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5 Covariant Derivative, Lie Derivative and Killings Equation

5.1 Definitions

5.1.1 Covariant derivative

[a]The covariant derivative of a covariant vector X_a

$$\nabla_b X_a = \partial_b X_a - \Gamma^c{}_{ab} X_c$$

[b]The covariant derivative of a contravariant vector X^a

$$\nabla_b X^a = \partial_b X^a + \Gamma^a{}_{bc} X^c$$

[c]The covariant derivative of a tensor

$$\nabla_c T^{ab} = \partial_c T^{ab} + \Gamma^a{}_{cd} T^{bd} + \Gamma^b{}_{cd} T^{ad}$$
$$\nabla_c T_{ab} = \partial_c T_{ab} - \Gamma^d{}_{ac} T_{bd} - \Gamma^d{}_{bc} T_{ad}$$
$$\nabla_c T^a{}_b = \partial_c T^a{}_b + \Gamma^a{}_{cd} T^d{}_b - \Gamma^d{}_{bc} T^a{}_d$$

The covariant derivative of a scalar function

$$\nabla_a f(x_a) = {}^{1}\frac{\partial f(x_a)}{\partial x_a}$$

[d]The covariant derivative of the metric tensor

$$\nabla_c g_{ab} = 0$$

5.1.2 [e] [2]The Lie Derivative

The Lie derivative of a covariant vector V_a with respect to a vector X

$$L_X V_a = X^b \partial_b V_a + V_b \partial_a X^b$$

The Lie derivative of a contravariant vector V^a with respect to a vector X

$$L_X V^a = X^b \partial_b V^a - V^b \partial_b X^a$$

The Lie derivative of a (0,2) tensor T_{ab} with respect to a vector X

$$L_X T_{ab} = X^c \partial_c T_{ab} + T_{bc} \partial_a X^c + T_{ac} \partial_b X^c$$

The Lie derivative of a (2,0) tensor T^{ab} with respect to a vector X

$$L_X T^{ab} = X^c \partial_c T^{ab} - T^{ac} \partial_c X^b + T^{bc} \partial_c X^b$$

The Lie derivative of a scalar function f with respect to a vector X

$$L_X f = X^a \partial_a f$$

[f]The Lie derivative of the metric tensor with respect to a Killing vector X

$$L_X g_{ab} = 0$$

5.1.3 [g]The Lie Derivatives in terms of the covariant derivative.

[3]It can sometimes be convenient to express the Lie-derivative in terms of the covariant derivative. We will show why this is valid for a (0,2) tensor.

We need:

$$\nabla_c T_{ab} = \partial_c T_{ab} - \Gamma^d{}_{ac} T_{bd} - \Gamma^d{}_{bc} T_{ad}$$
$$\nabla_a X^c = \partial_a X^c + \Gamma^c{}_{da} X^d$$
$$\nabla_b X^c = \partial_b X^c + \Gamma^c{}_{db} X^d$$
$$\Rightarrow \quad \partial_c T_{ab} = \nabla_c T_{ab} + \Gamma^d{}_{ac} T_{bd} + \Gamma^d{}_{bc} T_{ad}$$
$$\partial_a X^c = \nabla_a X^c - \Gamma^c{}_{da} X^d$$
$$\partial_b X^c = \nabla_b X^c - \Gamma^c{}_{db} X^d$$

The Lie derivative

$$\begin{aligned} L_X T_{ab} &= X^c \partial_c T_{ab} + T_{bc} \partial_a X^c + T_{ac} \partial_b X^c \\ &= X^c \left(\nabla_c T_{ab} + \Gamma^d{}_{ac} T_{bd} + \Gamma^d{}_{bc} T_{ad}\right) + T_{bc}\left(\nabla_a X^c - \Gamma^c{}_{da} X^d\right) + T_{ac}\left(\nabla_b X^c - \Gamma^c{}_{db} X^d\right) \end{aligned}$$

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[1] This equals the partial derivative

[2] The general formula is: $L_X T^{a\dots}_{b\dots} = X^c \partial_c T^{a\dots}_{b\dots} - T^{c\dots}_{b\dots} \partial_c X^a - \cdots + T^{a\dots}_{c\dots} \partial_b X^c + \cdots$

[3] An ordinary derivative can be replaced with a covariant derivative in a Lie derivative.
http://mathworld.wolfram.com/KillingVectors.html

http://physicssusan.mono.net/ logik.susan@gmail.com

$$= {}^{4\,5}X^c\nabla_c T_{ab} + T_{bc}\nabla_a X^c + T_{ac}\nabla_b X^c$$

5.1.4 Killings equation and derivatives of Killing vectors

Killing's equation

$$\nabla_b X_a + \nabla_a X_b = 0$$

[h]Derivatives of Killing vectors and the Riemann tensor:

$$\nabla_b\nabla_c X^a - \nabla_c\nabla_b X^a = {}^6 R^a{}_{bcd}X^d$$
$$\nabla_b\nabla_a X^a - \nabla_a\nabla_b X^a = {}^7 R_{bd}X^d$$
$$X^a\nabla_a R = {}^8 L_X R$$

5.2 Proofs

5.2.1 Derivatives of Killing vectors and the metric tensor

The covariant derivative of the metric tensor

$$\nabla_c g_{ab} = 0$$

Proof:

We need

$$\nabla_c g_{ab} = \partial_c g_{ab} - g_{db}\Gamma^d{}_{ac} - g_{ad}\Gamma^d{}_{cb}$$
$$\Gamma_{acb} = \Gamma_{cab}$$
$$\Gamma_{cab} + \Gamma_{cba} = \partial_c g_{ab}$$

If $g_{db} = g_{bd}$ (the metric tensor is symmetric):

$$\nabla_c g_{ab} = \partial_c g_{ab} - g_{bd}\Gamma^d{}_{ac} - g_{ad}\Gamma^d{}_{cb} = \partial_c g_{ab} - \Gamma_{acb} - \Gamma_{cba} = \partial_c g_{ab} - (\Gamma_{cab} - \Gamma_{cba})$$
$$= \partial_c g_{ab} - \partial_c g_{ab} = 0$$

$$\Rightarrow \quad \nabla_c g_{ab} = 0$$

5.2.2 [i]Show that if the Lie derivative of the metric tensor with respect to vector X vanishes ($\boldsymbol{L_X g_{ab} = 0}$), the vector X satisfies the Killing equation.

We need

$$\nabla_b X_a + \nabla_a X_b = 0$$
$$\nabla_c g_{ab} = 0$$
$$L_V T_{ab} = V^c\nabla_c T_{ab} + T_{cb}\nabla_a V^c + T_{ac}\nabla_b V^c$$
$$\Rightarrow \quad L_X g_{ab} = X^c\nabla_c g_{ab} + g_{cb}\nabla_a X^c + g_{ac}\nabla_b X^c = \nabla_a(g_{cb}X^c) + \nabla_b(g_{ac}X^c) = \nabla_a X_b + \nabla_b X_a$$

If $L_X g_{ab} = 0$ this implies that $\nabla_a X_b + \nabla_b X_a = 0$, which is the Killing equation.

5.2.3 [j]Constructing a Conserved Current with Killings Equation:

We need

$$J = \nabla_a J^a = T^{ab}(\nabla_a X_b)$