}\hat{t}} + R^{\hat{r}}{}_{\hat{z}\hat{r}\hat{t}} + R^{\hat{\Phi}}{}_{\hat{z}\hat{\Phi}\hat{t}} + R^{\hat{z}}{}_{\hat{z}\hat{z}\hat{t}} = 0$$

$$R_{\hat{r}\hat{r}} = R^{\hat{c}}{}_{\hat{r}\hat{c}\hat{r}} = R^{\hat{t}}{}_{\hat{r}\hat{t}\hat{r}} + R^{\hat{r}}{}_{\hat{r}\hat{r}\hat{r}} + R^{\hat{\Phi}}{}_{\hat{r}\hat{\Phi}\hat{r}} + R^{\hat{z}}{}_{\hat{r}\hat{z}\hat{r}} = -R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}} + R^{\hat{\Phi}}{}_{\hat{r}\hat{\Phi}\hat{r}} + R^{\hat{z}}{}_{\hat{r}\hat{z}\hat{r}}$$
$$= \frac{\ddot{L}}{L} + \frac{B'L' - B''L}{BL^3} + \frac{\dot{B}\dot{L}}{BL} + \frac{M'L' - M''L}{ML^3} + \frac{\dot{M}\dot{L}}{ML} = -A + F + G$$

$$R_{\hat{\Phi}\hat{\Phi}} = R^{\hat{c}}{}_{\hat{\Phi}\hat{c}\hat{\Phi}} = R^{\hat{t}}{}_{\hat{\Phi}\hat{t}\hat{\Phi}} + R^{\hat{r}}{}_{\hat{\Phi}\hat{r}\hat{\Phi}} + R^{\hat{\Phi}}{}_{\hat{\Phi}\hat{\Phi}\hat{\Phi}} + R^{\hat{z}}{}_{\hat{\Phi}\hat{z}\hat{\Phi}} = -R^{\hat{\Phi}}{}_{\hat{t}\hat{\Phi}\hat{t}} + R^{\hat{\Phi}}{}_{\hat{r}\hat{\Phi}\hat{r}} + R^{\hat{z}}{}_{\hat{\Phi}\hat{z}\hat{\Phi}}$$
$$= \frac{\ddot{B}}{B} + \frac{B'L' - B''L}{BL^3} + \frac{\dot{B}\dot{L}}{BL} + \frac{\dot{B}\dot{M}}{BM} - \frac{B'M'}{L^2BM} = -B + F + H$$

$$R_{\hat{z}\hat{z}} = R^{\hat{c}}{}_{\hat{z}\hat{c}\hat{z}} = R^{\hat{t}}{}_{\hat{z}\hat{t}\hat{z}} + R^{\hat{r}}{}_{\hat{z}\hat{r}\hat{z}} + R^{\hat{\Phi}}{}_{\hat{z}\hat{\Phi}\hat{z}} + R^{\hat{z}}{}_{\hat{z}\hat{z}\hat{z}} = -R^{\hat{z}}{}_{\hat{t}\hat{z}\hat{t}} + R^{\hat{z}}{}_{\hat{r}\hat{z}\hat{r}} + R^{\hat{z}}{}_{\hat{\Phi}\hat{z}\hat{\Phi}}$$
$$= \frac{\ddot{M}}{M} + \frac{M'L' - M''L}{ML^3} + \frac{\dot{M}\dot{L}}{ML} + \frac{\dot{B}\dot{M}}{BM} - \frac{B'M'}{L^2BM} = -C + G + H$$

Summarized in a matrix:

$$R_{\hat{a}\hat{b}} = \begin{pmatrix} -\frac{\ddot{L}}{L}-\frac{\ddot{B}}{B}-\frac{\ddot{M}}{M} & \frac{B'\dot{L}}{BL^2}-\frac{\dot{B}'}{BL}+\frac{M'\dot{L}}{ML^2}-\frac{\dot{M}'}{ML} & 0 & 0 \\ S & \begin{bmatrix}\frac{\ddot{L}}{L}+\frac{B'L'-B''L}{BL^3}+\frac{\dot{M}\dot{L}}{ML} \\ +\frac{\dot{B}\dot{L}}{BL}+\frac{M'L'-M''L}{ML^3}\end{bmatrix} & 0 & 0 \\ 0 & 0 & \begin{bmatrix}\frac{\ddot{B}}{B}+\frac{B'L'-B''L}{BL^3} \\ +\frac{\dot{B}\dot{L}}{BL}+\frac{\dot{B}\dot{M}}{BM}-\frac{B'M'}{L^2BM}\end{bmatrix} & 0 \\ 0 & 0 & 0 & \begin{bmatrix}\frac{\ddot{M}}{M}+\frac{M'L'-M''L}{ML^3} \\ +\frac{\dot{M}\dot{L}}{ML}+\frac{\dot{B}\dot{M}}{BM}-\frac{B'M'}{L^2BM}\end{bmatrix} \end{pmatrix}$$

Where $\hat{a}$ refers to column and $\hat{b}$ to row

7.8.5 The Ricci scalar

$$\begin{aligned}
R &= \eta^{\hat{a}\hat{b}}R_{\hat{a}\hat{b}} \\
R &= \eta^{\hat{t}\hat{t}}R_{\hat{t}\hat{t}} + \eta^{\hat{r}\hat{r}}R_{\hat{r}\hat{r}} + \eta^{\hat{\phi}\hat{\phi}}R_{\hat{\phi}\hat{\phi}} + \eta^{\hat{z}\hat{z}}R_{\hat{z}\hat{z}} = -R_{\hat{t}\hat{t}} + R_{\hat{r}\hat{r}} + R_{\hat{\phi}\hat{\phi}} + R_{\hat{z}\hat{z}} \\
&= -(A+B+C) - A + F + G - B + F + H - C + G + H \\
&= -2A - 2B - 2C + 2F + 2G + 2H \\
&= -2R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}} - 2R^{\hat{\phi}}{}_{\hat{t}\hat{\phi}\hat{t}} - 2R^{\hat{z}}{}_{\hat{t}z\hat{t}} + 2R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{r}} + 2R^{\hat{z}}{}_{\hat{r}\hat{z}\hat{r}} + 2R^{\hat{z}}{}_{\hat{\phi}\hat{z}\hat{\phi}} \\
&= 2\left(\frac{\ddot{L}}{L}+\frac{\ddot{B}}{B}+\frac{\ddot{M}}{M}+\frac{B'L'-B''L}{BL^3}+\frac{\dot{B}\dot{L}}{BL}+\frac{M'L'-M''L}{ML^3}+\frac{\dot{M}\dot{L}}{ML}+\frac{\dot{B}\dot{M}}{BM}-\frac{B'M'}{L^2BM}\right)
\end{aligned}$$

7.8.6 The Einstein tensor

$$\begin{aligned}
G_{\hat{a}\hat{b}} &= R_{\hat{a}\hat{b}} - \frac{1}{2}\eta_{\hat{a}\hat{b}}R \\
G_{\hat{t}\hat{t}} &= R_{\hat{t}\hat{t}} - \frac{1}{2}\eta_{\hat{t}\hat{t}}R = R_{\hat{t}\hat{t}} + \frac{1}{2}R = A + B + C + \frac{1}{2}(-2A - 2B - 2C + 2F + 2G + 2H) \\
&= F + G + H = R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{r}} + R^{\hat{z}}{}_{\hat{r}\hat{z}\hat{r}} + R^{\hat{z}}{}_{\hat{\phi}\hat{z}\hat{\phi}} \\
&= \frac{B'L'-B''L}{BL^3}+\frac{\dot{B}\dot{L}}{BL}+\frac{M'L'-M''L}{ML^3}+\frac{\dot{M}\dot{L}}{ML}+\frac{\dot{B}\dot{M}}{BM}-\frac{B'M'}{L^2BM} \\
G_{\hat{r}\hat{t}} &= R_{\hat{r}\hat{t}} - \frac{1}{2}\eta_{\hat{r}\hat{t}}R = R_{\hat{r}\hat{t}} = \frac{B'\dot{L}}{BL^2}-\frac{\dot{B}'}{BL}+\frac{M'\dot{L}}{ML^2}-\frac{\dot{M}'}{ML} \\
G_{\hat{\phi}\hat{t}} &= R_{\hat{\phi}\hat{t}} - \frac{1}{2}\eta_{\hat{\phi}\hat{t}}R = 0 \\
G_{\hat{z}\hat{t}} &= R_{\hat{z}\hat{t}} - \frac{1}{2}\eta_{\hat{z}\hat{t}}R = 0 \\
G_{\hat{r}\hat{r}} &= R_{\hat{r}\hat{r}} - \frac{1}{2}\eta_{\hat{r}\hat{r}}R = R_{\hat{r}\hat{r}} - \frac{1}{2}R \\
&= -A + F + G - \frac{1}{2}(-2A - 2B - 2C + 2F + 2G + 2H) = B + C + H \\
&= R^{\hat{\phi}}{}_{\hat{t}\hat{\phi}\hat{t}} + R^{\hat{z}}{}_{\hat{t}z\hat{t}} + R^{\hat{z}}{}_{\hat{\phi}\hat{z}\hat{\phi}} = -\frac{\ddot{B}}{B}-\frac{\ddot{M}}{M}+\frac{\dot{B}\dot{M}}{BM}-\frac{B'M'}{L^2BM} \\
G_{\hat{\phi}\hat{r}} &= R_{\hat{\phi}\hat{r}} - \frac{1}{2}\eta_{\hat{\phi}\hat{r}}R = 0
\end{aligned}$$

$$
\begin{aligned}
G_{\hat z\hat r} &= R_{\hat z\hat r} - \frac{1}{2}\eta_{\hat z\hat r}R = 0 \\
G_{\hat\phi\hat\phi} &= R_{\hat\phi\hat\phi} - \frac{1}{2}\eta_{\hat\phi\hat\phi}R = R_{\hat\phi\hat\phi} - \frac{1}{2}R \\
&= -B + F + H - \frac{1}{2}(-2A - 2B - 2C + 2F + 2G + 2H) \\
&= A + C + G = R^{\hat r}{}_{\hat t\hat r\hat t} + R^{\hat z}{}_{\hat t z\hat t} + R^{\hat z}{}_{\hat r\hat z\hat r} = -\frac{\ddot L}{L} - \frac{\ddot M}{M} + \frac{M'L' - M''L}{ML^3} + \frac{\dot M\dot L}{ML} \\
G_{\hat z\hat\phi} &= R_{\hat z\hat\phi} - \frac{1}{2}\eta_{\hat z\hat\phi}R = 0 \\
G_{\hat z\hat z} &= R_{\hat z\hat z} - \frac{1}{2}\eta_{\hat z\hat z}R = R_{\hat z\hat z} - \frac{1}{2}R = -C + G + H - \frac{1}{2}(-2A - 2B - 2C + 2F + 2G + 2H) \\
&= A + B + F = R^{\hat r}{}_{\hat t\hat r\hat t} + R^{\hat\phi}{}_{\hat t\hat\phi\hat t} + R^{\hat\phi}{}_{\hat r\hat\phi\hat r} = -\frac{\ddot L}{L} - \frac{\ddot B}{B} + \frac{B'L' - B''L}{BL^3} + \frac{\dot B\dot L}{BL}
\end{aligned}
$$

Summarized in a matrix:

$$
G_{\hat a\hat b} = \begin{Bmatrix}
\left[\begin{matrix}\frac{B'L' - B''L}{BL^3} + \frac{\dot B\dot L}{BL} + \frac{M'L' - M''L}{ML^3} \\ + \frac{\dot M\dot L}{ML} + \frac{\dot B\dot M}{BM} - \frac{B'M'}{L^2BM}\end{matrix}\right] & \left[\begin{matrix}\frac{B'\dot L}{BL^2} - \frac{\dot B'}{BL} \\ + \frac{M'\dot L}{ML^2} - \frac{\dot M'}{ML}\end{matrix}\right] & 0 & 0 \\
S & \left[\begin{matrix}-\frac{\ddot B}{B} - \frac{\ddot M}{M} \\ + \frac{\dot B\dot M}{BM} - \frac{B'M'}{L^2BM}\end{matrix}\right] & 0 & 0 \\
0 & 0 & \left[\begin{matrix}-\frac{\ddot L}{L} - \frac{\ddot M}{M} + \frac{\dot M\dot L}{ML} \\ + \frac{M'L' - M''L}{ML^3}\end{matrix}\right] & 0 \\
0 & 0 & 0 & \left[\begin{matrix}-\frac{\ddot L}{L} - \frac{\ddot B}{B} + \frac{\dot B\dot L}{BL} \\ + \frac{B'L' - B''L}{BL^3}\end{matrix}\right]
\end{Bmatrix}
$$

Where $\hat a$ refers to column and $\hat b$ to row

7.9 The Curvature of the Poincaré Half-Plane and the Poincaré Metric.

These examples emerged from mathematicians work in the 17th century and the discovery of the hyperbolic geometry and space with negative curvature.

7.9.1 [q]The Poincaré Half-Plane

The line element

$$ds^2 = \frac{1}{y^2}dx^2 + \frac{1}{y^2}dy^2$$

The Basis One-forms

$$
\begin{aligned}
\omega^{\hat x} &= \frac{1}{y}dx \qquad & dx &= y\omega^{\hat x} \\
\omega^{\hat y} &= \frac{1}{y}dy \qquad & dy &= y\omega^{\hat y} \\
\eta^{ij} &= \begin{Bmatrix} 1 & \\ & 1 \end{Bmatrix}
\end{aligned}
$$

The Ricci rotation Coefficients

$$d\omega^{\hat{a}} = -\Gamma^{\hat{a}}{}_{\hat{b}} \wedge \omega^{\hat{b}}$$

$$d\omega^{\hat{x}} = d\left(\frac{1}{y}dx\right) = -\frac{1}{y^2}dy \wedge dx = -\omega^{\hat{y}} \wedge \omega^{\hat{x}} = \omega^{\hat{x}} \wedge \omega^{\hat{y}}$$

$$d\omega^{\hat{y}} = 0$$

$$\Rightarrow \quad \Gamma^{\hat{x}}{}_{\hat{y}} = -\omega^{\hat{x}}$$

The Curvature Two-forms and The Riemann Tensor

$$\Omega^{\hat{a}}{}_{\hat{b}} = d\Gamma^{\hat{a}}{}_{\hat{b}} + \Gamma^{\hat{a}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{b}} = \frac{1}{2}R^{\hat{a}}{}_{\hat{b}\hat{c}\hat{d}}\omega^{\hat{c}} \wedge \omega^{\hat{d}}$$

$$d\Gamma^{\hat{x}}{}_{\hat{y}} = d\left(-\omega^{\hat{x}}\right) = -\omega^{\hat{x}} \wedge \omega^{\hat{y}}$$

$$\Rightarrow \quad \Omega^{\hat{x}}{}_{\hat{y}} = -\omega^{\hat{x}} \wedge \omega^{\hat{y}}$$

$$\Rightarrow \quad R^{\hat{x}}{}_{\hat{y}\hat{x}\hat{y}} = -1$$

The Ricci Tensor

$$R_{\hat{a}\hat{b}} = R^{\hat{c}}{}_{\hat{a}\hat{c}\hat{b}}$$

$$R_{\hat{x}\hat{x}} = R^{\hat{y}}{}_{\hat{x}\hat{y}\hat{x}} = -1$$

$$R_{\hat{y}\hat{y}} = R^{\hat{x}}{}_{\hat{y}\hat{x}\hat{y}} = -1$$

The Ricci scalar

$$R = R^{\hat{a}}{}_{\hat{a}} = R^{\hat{x}}{}_{\hat{x}} + R^{\hat{y}}{}_{\hat{y}} = -2$$

7.9.2 'The Poincaré metric

The line element

$$ds^2 = \frac{4}{1-x^2-y^2}dx^2 + \frac{4}{1-x^2-y^2}dy^2 \qquad (7.1.)$$

The Basis One-forms

$$\omega^{\hat{x}} = \frac{2}{\sqrt{1-x^2-y^2}}dx \qquad dx = \frac{1}{2}\sqrt{1-x^2-y^2}\,\omega^{\hat{x}}$$

$$\omega^{\hat{y}} = \frac{2}{\sqrt{1-x^2-y^2}}dy \qquad dy = \frac{1}{2}\sqrt{1-x^2-y^2}\,\omega^{\hat{y}}$$

$$\eta^{ij} = \begin{Bmatrix} 1 & \\ & 1 \end{Bmatrix}$$

The Ricci rotation Coefficients

$$d\omega^{\hat{a}} = -\Gamma^{\hat{a}}{}_{\hat{b}} \wedge \omega^{\hat{b}}$$

$$d\omega^{\hat{x}} = d\left(\frac{2}{\sqrt{1-x^2-y^2}}dx\right) = \frac{2y}{(1-x^2-y^2)^{\frac{3}{2}}}dy \wedge dx$$

$$= \frac{2y}{(1-x^2-y^2)^{\frac{3}{2}}}\left(\frac{1}{2}\sqrt{1-x^2-y^2}\,\omega^{\hat{y}}\right) \wedge \left(\frac{1}{2}\sqrt{1-x^2-y^2}\,\omega^{\hat{x}}\right)$$

$$= \frac{y}{2\sqrt{1-x^2-y^2}}\omega^{\hat{y}} \wedge \omega^{\hat{x}} = -\frac{y}{2\sqrt{1-x^2-y^2}}\omega^{\hat{x}} \wedge \omega^{\hat{y}}$$

$$d\omega^{\hat{y}} = d\left(\frac{2}{\sqrt{1-x^2-y^2}}dy\right) = \frac{2x}{(1-x^2-y^2)^{\frac{3}{2}}}dx \wedge dy = -\frac{x}{2\sqrt{1-x^2-y^2}}\omega^{\hat{y}} \wedge \omega^{\hat{x}}$$

$$\Rightarrow \quad \Gamma^{\hat{x}}{}_{\hat{y}} = \frac{y}{2\sqrt{1-x^2-y^2}}\omega^{\hat{x}} + {}^{11}Antisymmetric$$

[11] The curvature one-forms are antisymmetric: $\Gamma_{\hat{a}\hat{b}} = -\Gamma_{\hat{b}\hat{a}}$

$$\Gamma^{\hat{y}}{}_{\hat{x}} = \frac{x}{2\sqrt{1-x^2-y^2}}\omega^{\hat{y}} + {}^{12}Antisymmetric$$

$$\Rightarrow \quad \Gamma^{\hat{x}}{}_{\hat{y}} = \frac{y}{2\sqrt{1-x^2-y^2}}\omega^{\hat{x}} - \frac{x}{2\sqrt{1-x^2-y^2}}\omega^{\hat{y}}$$

$$\Gamma^{\hat{y}}{}_{\hat{x}} = \frac{x}{2\sqrt{1-x^2-y^2}}\omega^{\hat{y}} - \frac{y}{2\sqrt{1-x^2-y^2}}\omega^{\hat{x}}$$

The Curvature Two-forms and The Riemann Tensor

$$\Omega^{\hat{a}}{}_{\hat{b}} = d\Gamma^{\hat{a}}{}_{\hat{b}} + \Gamma^{\hat{a}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{b}} = {}^{13}d\Gamma^{\hat{a}}{}_{\hat{b}} = \frac{1}{2}R^{\hat{a}}{}_{\hat{b}\hat{c}\hat{d}}\omega^{\hat{c}} \wedge \omega^{\hat{d}}$$

$$\Omega^{\hat{x}}{}_{\hat{y}} = d\Gamma^{\hat{x}}{}_{\hat{y}} + \Gamma^{\hat{x}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{y}} = d\left(\frac{y}{2\sqrt{1-x^2-y^2}}\omega^{\hat{x}}\right) - d\left(\frac{x}{2\sqrt{1-x^2-y^2}}\omega^{\hat{y}}\right)$$

$$= d\left(\frac{y}{2\sqrt{1-x^2-y^2}}\frac{2}{\sqrt{1-x^2-y^2}}dx\right) - d\left(\frac{x}{2\sqrt{1-x^2-y^2}}\frac{2}{\sqrt{1-x^2-y^2}}dy\right)$$

$$= d\left(\frac{y}{1-x^2-y^2}dx\right) - d\left(\frac{x}{1-x^2-y^2}dy\right)$$

$$= \frac{1-x^2+y^2}{(1-x^2-y^2)^2}dy \wedge dx - d\frac{1+x^2-y^2}{(1-x^2-y^2)^2}dx \wedge dy = \frac{2}{(1-x^2-y^2)^2}dy \wedge dx$$

$$= \frac{2}{(1-x^2-y^2)^2}\left(\frac{1}{2}\sqrt{1-x^2-y^2}\omega^{\hat{y}}\right) \wedge \left(\frac{1}{2}\sqrt{1-x^2-y^2}\omega^{\hat{x}}\right)$$

$$= \frac{1}{2(1-x^2-y^2)}\omega^{\hat{y}} \wedge \omega^{\hat{x}}$$

$$\Rightarrow \quad \Omega^{\hat{x}}{}_{\hat{y}} = -\frac{1}{2(1-x^2-y^2)}\omega^{\hat{x}} \wedge \omega^{\hat{y}}$$

$$\Rightarrow \quad R^{\hat{x}}{}_{\hat{y}\hat{x}\hat{y}} = -\frac{1}{2(1-x^2-y^2)}$$

The Ricci Tensor

$$R_{\hat{a}\hat{b}} = R^{\hat{c}}{}_{\hat{a}\hat{c}\hat{b}}$$

$$R_{\hat{x}\hat{x}} = R^{\hat{y}}{}_{\hat{x}\hat{y}\hat{x}} = -\frac{1}{2(1-x^2-y^2)}$$

$$R_{\hat{y}\hat{y}} = R^{\hat{x}}{}_{\hat{y}\hat{x}\hat{y}} = -\frac{1}{2(1-x^2-y^2)}$$

The Ricci scalar

$$R = R^{\hat{a}}{}_{\hat{a}} = R^{\hat{x}}{}_{\hat{x}} + R^{\hat{y}}{}_{\hat{y}} = -\frac{1}{(1-x^2-y^2)}$$

7.10 The Tolman-Bondi- de Sitter metric (Spherical dust with a cosmological constant)

7.10.1 [s]Ricci rotation coefficients in the Tolman-Bondi- de Sitter metric

The line element:

$$ds^2 = dt^2 - e^{-2\psi(t,r)}dr^2 - R^2(t,r)d\theta^2 - R^2(t,r)\sin^2\theta\, d\phi^2$$

<u>The Basis one forms</u>

$$\omega^{\hat{t}} = dt$$

$$\omega^{\hat{r}} = e^{-\psi(t,r)}dr \qquad dr = e^{\psi(t,r)}\omega^{\hat{r}}$$

$$\omega^{\hat{\theta}} = R(t,r)d\theta \qquad d\theta = \frac{1}{R(2t,r)}\omega^{\hat{\theta}}$$

---

[12] The curvature one-forms are antisymmetric: $\Gamma_{\hat{a}\hat{b}} = -\Gamma_{\hat{b}\hat{a}}$

[13] $\Gamma^{\hat{a}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{b}} = 0$

$$\omega^{\hat\phi} = R(t,r)\sin\theta\, d\phi \qquad d\phi = \frac{1}{R(t,r)\sin\theta}\omega^{\hat\phi}$$

$$\eta^{ij} = \begin{Bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{Bmatrix}$$

$$\Lambda^{\hat a}{}_b = \begin{Bmatrix} 1 & & & \\ & e^{-\psi(t,r)} & & \\ & & R(t,r) & \\ & & & R(t,r)\sin\theta \end{Bmatrix}$$

$$\Lambda^b{}_{\hat a} = \begin{Bmatrix} 1 & & & \\ & e^{\psi(t,r)} & & \\ & & \frac{1}{R(t,r)} & \\ & & & \frac{1}{R(t,r)\sin\theta} \end{Bmatrix}$$

Cartan's First Structure equation and the calculation of the Ricci rotation coefficients $\Gamma^{\hat a}{}_{\hat b\hat c}$:

$$d\omega^{\hat a} = -\Gamma^{\hat a}{}_{\hat b} \wedge \omega^{\hat b}$$

$$\Gamma^{\hat a}{}_{\hat b} = \Gamma^{\hat a}{}_{\hat b\hat c}\omega^{\hat c}$$

$$\Rightarrow \quad d\omega^{\hat t} = 0$$

$$d\omega^{\hat r} = d\left(e^{-\psi(t,r)}dr\right) = -\dot\psi e^{-\psi(t,r)}dt \wedge dr = -\dot\psi\omega^{\hat t} \wedge \omega^{\hat r}$$

$$d\omega^{\hat\theta} = d(R(t,r)d\theta) = \dot R dt \wedge d\theta + R' dr \wedge d\theta = \frac{\dot R}{R}\omega^{\hat t} \wedge \omega^{\hat\theta} + \frac{R'}{R}e^{\psi(t,r)}\omega^{\hat r} \wedge \omega^{\hat\theta}$$

$$\begin{aligned} d\omega^{\hat\phi} &= d(R(t,r)\sin\theta\, d\phi) \\ &= \dot R\sin\theta\, dt \wedge d\phi + R'\sin\theta\, dr \wedge d\phi + R(t,r)\cos\theta\, d\theta \wedge d\phi \\ &= \frac{\dot R}{R}\omega^{\hat t} \wedge \omega^{\hat\phi} + \frac{R'}{R}e^{\psi(t,r)}\omega^{\hat r} \wedge \omega^{\hat\phi} + \frac{\cot\theta}{R}\omega^{\hat\theta} \wedge \omega^{\hat\phi} \end{aligned}$$

Summarizing the curvature one forms in a matrix:

$$\Gamma^{\hat a}{}_{\hat b} = \begin{Bmatrix} 0 & -\dot\psi\omega^{\hat r} & \frac{\dot R}{R}\omega^{\hat\theta} & \frac{\dot R}{R}\omega^{\hat\phi} \\ -\dot\psi\omega^{\hat r} & 0 & \frac{R'}{R}e^{\psi(t,r)}\omega^{\hat\theta} & \frac{R'}{R}e^{\psi(t,r)}\omega^{\hat\phi} \\ \frac{\dot R}{R}\omega^{\hat\theta} & -\frac{R'}{R}e^{\psi(t,r)}\omega^{\hat\theta} & 0 & \frac{\cot\theta}{R}\omega^{\hat\phi} \\ \frac{\dot R}{R}\omega^{\hat\phi} & -\frac{R'}{R}e^{\psi(t,r)}\omega^{\hat\phi} & -\frac{\cot\theta}{R}\omega^{\hat\phi} & 0 \end{Bmatrix}$$

Where $\hat a$ refers to column and $\hat b$ to row.

Now we can read off the Ricci rotation coefficients

$$\Gamma^{\hat t}{}_{\hat r\hat r} = -\dot\psi \qquad \Gamma^{\hat r}{}_{\hat t\hat r} = -\dot\psi$$

$$\Gamma^{\hat t}{}_{\hat\theta\hat\theta} = \frac{\dot R}{R} \qquad \Gamma^{\hat r}{}_{\hat\theta\hat\theta} = -\frac{R'}{R}e^{\psi(t,r)}$$

$$\Gamma^{\hat t}{}_{\hat\phi\hat\phi} = \frac{\dot R}{R} \qquad \Gamma^{\hat r}{}_{\hat\phi\hat\phi} = -\frac{R'}{R}e^{\psi(t,r)}$$

$$\Gamma^{\hat\theta}{}_{\hat t\hat\theta} = \frac{\dot R}{R} \qquad \Gamma^{\hat\phi}{}_{\hat t\hat\phi} = \frac{\dot R}{R}$$

$$\Gamma^{\hat\theta}{}_{\hat r\hat\theta} = \frac{R'}{R}e^{\psi(t,r)} \qquad \Gamma^{\hat\phi}{}_{\hat r\hat\phi} = \frac{R'}{R}e^{\psi(t,r)}$$

$$\Gamma^{\hat\theta}{}_{\hat\phi\hat\phi} = -\frac{\cot\theta}{R} \qquad \Gamma^{\hat\phi}{}_{\hat\theta\hat\phi} = \frac{\cot\theta}{R}$$

Transformation of the Ricci rotation coefficients $\Gamma^{\hat a}{}_{\hat b\hat c}$ into the Christoffel symbols $\Gamma^{a}{}_{bc}$

We have the transformation

$$\Gamma^a{}_{bc} = \Lambda^a{}_{\hat d}\Gamma^{\hat d}{}_{\hat e\hat f}\Lambda^{\hat e}{}_b\Lambda^{\hat f}{}_c$$

$$\Rightarrow \quad \Gamma^r{}_{tr} = \Lambda^r{}_{\hat d}\Gamma^{\hat d}{}_{\hat e\hat f}\Lambda^{\hat e}{}_t\Lambda^{\hat f}{}_r = \Gamma^{\hat r}{}_{\hat t\hat r}\Lambda^{\hat t}{}_t = -\dot\psi\cdot 1 = -\dot\psi$$

$$\Gamma^t{}_{rr} = \Lambda^t{}_{\hat d}\Gamma^{\hat d}{}_{\hat e\hat f}\Lambda^{\hat e}{}_r\Lambda^{\hat f}{}_r = \Lambda^t{}_{\hat t}\Gamma^{\hat t}{}_{\hat r\hat r}\left(\Lambda^{\hat r}{}_r\right)^2 = 1(-\dot\psi)\left(e^{-\psi(t,r)}\right)^2 = -\dot\psi e^{-2\psi(t,r)}$$

$$\Gamma^\theta{}_{t\theta} = \Lambda^\theta{}_{\hat d}\Gamma^{\hat d}{}_{\hat e\hat f}\Lambda^{\hat e}{}_t\Lambda^{\hat f}{}_\theta = \Gamma^{\hat\theta}{}_{\hat t\hat\theta}\Lambda^{\hat t}{}_t = \frac{\dot R}{R}\cdot 1 = \frac{\dot R}{R}$$

$$\Gamma^t{}_{\theta\theta} = \Lambda^t{}_{\hat d}\Gamma^{\hat d}{}_{\hat e\hat f}\Lambda^{\hat e}{}_\theta\Lambda^{\hat f}{}_\theta = \Lambda^t{}_{\hat t}\Gamma^{\hat t}{}_{\hat\theta\hat\theta}\left(\Lambda^{\hat\theta}{}_\theta\right)^2 = 1\cdot\frac{\dot R}{R}R^2 = R\dot R$$

$$\Gamma^\theta{}_{r\theta} = \Lambda^\theta{}_{\hat d}\Gamma^{\hat d}{}_{\hat e\hat f}\Lambda^{\hat e}{}_r\Lambda^{\hat f}{}_\theta = \Gamma^{\hat\theta}{}_{\hat r\hat\theta}\Lambda^{\hat r}{}_r = \frac{R'}{R}e^{\psi(t,r)}e^{-\psi(t,r)} = \frac{R'}{R}$$

$$\Gamma^r{}_{\theta\theta} = \Lambda^r{}_{\hat r}\Gamma^{\hat r}{}_{\hat\theta\hat\theta}\left(\Lambda^{\hat\theta}{}_\theta\right)^2 = e^{\psi(t,r)}\left(-\frac{R'}{R}e^{\psi(t,r)}\right)R^2 = -RR'e^{2\psi(t,r)}$$

$$\Gamma^\phi{}_{t\phi} = \Lambda^\phi{}_{\hat d}\Gamma^{\hat d}{}_{\hat e\hat f}\Lambda^{\hat e}{}_t\Lambda^{\hat f}{}_\phi = \Gamma^{\hat\phi}{}_{\hat t\hat\phi}\Lambda^{\hat t}{}_t = \frac{\dot R}{R}\cdot 1 = \frac{\dot R}{R}$$

$$\Gamma^t{}_{\phi\phi} = \Lambda^t{}_{\hat d}\Gamma^{\hat d}{}_{\hat e\hat f}\Lambda^{\hat e}{}_\phi\Lambda^{\hat f}{}_\phi = \Lambda^t{}_{\hat t}\Gamma^{\hat t}{}_{\hat\phi\hat\phi}\left(\Lambda^{\hat\phi}{}_\phi\right)^2 = 1\cdot\frac{\dot R}{R}R^2\sin^2\theta = R\dot R\sin^2\theta$$

$$\Gamma^\phi{}_{r\phi} = \Lambda^\phi{}_{\hat d}\Gamma^{\hat d}{}_{\hat e\hat f}\Lambda^{\hat e}{}_r\Lambda^{\hat f}{}_\phi = \Gamma^{\hat\phi}{}_{\hat r\hat\phi}\Lambda^{\hat r}{}_r = \frac{R'}{R}e^{\psi(t,r)}e^{-\psi(t,r)} = \frac{R'}{R}$$

$$\Gamma^r{}_{\phi\phi} = \Lambda^r{}_{\hat r}\Gamma^{\hat r}{}_{\hat\phi\hat\phi}\left(\Lambda^{\hat\phi}{}_\phi\right)^2 = e^{\psi(t,r)}\left(-\frac{R'}{R}e^{\psi(t,r)}\right)R^2\sin^2\theta = -RR'e^{2\psi(t,r)}\sin^2\theta$$

$$\Gamma^\phi{}_{\theta\phi} = \Lambda^\phi{}_{\hat d}\Gamma^{\hat d}{}_{\hat e\hat f}\Lambda^{\hat e}{}_\theta\Lambda^{\hat f}{}_\phi = \Gamma^{\hat\phi}{}_{\hat\theta\hat\phi}\Lambda^{\hat\theta}{}_\theta = \frac{\cot\theta}{R}R = \cot\theta$$

$$\Gamma^\theta{}_{\phi\phi} = \Lambda^\theta{}_{\hat\theta}\Gamma^{\hat\theta}{}_{\hat\phi\hat\phi}\left(\Lambda^{\hat\phi}{}_\phi\right)^2 = \frac{1}{R}\left(-\frac{\cot\theta}{R}\right)R^2\sin^2\theta = -\cos\theta\sin\theta$$

However by this method we do not obtain $\Gamma^r{}_{rr} = -\psi'$, because the Ricci rotaion coefficient $\Gamma^{\hat r}{}_{\hat r\hat r}$ does not exist. The conclusion must be, that this is not a reliable method to calculate Christoffel symbols.

To check we calculate the Christoffel symbols directly from the metric

The line element:

$$ds^2 = dt^2 - e^{-2\psi(t,r)}dr^2 - R^2(t,r)d\theta^2 - R^2(t,r)\sin^2\theta\, d\phi^2$$

The metric tensor and its inverse

$$g_{ab} = \begin{Bmatrix} 1 & 0 & 0 & 0 \\ 0 & -e^{-2\psi(t,r)} & 0 & 0 \\ 0 & 0 & -R^2(t,r) & 0 \\ 0 & 0 & 0 & -R^2(t,r)\sin^2\theta \end{Bmatrix}$$

$$g^{ab} = \begin{Bmatrix} 1 & 0 & 0 & 0 \\ 0 & -e^{2\psi(t,r)} & 0 & 0 \\ 0 & 0 & -\frac{1}{R^2(t,r)} & 0 \\ 0 & 0 & 0 & -\frac{1}{R^2(t,r)\sin^2\theta} \end{Bmatrix}$$

$$\Gamma_{abc} = \frac{1}{2}\left(\partial_c g_{ab} + \partial_b g_{ac} - \partial_a g_{bc}\right)$$

$$\Gamma^a{}_{bc} = g^{ad}\Gamma_{dbc}$$

$$\Gamma^r{}_{rr} = g^{rr}\Gamma_{rrr} = g^{rr}\frac{1}{2}\partial_r\left(-e^{-2\psi(t,r)}\right) = g^{rr}\psi'^{e^{-2\psi(t,r)}} = -e^{2\psi(t,r)}\psi' e^{-2\psi(t,r)} = -\psi'$$

$$\begin{aligned}
\Gamma^{t}{}_{rr} &= g^{tt}\Gamma_{trr} \\
&= g^{tt}\left(-\frac{1}{2}\partial_t\left(-e^{-2\psi(t,r)}\right)\right) = g^{tt}\left(-\dot{\psi}e^{-2\psi(t,r)}\right) = 1\left(-\dot{\psi}e^{-2\psi(t,r)}\right) = -\dot{\psi}e^{-2\psi(t,r)} \\
\Gamma^{r}{}_{tr} &= \Gamma^{r}{}_{rt} = g^{rr}\Gamma_{rtr} = g^{rr}\frac{1}{2}\partial_t\left(-e^{-2\psi(t,r)}\right) = g^{rr}\dot{\psi}e^{-2\psi(t,r)} = -e^{2\psi(t,r)}\dot{\psi}e^{-2\psi(t,r)} \\
&= -\dot{\psi} \\
\Gamma^{t}{}_{\theta\theta} &= g^{tt}\Gamma_{t\theta\theta} = g^{tt}\left(-\frac{1}{2}\partial_t\left(-R^2(t,r)\right)\right) = g^{tt}\dot{R}R = 1\cdot\dot{R}R = \dot{R}R \\
\Gamma^{\theta}{}_{t\theta} &= \Gamma^{\theta}{}_{\theta t} = g^{\theta\theta}\Gamma_{\theta t\theta} = g^{\theta\theta}\frac{1}{2}\partial_t\left(-R^2(t,r)\right) = g^{\theta\theta}\left(-\dot{R}R\right) = -\frac{1}{R^2}\left(-\dot{R}R\right) = \frac{\dot{R}}{R} \\
\Gamma^{r}{}_{\theta\theta} &= g^{rr}\Gamma_{r\theta\theta} = g^{rr}\left(-\frac{1}{2}\partial_r\left(-R^2(t,r)\right)\right) = g^{rr}R'R = -e^{2\psi(t,r)}R'R \\
\Gamma^{\theta}{}_{r\theta} &= \Gamma^{\theta}{}_{\theta r} = g^{\theta\theta}\Gamma_{\theta r\theta} = g^{\theta\theta}\left(\frac{1}{2}\partial_r\left(-R^2(t,r)\right)\right) = g^{\theta\theta}(-R'R) = -\frac{1}{R^2}(-R'R) = \frac{R'}{R} \\
\Gamma^{t}{}_{\phi\phi} &= g^{tt}\Gamma_{t\phi\phi} = g^{tt}\left(-\frac{1}{2}\partial_t(-R^2(t,r)\sin^2\theta\,)\right) = g^{tt}\dot{R}R\sin^2\theta = 1\cdot\dot{R}R\sin^2\theta \\
&= \dot{R}R\sin^2\theta \\
\Gamma^{\phi}{}_{t\phi} &= \Gamma^{\phi}{}_{\phi t} = g^{\phi\phi}\Gamma_{\phi t\phi} = g^{\phi\phi}\frac{1}{2}\partial_t(-R^2(t,r)\sin^2\theta\,) = g^{\phi\phi}\left(-\dot{R}R\sin^2\theta\right) \\
&= \left(-\frac{1}{R^2\sin^2\theta}\right)\left(-\dot{R}R\sin^2\theta\right) = \frac{\dot{R}}{R} \\
\Gamma^{r}{}_{\phi\phi} &= g^{rr}\Gamma_{r\phi\phi} = g^{rr}\left(-\frac{1}{2}\partial_r(-R^2(t,r)\sin^2\theta\,)\right) = g^{rr}R'R\sin^2\theta = -e^{2\psi(t,r)}R'R\sin^2\theta \\
\Gamma^{\phi}{}_{r\phi} &= \Gamma^{\phi}{}_{\phi r} = g^{\phi\phi}\Gamma_{\phi r\phi} = g^{\phi\phi}\frac{1}{2}\partial_r(-R^2(t,r)\sin^2\theta\,) = g^{\phi\phi}(-R'R\sin^2\theta) \\
&= \left(-\frac{1}{R^2\sin^2\theta}\right)(-R'R\sin^2\theta) = \frac{R'}{R} \\
\Gamma^{\theta}{}_{\phi\phi} &= g^{\theta\theta}\Gamma_{\theta\phi\phi} = g^{\theta\theta}\left(-\frac{1}{2}\partial_\theta(-R^2(t,r)\sin^2\theta\,)\right) = g^{\theta\theta}R^2\sin\theta\cos\theta \\
&= -\frac{1}{R^2}R^2\sin\theta\cos\theta = -\sin\theta\cos\theta \\
\Gamma^{\phi}{}_{\theta\phi} &= \Gamma^{\phi}{}_{\phi\theta} = g^{\phi\phi}\Gamma_{\phi\theta\phi} = g^{\phi\phi}\frac{1}{2}\partial_\theta(-R^2(t,r)\sin^2\theta\,) = g^{\phi\phi}(-R^2\sin\theta\cos\theta) \\
&= -\frac{1}{R^2\sin^2\theta}(-R^2\sin\theta\cos\theta) = \cot\theta
\end{aligned}$$

7.10.2 The curvature two forms

$$\begin{aligned}
\Omega^{\hat{a}}{}_{\hat{b}} &= d\Gamma^{\hat{a}}{}_{\hat{b}} + \Gamma^{\hat{a}}{}_{\hat{c}}\wedge\Gamma^{\hat{c}}{}_{\hat{b}} = \frac{1}{2}R^{\hat{a}}{}_{\hat{b}\hat{c}\hat{d}}\omega^{\hat{c}}\wedge\omega^{\hat{d}} \\
d\Gamma^{\hat{r}}{}_{\hat{t}} &= d\left(-\dot{\psi}\omega^{\hat{r}}\right) = d\left(-\dot{\psi}e^{-\psi(t,r)}dr\right) = [-\ddot{\psi}e^{-\psi(t,r)} + \left(\dot{\psi}\right)^2e^{-\psi(t,r)}]dt\wedge dr \\
&= [-\ddot{\psi} + \left(\dot{\psi}\right)^2]\omega^{\hat{t}}\wedge\omega^{\hat{r}} \\
\Gamma^{\hat{r}}{}_{\hat{c}}\wedge\Gamma^{\hat{c}}{}_{\hat{t}} &= \Gamma^{\hat{r}}{}_{\hat{t}}\wedge\Gamma^{\hat{t}}{}_{\hat{t}} + \Gamma^{\hat{r}}{}_{\hat{r}}\wedge\Gamma^{\hat{r}}{}_{\hat{t}} + \Gamma^{\hat{r}}{}_{\hat{\theta}}\wedge\Gamma^{\hat{\theta}}{}_{\hat{t}} + \Gamma^{\hat{r}}{}_{\hat{\phi}}\wedge\Gamma^{\hat{\phi}}{}_{\hat{t}} = 0 \\
\Rightarrow \quad \Omega^{\hat{r}}{}_{\hat{t}} &= [-\ddot{\psi} + \left(\dot{\psi}\right)^2]\omega^{\hat{t}}\wedge\omega^{\hat{r}} \\
d\Gamma^{\hat{\theta}}{}_{\hat{t}} &= d\left(\frac{\dot{R}}{R}\omega^{\hat{\theta}}\right) = d\left(\dot{R}(t,r)d\theta\right) = \ddot{R}dt\wedge d\theta + (\dot{R})'dr\wedge d\theta \\
&= \frac{\ddot{R}}{R}\omega^{\hat{t}}\wedge\omega^{\hat{\theta}} + \frac{(\dot{R})'}{R}e^{\psi(t,r)}\omega^{\hat{r}}\wedge\omega^{\hat{\theta}}
\end{aligned}$$

$$\Gamma^{\hat\theta}{}_{\hat c}\wedge\Gamma^{\hat c}{}_{\hat t} = \Gamma^{\hat\theta}{}_{\hat t}\wedge\Gamma^{\hat t}{}_{\hat t} + \Gamma^{\hat\theta}{}_{\hat r}\wedge\Gamma^{\hat r}{}_{\hat t} + \Gamma^{\hat\theta}{}_{\hat\theta}\wedge\Gamma^{\hat\theta}{}_{\hat t} + \Gamma^{\hat\theta}{}_{\hat\phi}\wedge\Gamma^{\hat\phi}{}_{\hat t} = \Gamma^{\hat\theta}{}_{\hat r}\wedge\Gamma^{\hat r}{}_{\hat t}$$
$$= \frac{R'}{R}e^{\psi(t,r)}\omega^{\hat\theta}\wedge\left(-\dot\psi\omega^{\hat r}\right)$$
$$\Rightarrow \quad \Omega^{\hat\theta}{}_{\hat t} = \frac{\ddot R}{R}\omega^{\hat t}\wedge\omega^{\hat\theta} + \frac{\left(\dot R\right)' + R'\dot\psi}{R}e^{\psi(t,r)}\omega^{\hat r}\wedge\omega^{\hat\theta}$$
$$d\Gamma^{\hat\phi}{}_{\hat t} = d\left(\frac{\dot R}{R}\omega^{\hat\phi}\right) = d\left(\dot R(t,r)\sin\theta\, d\phi\right)$$
$$= \ddot R\sin\theta\, dt\wedge d\phi + \left(\dot R\right)'\sin\theta\, dr\wedge d\phi + \dot R\cos\theta\, d\theta\wedge d\phi$$
$$= \frac{\ddot R}{R}\omega^{\hat t}\wedge\omega^{\hat\phi} + \frac{\left(\dot R\right)'}{R}e^{\psi(t,r)}\omega^{\hat r}\wedge\omega^{\hat\phi} + \frac{\dot R}{R^2}\cot\theta\,\omega^{\hat\theta}\wedge\omega^{\hat\phi}$$
$$\Gamma^{\hat\phi}{}_{\hat c}\wedge\Gamma^{\hat c}{}_{\hat t} = \Gamma^{\hat\phi}{}_{\hat t}\wedge\Gamma^{\hat t}{}_{\hat t} + \Gamma^{\hat\phi}{}_{\hat r}\wedge\Gamma^{\hat r}{}_{\hat t} + \Gamma^{\hat\phi}{}_{\hat\theta}\wedge\Gamma^{\hat\theta}{}_{\hat t} + \Gamma^{\hat\phi}{}_{\hat\phi}\wedge\Gamma^{\hat\phi}{}_{\hat t} = \Gamma^{\hat\phi}{}_{\hat r}\wedge\Gamma^{\hat r}{}_{\hat t} + \Gamma^{\hat\phi}{}_{\hat\theta}\wedge\Gamma^{\hat\theta}{}_{\hat t}$$
$$= \frac{R'}{R}e^{\psi(t,r)}\omega^{\hat\phi}\wedge\left(-\dot\psi\omega^{\hat r}\right) + \frac{\cot\theta}{R}\omega^{\hat\phi}\wedge\frac{\dot R}{R}\omega^{\hat\theta}$$
$$\Rightarrow \quad \Omega^{\hat\phi}{}_{\hat t} = \frac{\ddot R}{R}\omega^{\hat t}\wedge\omega^{\hat\phi} + \frac{\left(\dot R\right)' + R'\dot\psi}{R}e^{\psi(t,r)}\omega^{\hat r}\wedge\omega^{\hat\phi}$$
$$d\Gamma^{\hat\theta}{}_{\hat r} = d\left(\frac{R'}{R}e^{\psi(t,r)}\omega^{\hat\theta}\right) = d\left(R'e^{\psi(t,r)}d\theta\right)$$
$$= \left[\left(\dot R\right)'e^{\psi(t,r)} + R'\dot\psi e^{\psi(t,r)}\right]dt\wedge d\theta + \left[R''e^{\psi(t,r)} + R'\psi'e^{\psi(t,r)}\right]dr\wedge d\theta$$
$$= \left[\left(\dot R\right)' + R'\dot\psi\right]\frac{e^{\psi(t,r)}}{R}\,\omega^{\hat t}\wedge\omega^{\hat\theta} + \left[R'' + R'\psi'\right]\frac{e^{2\psi(t,r)}}{R}\omega^{\hat r}\wedge\omega^{\hat\theta}$$
$$\Gamma^{\hat\theta}{}_{\hat c}\wedge\Gamma^{\hat c}{}_{\hat r} = \Gamma^{\hat\theta}{}_{\hat t}\wedge\Gamma^{\hat t}{}_{\hat r} + \Gamma^{\hat\theta}{}_{\hat r}\wedge\Gamma^{\hat r}{}_{\hat r} + \Gamma^{\hat\theta}{}_{\hat\theta}\wedge\Gamma^{\hat\theta}{}_{\hat r} + \Gamma^{\hat\theta}{}_{\hat\phi}\wedge\Gamma^{\hat\phi}{}_{\hat r} = \Gamma^{\hat\theta}{}_{\hat t}\wedge\Gamma^{\hat t}{}_{\hat r}$$
$$= \frac{\dot R}{R}\omega^{\hat\theta}\wedge\left(-\dot\psi\omega^{\hat r}\right)$$
$$\Rightarrow \quad \Omega^{\hat\theta}{}_{\hat r} = \left[\left(\dot R\right)' + R'\dot\psi\right]\frac{e^{\psi(t,r)}}{R}\,\omega^{\hat t}\wedge\omega^{\hat\theta} + \left[\left(R'' + R'\psi'\right)\frac{e^{2\psi(t,r)}}{R} + \frac{\dot R\dot\psi}{R}\right]\omega^{\hat r}\wedge\omega^{\hat\theta}$$
$$d\Gamma^{\hat\phi}{}_{\hat r} = d\left(\frac{R'}{R}e^{\psi(t,r)}\omega^{\hat\phi}\right) = d\left(R'e^{\psi(t,r)}\sin\theta\, d\phi\right)$$
$$=^{14}\left[\left(\dot R\right)' + R'\dot\psi\right]\frac{e^{\psi(t,r)}}{R}\omega^{\hat t}\wedge\omega^{\hat\phi} + \left[R'' + R'\psi'\right]\frac{e^{2\psi(t,r)}}{R}\omega^{\hat r}\wedge\omega^{\hat\phi} + \frac{R'}{R^2}e^{\psi(t,r)}\cot\theta\,\omega^{\hat\theta}\wedge\omega^{\hat\phi}$$
$$\Gamma^{\hat\phi}{}_{\hat c}\wedge\Gamma^{\hat c}{}_{\hat r} = \Gamma^{\hat\phi}{}_{\hat t}\wedge\Gamma^{\hat t}{}_{\hat r} + \Gamma^{\hat\phi}{}_{\hat r}\wedge\Gamma^{\hat r}{}_{\hat r} + \Gamma^{\hat\phi}{}_{\hat\theta}\wedge\Gamma^{\hat\theta}{}_{\hat r} + \Gamma^{\hat\phi}{}_{\hat\phi}\wedge\Gamma^{\hat\phi}{}_{\hat r} = \Gamma^{\hat\phi}{}_{\hat t}\wedge\Gamma^{\hat t}{}_{\hat r} + \Gamma^{\hat\phi}{}_{\hat\theta}\wedge\Gamma^{\hat\theta}{}_{\hat r}$$
$$= \frac{\dot R}{R}\omega^{\hat\phi}\wedge\left(-\dot\psi\omega^{\hat r}\right) + \frac{\cot\theta}{R}\omega^{\hat\phi}\wedge\frac{R'}{R}e^{\psi(t,r)}\omega^{\hat\theta}$$
$$\Rightarrow \quad \Omega^{\hat\phi}{}_{\hat r} = \left[\left(\dot R\right)' + R'\dot\psi\right]\frac{e^{\psi(t,r)}}{R}\omega^{\hat t}\wedge\omega^{\hat\phi} + \left[\left(R'' + R'\psi'\right)\frac{e^{2\psi(t,r)}}{R} + \frac{\dot R\dot\psi}{R}\right]\omega^{\hat r}\wedge\omega^{\hat\phi}$$
$$d\Gamma^{\hat\phi}{}_{\hat\theta} = d\left(\frac{\cot\theta}{R}\omega^{\hat\phi}\right) = d(\cos\theta\, d\phi) = -\sin\theta\, d\theta\wedge d\phi = -\frac{1}{R^2}\omega^{\hat\theta}\wedge\omega^{\hat\phi}$$
$$\Gamma^{\hat\phi}{}_{\hat c}\wedge\Gamma^{\hat c}{}_{\hat\theta} = \Gamma^{\hat\phi}{}_{\hat t}\wedge\Gamma^{\hat t}{}_{\hat\theta} + \Gamma^{\hat\phi}{}_{\hat r}\wedge\Gamma^{\hat r}{}_{\hat\theta} + \Gamma^{\hat\phi}{}_{\hat\theta}\wedge\Gamma^{\hat\theta}{}_{\hat\theta} + \Gamma^{\hat\phi}{}_{\hat\phi}\wedge\Gamma^{\hat\phi}{}_{\hat\theta} = \Gamma^{\hat\phi}{}_{\hat t}\wedge\Gamma^{\hat t}{}_{\hat\theta} + \Gamma^{\hat\phi}{}_{\hat r}\wedge\Gamma^{\hat r}{}_{\hat\theta}$$
$$= \frac{\dot R}{R}\omega^{\hat\phi}\wedge\frac{\dot R}{R}\omega^{\hat\theta} + \frac{R'}{R}e^{\psi(t,r)}\omega^{\hat\phi}\wedge\left(-\frac{R'}{R}e^{\psi(t,r)}\omega^{\hat\theta}\right)$$
$$\Rightarrow \quad \Omega^{\hat\phi}{}_{\hat\theta} = \left[\frac{1}{R^2} + \frac{\left(\dot R\right)^2}{R^2} - \frac{(R')^2}{R^2}e^{2\psi(t,r)}\right]\omega^{\hat\phi}\wedge\omega^{\hat\theta}$$

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[14] $= \left[\left(\dot R\right)'e^{\psi(t,r)}\sin\theta + R'\dot\psi e^{\psi(t,r)}\sin\theta\right]dt\wedge d\phi + \left[R''e^{\psi(t,r)}\sin\theta + R'\psi'e^{\psi(t,r)}\sin\theta\right]dr\wedge d\phi + R'e^{\psi(t,r)}\cos\theta\, d\theta\wedge d\phi =$

Summarized in a matrix:

$$\Omega^{\hat{a}}{}_{\hat{b}} = \begin{Bmatrix} 0 & [-\ddot{\psi}+(\dot{\psi})^2]\omega^{\hat{t}}\wedge\omega^{\hat{r}} & \begin{bmatrix} \frac{\ddot{R}}{R}\omega^{\hat{t}}\wedge\omega^{\hat{\theta}} \\ +\frac{(\dot{R})'+R'\dot{\psi}}{R}e^{\psi}\omega^{\hat{r}}\wedge\omega^{\hat{\theta}} \end{bmatrix} & \begin{bmatrix} \frac{\ddot{R}}{R}\omega^{\hat{t}}\wedge\omega^{\hat{\phi}} \\ +\frac{(\dot{R})'+R'\dot{\psi}}{R}e^{\psi}\omega^{\hat{r}}\wedge\omega^{\hat{\phi}} \end{bmatrix} \\ S & 0 & \begin{bmatrix} [(\dot{R})'+R'\dot{\psi}]\frac{e^{\psi}}{R}\omega^{\hat{t}}\wedge\omega^{\hat{\theta}} \\ +\left[(R''+R'\psi')\frac{e^{2\psi}}{R}+\frac{\dot{R}\dot{\psi}}{R}\right]\omega^{\hat{r}}\wedge\omega^{\hat{\theta}} \end{bmatrix} & \begin{bmatrix} [(\dot{R})'+R'\dot{\psi}]\frac{e^{\psi}}{R}\omega^{\hat{t}}\wedge\omega^{\hat{\phi}} \\ +\left[(R''+R'\psi')\frac{e^{2\psi}}{R}+\frac{\dot{R}\dot{\psi}}{R}\right]\omega^{\hat{r}}\wedge\omega^{\hat{\phi}} \end{bmatrix} \\ S & AS & 0 & \left[\frac{1}{R^2}+\frac{(\dot{R})^2}{R^2}-\frac{(R')^2}{R^2}e^{2\psi(t,r)}\right]\omega^{\hat{\phi}}\wedge\omega^{\hat{\theta}} \\ S & AS & AS & 0 \end{Bmatrix}$$

Where $\hat{a}$ refers to column and $\hat{b}$ to row.

Now we can find the independent elements of the Riemann tensor in the non-coordinate basis:

$$\begin{aligned}
R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}}(A) &= \left[\ddot{\psi}-(\dot{\psi})^2\right] \\
R^{\hat{\theta}}{}_{\hat{t}\hat{\theta}\hat{t}}(B) &= -\frac{\ddot{R}}{R} \\
R^{\hat{\theta}}{}_{\hat{t}\hat{\theta}\hat{r}}(C) &= -\left[(\dot{R})'+R'\dot{\psi}\right]\frac{e^{\psi(t,r)}}{R} \\
R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}}(D) &= -\left[(R''+R'\psi')\frac{e^{2\psi(t,r)}}{R}+\frac{\dot{R}\dot{\psi}}{R}\right] \\
R^{\hat{\phi}}{}_{\hat{t}\hat{\phi}\hat{t}}(B) &= -\frac{\ddot{R}}{R} \\
R^{\hat{\phi}}{}_{\hat{t}\hat{\phi}\hat{r}}(C) &= -\left[(\dot{R})'+R'\dot{\psi}\right]\frac{e^{\psi(t,r)}}{R} \\
R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{r}}(D) &= -\left[(R''+R'\psi')\frac{e^{2\psi(t,r)}}{R}+\frac{\dot{R}\dot{\psi}}{R}\right] \\
R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{\theta}}(E) &= \left[\frac{1}{R^2}+\frac{(\dot{R})^2}{R^2}-\frac{(R')^2}{R^2}e^{2\psi(t,r)}\right]
\end{aligned}$$

Where A,B,C,D,E will be used later, to make the calculations easier.

7.10.3 The Ricci Tensor

$$R_{\hat{a}\hat{b}} = R^{\hat{c}}{}_{\hat{a}\hat{c}\hat{b}}$$

$$\begin{aligned}
\Rightarrow \quad R_{\hat{t}\hat{t}} &= R^{\hat{c}}{}_{\hat{t}\hat{c}\hat{t}} = R^{\hat{t}}{}_{\hat{t}\hat{t}\hat{t}} + R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}} + R^{\hat{\theta}}{}_{\hat{t}\hat{\theta}\hat{t}} + R^{\hat{\phi}}{}_{\hat{t}\hat{\phi}\hat{t}} = R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}} + R^{\hat{\theta}}{}_{\hat{t}\hat{\theta}\hat{t}} + R^{\hat{\phi}}{}_{\hat{t}\hat{\phi}\hat{t}} = \left[\ddot{\psi}-(\dot{\psi})^2\right]-2\frac{\ddot{R}}{R} \\
&= A+2B \\
R_{\hat{r}\hat{t}} &= R^{\hat{c}}{}_{\hat{r}\hat{c}\hat{t}} = R^{\hat{t}}{}_{\hat{r}\hat{t}\hat{t}} + R^{\hat{r}}{}_{\hat{r}\hat{r}\hat{t}} + R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{t}} + R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{t}} = R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{t}} + R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{t}} \\
&= -\left[(\dot{R})'+R'\dot{\psi}\right]\frac{e^{\psi(t,r)}}{R} - \left[(\dot{R})'+R'\dot{\psi}\right]\frac{e^{\psi(t,r)}}{R} = -2\left[(\dot{R})'+R'\dot{\psi}\right]\frac{e^{\psi(t,r)}}{R} = 2C \\
R_{\hat{\theta}\hat{t}} &= R^{\hat{c}}{}_{\hat{\theta}\hat{c}\hat{t}} = R^{\hat{t}}{}_{\hat{\theta}\hat{t}\hat{t}} + R^{\hat{r}}{}_{\hat{\theta}\hat{r}\hat{t}} + R^{\hat{\theta}}{}_{\hat{\theta}\hat{\theta}\hat{t}} + R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{t}} = 0 \\
R_{\hat{\phi}\hat{t}} &= R^{\hat{c}}{}_{\hat{\phi}\hat{c}\hat{t}} = R^{\hat{t}}{}_{\hat{\phi}\hat{t}\hat{t}} + R^{\hat{r}}{}_{\hat{\phi}\hat{r}\hat{t}} + R^{\hat{\theta}}{}_{\hat{\phi}\hat{\theta}\hat{t}} + R^{\hat{\phi}}{}_{\hat{\phi}\hat{\phi}\hat{t}} = 0 \\
R_{\hat{r}\hat{r}} &= R^{\hat{c}}{}_{\hat{r}\hat{c}\hat{r}} = R^{\hat{t}}{}_{\hat{r}\hat{t}\hat{r}} + R^{\hat{r}}{}_{\hat{r}\hat{r}\hat{r}} + R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} + R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{r}} = -R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}} + R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} + R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{r}}
\end{aligned}$$

$$= -\left[\ddot{\psi} - (\dot{\psi})^2\right] - 2\left[(R'' + R'\psi')\frac{e^{2\psi(t,r)}}{R} + \frac{\dot{R}\dot{\psi}}{R}\right] = -A + 2D$$

$$R_{\hat{\theta}\hat{\theta}} = R^{\hat{c}}{}_{\hat{\theta}\hat{c}\hat{\theta}} = R^{\hat{t}}{}_{\hat{\theta}\hat{t}\hat{\theta}} + R^{\hat{r}}{}_{\hat{\theta}\hat{r}\hat{\theta}} + R^{\hat{\theta}}{}_{\hat{\theta}\hat{\theta}\hat{\theta}} + R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{\theta}} = -R^{\hat{\theta}}{}_{\hat{t}\hat{\theta}\hat{t}} + R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} + R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{\theta}}$$

$$= \frac{\ddot{R}}{R} - \left[(R'' + R'\psi')\frac{e^{2\psi(t,r)}}{R} + \frac{\dot{R}\dot{\psi}}{R}\right] + \left[\frac{1}{R^2} + \frac{(\dot{R})^2}{R^2} - \frac{(R')^2}{R^2}e^{2\psi(t,r)}\right] = -B + D + E$$

$$R_{\hat{\phi}\hat{\phi}} = R^{\hat{c}}{}_{\hat{\phi}\hat{c}\hat{\phi}} = R^{\hat{t}}{}_{\hat{\phi}\hat{t}\hat{\phi}} + R^{\hat{r}}{}_{\hat{\phi}\hat{r}\hat{\phi}} + R^{\hat{\theta}}{}_{\hat{\phi}\hat{\theta}\hat{\phi}} + R^{\hat{\phi}}{}_{\hat{\phi}\hat{\phi}\hat{\phi}} = -R^{\hat{\phi}}{}_{\hat{t}\hat{\phi}\hat{t}} + R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{r}} + R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{\theta}}$$

$$= \frac{\ddot{R}}{R} - \left[(R'' + R'\psi')\frac{e^{2\psi(t,r)}}{R} + \frac{\dot{R}\dot{\psi}}{R}\right] + \left[\frac{1}{R^2} + \frac{(\dot{R})^2}{R^2} - \frac{(R')^2}{R^2}e^{2\psi(t,r)}\right] = -B + D + E$$

$$= R_{\hat{\theta}\hat{\theta}}$$

Summarized in a matrix:

$$R_{\hat{a}\hat{b}} = \begin{Bmatrix} \begin{bmatrix}\ddot{\psi} - (\dot{\psi})^2 \\ -2\frac{\ddot{R}}{R}\end{bmatrix} & -2\left[(\dot{R})' + R'\dot{\psi}\right]\frac{e^{\psi}}{R} & 0 & 0 \\ S & \begin{bmatrix} -\left[\ddot{\psi} - (\dot{\psi})^2\right] \\ -2\left[(R'' + R'\psi')\frac{e^{2\psi}}{R} + \frac{\dot{R}\dot{\psi}}{R}\right]\end{bmatrix} & 0 & 0 \\ 0 & 0 & \begin{bmatrix}\frac{\ddot{R}}{R} - \left[(R'' + R'\psi')\frac{e^{2\psi}}{R} + \frac{\dot{R}\dot{\psi}}{R}\right] \\ + \frac{1}{R^2} + \frac{(\dot{R})^2}{R^2} - \frac{(R')^2}{R^2}e^{2\psi}\end{bmatrix} & 0 \\ 0 & 0 & 0 & \begin{bmatrix}\frac{\ddot{R}}{R} - \left[(R'' + R'\psi')\frac{e^{2\psi}}{R} + \frac{\dot{R}\dot{\psi}}{R}\right] \\ + \frac{1}{R^2} + \frac{(\dot{R})^2}{R^2} - \frac{(R')^2}{R^2}e^{2\psi}\end{bmatrix} \end{Bmatrix}$$

Where $\hat{a}$ refers to column and $\hat{b}$ to row.

7.10.4 The Ricci Scalar

$$R = \eta^{\hat{a}\hat{b}}R_{\hat{a}\hat{b}}$$

$$R = \eta^{\hat{t}\hat{t}}R_{\hat{t}\hat{t}} + \eta^{\hat{r}\hat{r}}R_{\hat{r}\hat{r}} + \eta^{\hat{\theta}\hat{\theta}}R_{\hat{\theta}\hat{\theta}} + \eta^{\hat{\phi}\hat{\phi}}R_{\hat{\phi}\hat{\phi}} = R_{\hat{t}\hat{t}} - R_{\hat{r}\hat{r}} - R_{\hat{\theta}\hat{\theta}} - R_{\hat{\phi}\hat{\phi}}$$

$$= A + 2B - (-A + 2D) - 2(-B + D + E) = 2A + 4B - 4D - 2E$$

$$= 2R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}} + 4R^{\hat{\theta}}{}_{\hat{t}\hat{\theta}\hat{t}} - 4R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} - 2R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{\theta}}$$

$$= 2\left[\ddot{\psi} - (\dot{\psi})^2\right] - 4\frac{\ddot{R}}{R} + 4\left[(R'' + R'\psi')\frac{e^{2\psi(t,r)}}{R} + \frac{\dot{R}\dot{\psi}}{R}\right] - 2\left[\frac{1}{R^2} + \frac{(\dot{R})^2}{R^2} - \frac{(R')^2}{R^2}e^{2\psi(t,r)}\right]$$

7.10.5 tThe Einstein Tensor

$$G_{\hat{a}\hat{b}} = R_{\hat{a}\hat{b}} - \frac{1}{2}\eta_{\hat{a}\hat{b}}R$$

$$G_{\hat{t}\hat{t}} = R_{\hat{t}\hat{t}} - \frac{1}{2}\eta_{\hat{t}\hat{t}}R = R_{\hat{t}\hat{t}} - \frac{1}{2}R = A + 2B - \frac{1}{2}(2A + 4B - 4D - 2E) = 2D + E$$

$$= 2R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} + R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{\theta}} = -2\left[(R'' + R'\psi')\frac{e^{2\psi(t,r)}}{R} + \frac{\dot{R}\dot{\psi}}{R}\right] + \left[\frac{1}{R^2} + \frac{(\dot{R})^2}{R^2} - \frac{(R')^2}{R^2}e^{2\psi(t,r)}\right]$$

$$= \frac{1}{R^2}\left[1 - 2R\dot{R}\dot{\psi} + (\dot{R})^2 - (2RR'' + 2RR'\psi' + (R')^2)e^{2\psi(t,r)}\right]$$

$$
\begin{aligned}
G_{\hat{r}\hat{t}} &= R_{\hat{r}\hat{t}} - \frac{1}{2}\eta_{\hat{r}\hat{t}}R = R_{\hat{r}\hat{t}} = R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{t}} + R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{t}} = -2\left[\left(\dot{R}\right)' + R'\dot{\psi}\right]\frac{e^{\psi(t,r)}}{R} \\
G_{\hat{\theta}\hat{t}} &= R_{\hat{\theta}\hat{t}} - \frac{1}{2}\eta_{\hat{\theta}\hat{t}}R = 0 \\
G_{\hat{r}\hat{r}} &= R_{\hat{r}\hat{r}} - \frac{1}{2}\eta_{\hat{r}\hat{r}}R = R_{\hat{r}\hat{r}} + \frac{1}{2}R = -A + 2D + \frac{1}{2}(2A + 4B - 4D - 2E) = 2B - E \\
&= 2R^{\hat{\theta}}{}_{\hat{t}\hat{\theta}\hat{t}} - R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{\theta}} = -2\frac{\ddot{R}}{R} - \left[\frac{1}{R^2} + \frac{\left(\dot{R}\right)^2}{R^2} - \frac{(R')^2}{R^2}e^{2\psi(t,r)}\right] \\
&= \frac{1}{R^2}\left[(R')^2 e^{2\psi(t,r)} - 2R\ddot{R} - 1 - \left(\dot{R}\right)^2\right] \\
G_{\hat{\theta}\hat{\theta}} &= R_{\hat{\theta}\hat{\theta}} - \frac{1}{2}\eta_{\hat{\theta}\hat{\theta}}R = R_{\hat{\theta}\hat{\theta}} + \frac{1}{2}R = -B + D + E + \frac{1}{2}(2A + 4B - 4D - 2E) = A + B - D \\
&= R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}} + R^{\hat{\theta}}{}_{\hat{t}\hat{\theta}\hat{t}} - R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} = \left[\ddot{\psi} - \left(\dot{\psi}\right)^2\right] - \frac{\ddot{R}}{R} + \left[(R'' + R'\psi')\frac{e^{2\psi(t,r)}}{R} + \frac{\dot{R}\dot{\psi}}{R}\right] \\
&= \left[\ddot{\psi} - \left(\dot{\psi}\right)^2\right] + \frac{1}{R}\left[(R'' + R'\psi')e^{2\psi(t,r)} + \dot{R}\dot{\psi} - \ddot{R}\right] \\
G_{\hat{\phi}\hat{\phi}} &= G_{\hat{\theta}\hat{\theta}} = \left[\ddot{\psi} - \left(\dot{\psi}\right)^2\right] + \frac{1}{R}\left[(R'' + R'\psi')e^{2\psi(t,r)} + \dot{R}\dot{\psi} - \ddot{R}\right]
\end{aligned}
$$

Summarized in a matrix:

$$
G_{\hat{a}\hat{b}} = \begin{Bmatrix}
\frac{1}{R^2}\begin{bmatrix} 1 - 2R\dot{R}\dot{\psi} + \left(\dot{R}\right)^2 \\ -(2RR'' + 2RR'\psi' + (R')^2)e^{2\psi} \end{bmatrix} & -2\left[\left(\dot{R}\right)' + R'\dot{\psi}\right]\frac{e^{\psi}}{R} & 0 & 0 \\
S & \frac{1}{R^2}\begin{bmatrix} (R')^2 e^{2\psi} \\ -2R\ddot{R} - 1 - \left(\dot{R}\right)^2 \end{bmatrix} & 0 & 0 \\
0 & 0 & \begin{bmatrix} \ddot{\psi} - \left(\dot{\psi}\right)^2 \\ +\frac{1}{R}\left[(R'' + R'\psi')e^{2\psi} + \dot{R}\dot{\psi} - \ddot{R}\right] \end{bmatrix} & 0 \\
0 & 0 & 0 & \begin{bmatrix} \ddot{\psi} - \left(\dot{\psi}\right)^2 \\ +\frac{1}{R}\left[(R'' + R'\psi')e^{2\psi} + \dot{R}\dot{\psi} - \ddot{R}\right] \end{bmatrix}
\end{Bmatrix}
$$

Where $\hat{a}$ refers to column and $\hat{b}$ to row.

7.10.6 The Einstein tensor in the coordinate basis:

$$
\begin{aligned}
G_{ab} &= \Lambda^{\hat{c}}{}_a \Lambda^{\hat{d}}{}_b G_{\hat{c}\hat{d}} \\
G_{tt} &= \Lambda^{\hat{c}}{}_t \Lambda^{\hat{d}}{}_t G_{\hat{c}\hat{d}} = \left(\Lambda^{\hat{t}}{}_t\right)^2 G_{\hat{t}\hat{t}} = \frac{1}{R^2}\left[1 - 2R\dot{R}\dot{\psi} + \left(\dot{R}\right)^2 - (2RR'' + 2RR'\psi' + (R')^2)e^{2\psi}\right] \\
G_{rt} &= \Lambda^{\hat{c}}{}_r \Lambda^{\hat{d}}{}_t G_{\hat{c}\hat{d}} = \Lambda^{\hat{r}}{}_r \Lambda^{\hat{t}}{}_t G_{\hat{r}\hat{t}} = -2\left[\frac{\left(\dot{R}\right)'}{R} + \frac{R'\dot{\psi}}{R}\right] \\
G_{rr} &= \Lambda^{\hat{c}}{}_r \Lambda^{\hat{d}}{}_r G_{\hat{c}\hat{d}} = \left(\Lambda^{\hat{r}}{}_r\right)^2 G_{\hat{r}\hat{r}} = \frac{1}{R^2}\left[(R')^2 - \left(2R\ddot{R} + 1 + \left(\dot{R}\right)^2\right)e^{-2\psi}\right] \\
G_{\theta\theta} &= \Lambda^{\hat{c}}{}_\theta \Lambda^{\hat{d}}{}_\theta G_{\hat{c}\hat{d}} = \left(\Lambda^{\hat{\theta}}{}_\theta\right)^2 G_{\hat{\theta}\hat{\theta}} = R^2\left(\left[\ddot{\psi} - \left(\dot{\psi}\right)^2\right] + \frac{1}{R}\left[(R'' + R'\psi')e^{2\psi} + \dot{R}\dot{\psi} - \ddot{R}\right]\right) \\
G_{\phi\phi} &= \Lambda^{\hat{c}}{}_\phi \Lambda^{\hat{d}}{}_\phi G_{\hat{c}\hat{d}} = \left(\Lambda^{\hat{\phi}}{}_\phi\right)^2 G_{\hat{\phi}\hat{\phi}} = R^2 \sin^2\theta\left(\left[\ddot{\psi} - \left(\dot{\psi}\right)^2\right] + \frac{1}{R}\left[(R'' + R'\psi')e^{2\psi} + \dot{R}\dot{\psi} - \ddot{R}\right]\right)
\end{aligned}
$$

Summarized in a matrix:

$$G_{ab} = \left\{\begin{matrix} \frac{1}{R^2}\begin{bmatrix} 1-2R\dot{R}\dot{\psi}+(\dot{R})^2 \\ -(2RR''+2RR'\psi'+(R')^2)e^{2\psi} \end{bmatrix} & -2\left[\frac{(\dot{R})'}{R}+\frac{R'\dot{\psi}}{R}\right] & 0 & 0 \\ S & \frac{1}{R^2}\begin{bmatrix} (R')^2 \\ -\left(2R\ddot{R}+1+(\dot{R})^2\right)e^{-2\psi} \end{bmatrix} & 0 & 0 \\ 0 & 0 & R^2\begin{bmatrix} \ddot{\psi}-(\dot{\psi})^2 \\ +\frac{1}{R}[(R''+R'\psi')e^{2\psi}+\dot{R}\dot{\psi}-\ddot{R}] \end{bmatrix} & 0 \\ 0 & 0 & 0 & R^2\sin^2\theta\begin{bmatrix} \ddot{\psi}-(\dot{\psi})^2 \\ +\frac{1}{R}[(R''+R'\psi')e^{2\psi}+\dot{R}\dot{\psi}-\ddot{R}] \end{bmatrix} \end{matrix}\right\}$$

Where a refers to column and b to row

7.11 [u]The Taub-Nut Spacetime

[15]The Taub-Nut space-time is an exact solution to the Einstein equation.

<u>The line element</u>

$$ds^2 = -\frac{dt^2}{U^2(t)} + (2l)^2U^2(t)(dr+\cos\theta\, d\phi)^2 + V^2(t)(d\theta^2+\sin^2\theta\, d\phi^2)$$

If $U^2(t) > 0$ t is a time variable. If $U^2(t) < 0$ t is a space variable.

Defining

$$\begin{aligned}
U^2(t) &= -1 + \frac{2(mt+l^2)}{t^2+l^2} = \frac{2mt-t^2+l^2}{t^2+l^2} = \frac{P^2(t)}{V^2(t)} \\
V^2(t) &= t^2+l^2 \\
P^2(t) &= 2mt-t^2+l^2 \\
\Rightarrow \qquad \dot{V}(t) &= \frac{1}{2V(t)}\frac{d}{dt}\left(V^2(t)\right) = \frac{t}{V(t)} \\
\ddot{V}(t) &= \frac{d}{dt}\left(\dot{V}(t)\right) = \frac{d}{dt}\left(\frac{t}{V(t)}\right) = \frac{V(t)-t\dot{V}(t)}{V^2(t)} = \frac{1}{V(t)} - \frac{t}{V^2(t)}\left(\frac{t}{V(t)}\right) \\
&= \frac{1}{V(t)} - \frac{t^2}{V^3(t)} \\
\dot{U}(t) &= \frac{1}{2U(t)}\frac{d}{dt}\left(U^2(t)\right) = \frac{1}{2U(t)}\frac{d}{dt}\left(\frac{P^2(t)}{V^2(t)}\right) \\
&= \frac{1}{2U(t)}\left(\frac{V^2(t)\frac{d}{dt}P^2(t) - P^2(t)\frac{d}{dt}V^2(t)}{V^4(t)}\right) = \frac{1}{U(t)}\left(\frac{(m-t)V^2(t)-tP^2(t)}{V^4(t)}\right) \\
&= \frac{1}{U(t)}\left(\frac{m-t}{V^2(t)} - \frac{tU^2(t)}{V^2(t)}\right) \\
\frac{d}{dt}\left(U(t)\dot{U}(t)\right) &= \frac{d}{dt}\left(\frac{m-t}{V^2(t)} - \frac{tU^2(t)}{V^2(t)}\right) = \frac{d}{dt}\left(\frac{m-t}{V^2(t)}\right) - \frac{d}{dt}\left(\frac{tU^2(t)}{V^2(t)}\right) \\
&= \frac{-V^2(t)-2(m-t)V(t)\dot{V}(t)}{V^4(t)} - \frac{\frac{d}{dt}\left(tU^2(t)\right)V^2(t) - 2tU^2(t)V(t)\dot{V}(t)}{V^4(t)} \\
&= -\frac{1}{V^2(t)} - \frac{2t(m-t)}{V^4(t)} - \frac{\left(U^2(t)+2tU(t)\dot{U}(t)\right)V^2(t) - 2t^2U^2(t)}{V^4(t)} \\
&= -\frac{1}{V^2(t)} - \frac{2t(m-t)}{V^4(t)} - \frac{U^2(t)}{V^2(t)} - \frac{2tV^2(t)}{V^4(t)}\left(\frac{m-t}{V^2(t)} - \frac{tU^2(t)}{V^2(t)}\right) + \frac{2t^2U^2(t)}{V^4(t)} \\
&= -\frac{1}{V^2(t)} - \frac{4t(m-t)}{V^4(t)} - \frac{U^2(t)}{V^2(t)} + \frac{4t^2U^2(t)}{V^4(t)}
\end{aligned}$$

[15] https://en.wikipedia.org/wiki/Taub%E2%80%93NUT_space

7.11.1 The Riemann tensor

[16]The TAUB-Nut space-time is a solution of the vacuum Einstein equation if the Ricci scalar $R = 0$.
We choose to work in the non-coordinate basis.
The basis one forms

$$
\begin{aligned}
\omega^{\hat{t}} &= \frac{1}{U(t)}dt & dt &= U(t)\omega^{\hat{t}} \\
\omega^{\hat{r}} &= 2lU(t)(dr + \cos\theta\, d\phi) & dr &= \frac{1}{2lU(t)}\omega^{\hat{r}} - \frac{\cot\theta}{V(t)}\omega^{\hat{\Phi}} \\
\omega^{\hat{\theta}} &= V(t)d\theta & d\theta &= \frac{1}{V(t)}\omega^{\hat{\theta}} \\
\omega^{\hat{\Phi}} &= \sin\theta\, V(t)d\phi & d\phi &= \frac{1}{\sin\theta\, V(t)}\omega^{\hat{\Phi}}
\end{aligned}
$$

$$
\eta^{ij} = \begin{Bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{Bmatrix}
$$

7.11.1.1 Cartan's First Structure equation

$$
\begin{aligned}
d\omega^{\hat{a}} &= -\Gamma^{\hat{a}}{}_{\hat{b}} \wedge \omega^{\hat{b}} \\
d\omega^{\hat{t}} &= d\left(\frac{1}{U(t)}dt\right) = 0 \\
d\omega^{r} &= -\dot{U}(t)\omega^{\hat{r}} \wedge \omega^{\hat{t}} - \frac{lU(t)}{V^2(t)}\omega^{\hat{\theta}} \wedge \omega^{\hat{\Phi}} + \frac{lU(t)}{V^2(t)}\omega^{\hat{\Phi}} \wedge \omega^{\hat{\theta}} \\
&= d(2lU(t)dr) + d(2lU(t)\cos\theta\, d\phi) \\
&= {}^{17\ 18\ 19\ 20} - \Gamma^{\hat{r}}{}_{\hat{t}} \wedge \omega^{\hat{t}} - \Gamma^{\hat{r}}{}_{\hat{\Phi}} \wedge \omega^{\hat{\Phi}} - \Gamma^{\hat{r}}{}_{\hat{\theta}} \wedge \omega^{\hat{\theta}} \\
d\omega^{\hat{\theta}} &= d(V(t)d\theta) = \dot{V}(t)dt \wedge d\theta = -\frac{U(t)\dot{V}(t)}{V(t)}\omega^{\hat{\theta}} \wedge \omega^{\hat{t}} = -\Gamma^{\hat{\theta}}{}_{\hat{t}} \wedge \omega^{\hat{t}} \\
d\omega^{\hat{\Phi}} &= d(\sin\theta\, V(t)d\phi) = \sin\theta\, \dot{V}(t)dt \wedge d\phi + \cos\theta\, V(t)d\theta \wedge d\phi \\
&= \frac{U(t)\dot{V}(t)}{V(t)}\omega^{\hat{t}} \wedge \omega^{\hat{\Phi}} + \frac{\cot\theta}{V(t)}\omega^{\hat{\theta}} \wedge \omega^{\hat{\Phi}} = -\frac{U(t)\dot{V}(t)}{V(t)}\omega^{\hat{\Phi}} \wedge \omega^{\hat{t}} - \frac{\cot\theta}{V(t)}\omega^{\hat{\Phi}} \wedge \omega^{\hat{\theta}} \\
&= -\Gamma^{\hat{\Phi}}{}_{\hat{t}} \wedge \omega^{\hat{t}} - \Gamma^{\hat{\Phi}}{}_{\hat{\theta}} \wedge \omega^{\hat{\theta}}
\end{aligned}
$$

Summarizing the curvature one forms in a matrix:

---

[16] $G_{\hat{a}\hat{b}} = R_{\hat{a}\hat{b}} - \frac{1}{2}\eta_{\hat{a}\hat{b}}R = 0 \Rightarrow \eta^{\hat{a}\hat{b}}R_{\hat{a}\hat{b}} - \frac{1}{2}\eta^{\hat{a}\hat{b}}\eta_{\hat{a}\hat{b}}R = R - \frac{1}{2}4R = 0 \Rightarrow R = 0$

[17] $= 2l\dot{U}(t)dt \wedge dr + 2l\dot{U}(t)\cos\theta\, dt \wedge d\phi - 2lU(t)\sin\theta\, d\theta \wedge d\phi =$

[18] $= 2l\dot{U}(t)\big(U(t)\omega^{\hat{t}}\big) \wedge \left(\frac{1}{2lU(t)}\omega^{\hat{r}} - \frac{\cot\theta}{V(t)}\omega^{\hat{\Phi}}\right) + 2l\dot{U}(t)\cos\theta\,\big(U(t)\omega^{\hat{t}}\big) \wedge \left(\frac{1}{\sin\theta V(t)}\omega^{\hat{\Phi}}\right) - 2lU(t)\sin\theta\left(\frac{1}{V(t)}\omega^{\hat{\theta}}\right) \wedge \left(\frac{1}{\sin\theta V(t)}\omega^{\hat{\Phi}}\right) =$

[19] $= \left(\dot{U}(t)\omega^{\hat{t}} \wedge \omega^{\hat{r}} - 2l\dot{U}(t)\big(U(t)\omega^{\hat{t}}\big) \wedge \frac{\cot\theta}{V(t)}\omega^{\hat{\Phi}}\right) + 2l\dot{U}(t)\cos\theta\,\big(U(t)\omega^{\hat{t}}\big) \wedge \left(\frac{1}{\sin\theta V(t)}\omega^{\hat{\Phi}}\right) - 2lU(t)\sin\theta\left(\frac{1}{V(t)}\omega^{\hat{\theta}}\right) \wedge \left(\frac{1}{\sin\theta V(t)}\omega^{\hat{\Phi}}\right) =$

[20] $= \dot{U}(t)\omega^{\hat{t}} \wedge \omega^{\hat{r}} - \frac{2l\dot{U}(t)U(t)\cot\theta}{V(t)}\omega^{\hat{t}} \wedge \omega^{\hat{\Phi}} + \frac{2l\dot{U}(t)U(t)\cot\theta}{V(t)}\omega^{\hat{t}} \wedge \omega^{\hat{\Phi}} - \frac{2lU(t)}{V^2(t)}\omega^{\hat{\theta}} \wedge \omega^{\hat{\Phi}} =$

$$\Gamma^{\hat{a}}{}_{\hat{b}} = \begin{Bmatrix} 0 & \dot{U}(t)\omega^{\hat{r}} & \frac{U(t)\dot{V}(t)}{V(t)}\omega^{\hat{\theta}} & \frac{U(t)\dot{V}(t)}{V(t)}\omega^{\hat{\Phi}} \\ S & 0 & \frac{lU(t)}{V^2(t)}\omega^{\hat{\Phi}} & -\frac{lU(t)}{V^2(t)}\omega^{\hat{\theta}} \\ S & AS & 0 & \frac{\cot\theta}{V(t)}\omega^{\hat{\Phi}} \\ S & AS & AS & 0 \end{Bmatrix}$$

Where $\hat{a}$ refers to column and $\hat{b}$ to row.

7.11.1.2 The curvature two forms

$$\Omega^{\hat{a}}{}_{\hat{b}} = d\Gamma^{\hat{a}}{}_{\hat{b}} + \Gamma^{\hat{a}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{b}}$$

$$d\Gamma^{\hat{r}}{}_{\hat{t}} = d\left(\dot{U}(t)\omega^{\hat{r}}\right) = d\left(\dot{U}(t)2lU(t)(dr + \cos\theta\, d\phi)\right)$$

$$= {}^{21\ 22\ 23}\left(\ddot{U}(t)U(t) + \dot{U}^2(t)\right)\omega^{\hat{t}} \wedge \omega^{\hat{r}} - 2\frac{l\dot{U}(t)U(t)}{V^2(t)}\omega^{\hat{\theta}} \wedge \omega^{\hat{\Phi}}$$

$$\Gamma^{\hat{r}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{t}} = \Gamma^{\hat{r}}{}_{\hat{t}} \wedge \Gamma^{\hat{t}}{}_{\hat{t}} + \Gamma^{\hat{r}}{}_{\hat{r}} \wedge \Gamma^{\hat{r}}{}_{\hat{t}} + \Gamma^{\hat{r}}{}_{\hat{\theta}} \wedge \Gamma^{\hat{\theta}}{}_{\hat{t}} + \Gamma^{\hat{r}}{}_{\hat{\Phi}} \wedge \Gamma^{\hat{\Phi}}{}_{\hat{t}}$$

$$= \left(-\frac{lU(t)}{V^2(t)}\omega^{\hat{\Phi}}\right) \wedge \left(\frac{U(t)\dot{V}(t)}{V(t)}\omega^{\hat{\theta}}\right) + \left(\frac{lU(t)}{V^2(t)}\omega^{\hat{\theta}}\right) \wedge \left(\frac{U(t)\dot{V}(t)}{V(t)}\omega^{\hat{\Phi}}\right)$$

$$= -\frac{2lU^2(t)\dot{V}(t)}{V^3(t)}\omega^{\hat{\Phi}} \wedge \omega^{\hat{\theta}}$$

$$\Rightarrow \quad \Omega^{\hat{r}}{}_{\hat{t}} = \left(\ddot{U}(t)U(t) + \dot{U}^2(t)\right)\omega^{\hat{t}} \wedge \omega^{\hat{r}} - 2\frac{l\dot{U}(t)U(t)}{V^2(t)}\omega^{\hat{\theta}} \wedge \omega^{\hat{\Phi}} - \frac{2lU^2(t)\dot{V}(t)}{V^3(t)}\omega^{\hat{\Phi}} \wedge \omega^{\hat{\theta}}$$

$$= \left(\ddot{U}(t)U(t) + \dot{U}^2(t)\right)\omega^{\hat{t}} \wedge \omega^{\hat{r}} - \frac{2lU(t)}{V^3(t)}\left(\dot{U}(t)V(t) - U(t)\dot{V}(t)\right)\omega^{\hat{\theta}} \wedge \omega^{\hat{\Phi}}$$

$$d\Gamma^{\hat{\theta}}{}_{\hat{t}} = d\left(\frac{U(t)\dot{V}(t)}{V(t)}\omega^{\hat{\theta}}\right) = d\left(U(t)\dot{V}(t)d\theta\right) = \left(\dot{U}(t)\dot{V}(t) + U(t)\ddot{V}(t)\right)dt \wedge d\theta$$

$$= \frac{U(t)}{V(t)}\left(\dot{U}(t)\dot{V}(t) + U(t)\ddot{V}(t)\right)\omega^{\hat{t}} \wedge \omega^{\hat{\theta}}$$

$$\Gamma^{\hat{\theta}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{t}} = \Gamma^{\hat{\theta}}{}_{\hat{t}} \wedge \Gamma^{\hat{t}}{}_{\hat{t}} + \Gamma^{\hat{\theta}}{}_{\hat{r}} \wedge \Gamma^{\hat{r}}{}_{\hat{t}} + \Gamma^{\hat{\theta}}{}_{\hat{\theta}} \wedge \Gamma^{\hat{\theta}}{}_{\hat{t}} + \Gamma^{\hat{\theta}}{}_{\hat{\Phi}} \wedge \Gamma^{\hat{\Phi}}{}_{\hat{t}}$$

$$= \left(\frac{lU(t)}{V^2(t)}\omega^{\hat{\Phi}}\right) \wedge \left(\dot{U}(t)\omega^{\hat{r}}\right) + \left(-\frac{\cot\theta}{V(t)}\omega^{\hat{\Phi}}\right) \wedge \left(\frac{U(t)\dot{V}(t)}{V(t)}\omega^{\hat{\Phi}}\right)$$

$$= \frac{lU(t)\dot{U}(t)}{V^2(t)}\omega^{\hat{\Phi}} \wedge \omega^{\hat{r}}$$

$$\Rightarrow \quad \Omega^{\hat{\theta}}{}_{\hat{t}} = \frac{U(t)}{V(t)}\left(\dot{U}(t)\dot{V}(t) + U(t)\ddot{V}(t)\right)\omega^{\hat{t}} \wedge \omega^{\hat{\theta}} + \frac{lU(t)\dot{U}(t)}{V^2(t)}\omega^{\hat{\Phi}} \wedge \omega^{\hat{r}}$$

$$d\Gamma^{\hat{\Phi}}{}_{\hat{t}} = d\left(\frac{U(t)\dot{V}(t)}{V(t)}\omega^{\hat{\Phi}}\right) = d\left(U(t)\dot{V}(t)\sin\theta\, d\phi\right)$$

$$= \left(\dot{U}(t)\dot{V}(t) + U(t)\ddot{V}(t)\right)\sin\theta\, dt \wedge d\phi + U(t)\dot{V}(t)\cos\theta\, d\theta \wedge d\phi$$

---

[21] $= 2l\left(\ddot{U}(t)U(t) + \dot{U}^2(t)\right)dt \wedge dr + 2l\left(\ddot{U}(t)U(t) + \dot{U}^2(t)\right)dt \wedge \cos\theta\, d\phi - \dot{U}(t)2lU(t)\sin\theta\, d\theta \wedge d\phi =$

[22] $= 2l\left(\ddot{U}(t)U(t) + \dot{U}^2(t)\right)\left(U(t)\omega^{\hat{t}}\right) \wedge \left(\frac{1}{2lU(t)}\omega^{\hat{r}} - \frac{\cot\theta}{V(t)}\omega^{\hat{\Phi}}\right) + 2l\left(\ddot{U}(t)U(t) + \dot{U}^2(t)\right)\left(U(t)\omega^{\hat{t}}\right) \wedge \cos\theta\left(\frac{1}{\sin\theta V(t)}\omega^{\hat{\Phi}}\right) - \dot{U}(t)2lU(t)\sin\theta\left(\frac{1}{V(t)}\omega^{\hat{\theta}}\right) \wedge \left(\frac{1}{\sin\theta V(t)}\omega^{\hat{\Phi}}\right) =$

[23] $= 2l\left(\ddot{U}(t)U(t) + \dot{U}^2(t)\right)\left(U(t)\omega^{\hat{t}}\right) \wedge \left(\frac{1}{2lU(t)}\omega^{\hat{r}}\right) - 2l\left(\ddot{U}(t)U(t) + \dot{U}^2(t)\right)\left(U(t)\omega^{\hat{t}}\right) \wedge \left(\frac{\cot\theta}{V(t)}\omega^{\hat{\Phi}}\right) + 2l\left(\ddot{U}(t)U(t) + \dot{U}^2(t)\right)\left(U(t)\omega^{\hat{t}}\right) \wedge \cos\theta\left(\frac{1}{\sin\theta V(t)}\omega^{\hat{\Phi}}\right) - \dot{U}(t)2lU(t)\sin\theta\left(\frac{1}{V(t)}\omega^{\hat{\theta}}\right) \wedge \left(\frac{1}{\sin\theta V(t)}\omega^{\hat{\Phi}}\right) =$

$$= \frac{U(t)}{V(t)}\left(\dot{U}(t)\dot{V}(t) + U(t)\ddot{V}(t)\right)\omega^{\hat{t}} \wedge \omega^{\hat{\phi}} + \frac{U(t)\dot{V}(t)\cot\theta}{V^2(t)}\omega^{\hat{\theta}} \wedge \omega^{\hat{\phi}}$$

$$\Gamma^{\hat{\phi}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{t}} = \Gamma^{\hat{\phi}}{}_{\hat{t}} \wedge \Gamma^{\hat{t}}{}_{\hat{t}} + \Gamma^{\hat{\phi}}{}_{\hat{r}} \wedge \Gamma^{\hat{r}}{}_{\hat{t}} + \Gamma^{\hat{\phi}}{}_{\hat{\theta}} \wedge \Gamma^{\hat{\theta}}{}_{\hat{t}} + \Gamma^{\hat{\phi}}{}_{\hat{\phi}} \wedge \Gamma^{\hat{\phi}}{}_{\hat{t}}$$

$$= -\frac{lU(t)\dot{U}(t)}{V^2(t)}\omega^{\hat{\theta}} \wedge \omega^{\hat{r}} + \frac{U(t)\dot{V}(t)\cot\theta}{V^2(t)}\omega^{\hat{\phi}} \wedge \omega^{\hat{\theta}}$$

$$\Rightarrow \quad \Omega^{\hat{\phi}}{}_{\hat{t}} = \frac{U(t)}{V(t)}\left(\dot{U}(t)\dot{V}(t) + U(t)\ddot{V}(t)\right)\omega^{\hat{t}} \wedge \omega^{\hat{\phi}} - \frac{lU(t)\dot{U}(t)}{V^2(t)}\omega^{\hat{\theta}} \wedge \omega^{\hat{r}}$$

$$d\Gamma^{\hat{\theta}}{}_{\hat{r}} = d\left(\frac{lU(t)}{V^2(t)}\omega^{\hat{\phi}}\right) = d\left(\frac{lU(t)}{V(t)}\sin\theta\, d\phi\right)$$

$$= l\left(\frac{\dot{U}(t)V(t) - U(t)\dot{V}(t)}{V^2(t)}\right)\sin\theta\, dt \wedge d\phi + \frac{lU(t)}{V(t)}\cos\theta\, d\theta \wedge d\phi$$

$$= l\frac{U(t)}{V(t)}\left(\frac{\dot{U}(t)V(t) - U(t)\dot{V}(t)}{V^2(t)}\right)\omega^{\hat{t}} \wedge \omega^{\hat{\phi}} + \frac{lU(t)}{V^3(t)}\cot\theta\, \omega^{\hat{\theta}} \wedge \omega^{\hat{\phi}}$$

$$\Gamma^{\hat{\theta}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{r}} = \Gamma^{\hat{\theta}}{}_{\hat{t}} \wedge \Gamma^{\hat{t}}{}_{\hat{r}} + \Gamma^{\hat{\theta}}{}_{\hat{r}} \wedge \Gamma^{\hat{r}}{}_{\hat{r}} + \Gamma^{\hat{\theta}}{}_{\hat{\theta}} \wedge \Gamma^{\hat{\theta}}{}_{\hat{r}} + \Gamma^{\hat{\theta}}{}_{\hat{\phi}} \wedge \Gamma^{\hat{\phi}}{}_{\hat{r}}$$

$$= \frac{U(t)\dot{U}(t)\dot{V}(t)}{V(t)}\omega^{\hat{\theta}} \wedge \omega^{\hat{r}} + \frac{lU(t)}{V^3(t)}\cot\theta\, \omega^{\hat{\phi}} \wedge \omega^{\hat{\theta}}$$

$$\Rightarrow \quad \Omega^{\hat{\theta}}{}_{\hat{r}} = l\frac{U(t)}{V(t)}\left(\frac{\dot{U}(t)V(t) - U(t)\dot{V}(t)}{V^2(t)}\right)\omega^{\hat{t}} \wedge \omega^{\hat{\phi}} + \frac{U(t)\dot{U}(t)\dot{V}(t)}{V(t)}\omega^{\hat{\theta}} \wedge \omega^{\hat{r}}$$

$$d\Gamma^{\hat{\phi}}{}_{\hat{r}} = d\left(-\frac{lU(t)}{V^2(t)}\omega^{\hat{\theta}}\right) = -l\left(\frac{\dot{U}(t)V(t) - U(t)\dot{V}(t)}{V^2(t)}\right)dt \wedge d\theta$$

$$= -l\frac{U(t)}{V(t)}\left(\frac{\dot{U}(t)V(t) - U(t)\dot{V}(t)}{V^2(t)}\right)\omega^{\hat{t}} \wedge \omega^{\hat{\theta}}$$

$$\Gamma^{\hat{\phi}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{r}} = \Gamma^{\hat{\phi}}{}_{\hat{t}} \wedge \Gamma^{\hat{t}}{}_{\hat{r}} + \Gamma^{\hat{\phi}}{}_{\hat{r}} \wedge \Gamma^{\hat{r}}{}_{\hat{r}} + \Gamma^{\hat{\phi}}{}_{\hat{\theta}} \wedge \Gamma^{\hat{\theta}}{}_{\hat{r}} + \Gamma^{\hat{\phi}}{}_{\hat{\phi}} \wedge \Gamma^{\hat{\phi}}{}_{\hat{r}} = \frac{U(t)\dot{U}(t)\dot{V}(t)}{V(t)}\omega^{\hat{\phi}} \wedge \omega^{\hat{r}}$$

$$\Rightarrow \quad \Omega^{\hat{\phi}}{}_{\hat{r}} = -l\frac{U(t)}{V(t)}\left(\frac{\dot{U}(t)V(t) - U(t)\dot{V}(t)}{V^2(t)}\right)\omega^{\hat{t}} \wedge \omega^{\hat{\theta}} + \frac{U(t)\dot{U}(t)\dot{V}(t)}{V(t)}\omega^{\hat{\phi}} \wedge \omega^{\hat{r}}$$

$$d\Gamma^{\hat{\phi}}{}_{\hat{\theta}} = d(\cos\theta\, d\phi) = -\sin\theta\, d\theta \wedge d\phi = -\frac{1}{V^2(t)}\omega^{\hat{\theta}} \wedge \omega^{\hat{\phi}}$$

$$\Gamma^{\hat{\phi}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{\theta}} = \Gamma^{\hat{\phi}}{}_{\hat{t}} \wedge \Gamma^{\hat{t}}{}_{\hat{\theta}} + \Gamma^{\hat{\phi}}{}_{\hat{r}} \wedge \Gamma^{\hat{r}}{}_{\hat{\theta}} + \Gamma^{\hat{\phi}}{}_{\hat{\theta}} \wedge \Gamma^{\hat{\theta}}{}_{\hat{\theta}} + \Gamma^{\hat{\phi}}{}_{\hat{\phi}} \wedge \Gamma^{\hat{\phi}}{}_{\hat{\theta}}$$

$$= \left(\frac{U(t)\dot{V}(t)}{V(t)}\omega^{\hat{\phi}}\right) \wedge \left(\frac{U(t)\dot{V}(t)}{V(t)}\omega^{\hat{\theta}}\right) + \left(-\frac{lU(t)}{V^2(t)}\omega^{\hat{\theta}}\right) \wedge \left(-\frac{lU(t)}{V^2(t)}\omega^{\hat{\phi}}\right)$$

$$= \left[\left(\frac{U(t)\dot{V}(t)}{V(t)}\right)^2 - \left(\frac{lU(t)}{V^2(t)}\right)^2\right]\omega^{\hat{\phi}} \wedge \omega^{\hat{\theta}}$$

$$\Rightarrow \quad \Omega^{\hat{\phi}}{}_{\hat{\theta}} = \left[\left(\frac{U(t)\dot{V}(t)}{V(t)}\right)^2 - \left(\frac{lU(t)}{V^2(t)}\right)^2 + \frac{1}{V^2(t)}\right]\omega^{\hat{\phi}} \wedge \omega^{\hat{\theta}}$$

7.11.1.3 The Riemann tensor:

$$\Omega^{\hat{a}}{}_{\hat{b}} = \frac{1}{2}R^{\hat{a}}{}_{\hat{b}\hat{c}\hat{d}}\omega^{\hat{c}} \wedge \omega^{\hat{d}}$$

$$R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}} = -\ddot{U}(t)U(t) - \dot{U}^2(t) \qquad R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{\theta}} = \left(\frac{U(t)\dot{V}(t)}{V(t)}\right)^2 - \left(\frac{lU(t)}{V^2(t)}\right)^2 + \frac{1}{V^2(t)}$$

$$R^{\hat{r}}{}_{\hat{t}\hat{\theta}\hat{\phi}} = -\frac{2lU(t)}{V^3(t)}\left(\dot{U}(t)V(t) - U(t)\dot{V}(t)\right)$$

$$
\begin{aligned}
R^{\hat{\theta}}{}_{\hat{t}\hat{\theta}\hat{t}} &= -\frac{U(t)}{V(t)}\left(\dot{U}(t)\dot{V}(t) + U(t)\ddot{V}(t)\right) & &= R^{\hat{\phi}}{}_{\hat{t}\hat{\phi}\hat{t}} = -\frac{U(t)}{V(t)}\left(\dot{U}(t)\dot{V}(t) + U(t)\ddot{V}(t)\right) \\
R^{\hat{\theta}}{}_{\hat{t}\hat{\phi}\hat{r}} &= \frac{lU(t)\dot{U}(t)}{V^2(t)} & R^{\hat{\phi}}{}_{\hat{t}\hat{\theta}\hat{r}} &= -\frac{lU(t)\dot{U}(t)}{V^2(t)} \\
R^{\hat{\theta}}{}_{\hat{r}\hat{t}\hat{\phi}} &= l\frac{U(t)}{V(t)}\left(\frac{\dot{U}(t)V(t) - U(t)\dot{V}(t)}{V^2(t)}\right) & R^{\hat{\phi}}{}_{\hat{r}\hat{t}\hat{\theta}} &= -l\frac{U(t)}{V(t)}\left(\frac{\dot{U}(t)V(t) - U(t)\dot{V}(t)}{V^2(t)}\right) \\
R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} &= U(t)\dot{U}(t)\frac{\dot{V}(t)}{V(t)} & &= R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{r}} = U(t)\dot{U}(t)\frac{\dot{V}(t)}{V(t)}
\end{aligned}
$$

7.11.2 The Ricci tensor

The diagonal elements of the Ricci tensor:

$$
\begin{aligned}
R_{\hat{a}\hat{b}} &= R^{\hat{c}}{}_{\hat{a}\hat{c}\hat{b}} \\
R_{\hat{t}\hat{t}} &= R^{\hat{t}}{}_{\hat{t}\hat{t}\hat{t}} + R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}} + R^{\hat{\theta}}{}_{\hat{t}\hat{\theta}\hat{t}} + R^{\hat{\phi}}{}_{\hat{t}\hat{\phi}\hat{t}} = R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}} + 2R^{\hat{\theta}}{}_{\hat{t}\hat{\theta}\hat{t}} \\
R_{\hat{r}\hat{r}} &= R^{\hat{t}}{}_{\hat{r}\hat{t}\hat{r}} + R^{\hat{r}}{}_{\hat{r}\hat{r}\hat{r}} + R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} + R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{r}} = -R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}} + 2R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} \\
R_{\hat{\theta}\hat{\theta}} &= R^{\hat{t}}{}_{\hat{\theta}\hat{t}\hat{\theta}} + R^{\hat{r}}{}_{\hat{\theta}\hat{r}\hat{\theta}} + R^{\hat{\theta}}{}_{\hat{\theta}\hat{\theta}\hat{\theta}} + R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{\theta}} = -R^{\hat{\theta}}{}_{\hat{t}\hat{\theta}\hat{t}} + R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} + R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{\theta}} = R_{\hat{\phi}\hat{\phi}}
\end{aligned}
$$

7.11.3 The Ricci Scalar

$$
\begin{aligned}
R &= \eta^{\hat{a}\hat{b}}R_{\hat{a}\hat{b}} = \eta^{\hat{t}\hat{t}}R_{\hat{t}\hat{t}} + \eta^{\hat{r}\hat{r}}R_{\hat{r}\hat{r}} + \eta^{\hat{\theta}\hat{\theta}}R_{\hat{\theta}\hat{\theta}} + \eta^{\hat{\phi}\hat{\phi}}R_{\hat{\phi}\hat{\phi}} = -R_{\hat{t}\hat{t}} + R_{\hat{r}\hat{r}} + R_{\hat{\theta}\hat{\theta}} + R_{\hat{\phi}\hat{\phi}} \\
&= -\left(R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}} + 2R^{\hat{\theta}}{}_{\hat{t}\hat{\theta}\hat{t}}\right) - R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}} + 2R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} - R^{\hat{\theta}}{}_{\hat{t}\hat{\theta}\hat{t}} + R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} + R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{\theta}} - R^{\hat{\phi}}{}_{\hat{t}\hat{\phi}\hat{t}} + R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{r}} + R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{\theta}} \\
&= 2\left(-R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}} + 2R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} - 2R^{\hat{\theta}}{}_{\hat{t}\hat{\theta}\hat{t}} + R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{\theta}}\right) \\
&= {}^{24\ 25\ 26\ 27} 2\left(\frac{U^2(t)}{V^2(t)} - \frac{U^2(t)}{V^4(t)}(t^2 + l^2)\right) \\
&= 0
\end{aligned}
$$

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[24] $= 2\left(\frac{d}{dt}\left(U(t)\dot{U}(t)\right) + 2U(t)\dot{U}(t)\frac{\dot{V}(t)}{V(t)} + 2\frac{U(t)}{V(t)}\left(\dot{U}(t)\dot{V}(t) + U(t)\ddot{V}(t)\right) + \left(\frac{U(t)\dot{V}(t)}{V(t)}\right)^2 - \left(\frac{lU(t)}{V^2(t)}\right)^2 + \frac{1}{V^2(t)}\right) =$

[25] $= 2\left(\frac{d}{dt}\left(U(t)\dot{U}(t)\right) + 4U(t)\dot{U}(t)\frac{\dot{V}(t)}{V(t)} + 2U^2(t)\frac{\ddot{V}(t)}{V(t)} + \left(\frac{U(t)\dot{V}(t)}{V(t)}\right)^2 - \left(\frac{lU(t)}{V^2(t)}\right)^2 + \frac{1}{V^2(t)}\right) =$

[26] $= 2\left(-\frac{1}{V^2(t)} - \frac{4t(m-t)}{V^4(t)} - \frac{U^2(t)}{V^2(t)} + \frac{4t^2U^2(t)}{V^4(t)} + 4\left(\frac{m-t}{V^2(t)} - \frac{tU^2(t)}{V^2(t)}\right)\frac{t}{V^2(t)} + 2U^2(t)\left(\frac{1}{V^2(t)} - \frac{t^2}{V^4(t)}\right) + \left(t\frac{U(t)}{V^2(t)}\right)^2 - \left(\frac{lU(t)}{V^2(t)}\right)^2 + \frac{1}{V^2(t)}\right) =$

[27] $= 2\left(-\frac{4t(m-t)}{V^4(t)} - \frac{U^2(t)}{V^2(t)} + \frac{4t^2U^2(t)}{V^4(t)} + 4\frac{t(m-t)}{V^4(t)} - 4\frac{t^2U^2(t)}{V^4(t)} + 2\frac{U^2(t)}{V^2(t)} - 2\frac{t^2U^2(t)}{V^4(t)} + t^2\frac{U^2(t)}{V^4(t)} - \frac{l^2U^2(t)}{V^4(t)}\right) =$

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[a] (McMahon, 2006, s. 80), (Penrose, 2004, s. 231)
[b] (McMahon, 2006, s. 80)
[c] (McMahon, 2006, s. 113)
[d] (McMahon, 2006, s. 87)
[e] (McMahon, 2006, s. 89)
[f] (McMahon, 2006, s. 113)
[g] (McMahon, 2006, s. 91), (Kay, 1988, s. 75)
[h] (McMahon, 2006, s. 91), (Kay, 1988, s. 102)
[i] (McMahon, 2006, s. 120)
[j] (McMahon, 2006, s. 120)
[k] (McMahon, 2006, s. 84), (Hartle, An Introduction to Einstein's General Relativity, 2003, s. 143, 165, 184), (Kay, 1988, s. 126)
[l] (McMahon, 2006, s. 120)
[m] (Hartle, An Introduction to Einstein's General Relativity, 2003, s. 160)
[n] (McMahon, 2006, s. 325)
[o] (McMahon, 2006, s. 325)
[p] (McMahon, 2006, s. 152-53)
[q] (Penrose, 2004, s. 50), (Hartle, Gravity - An introduction to Einstein's General Relativity, 2003, p. 184)
[r] (Penrose, 2004, s. 50)
[s] (McMahon, 2006, s. 106)
[t] (McMahon, 2006, s. 121)
[u] (Choquet-Bruhat, 2015, s. 98), (Ellis, 1973, p. 170)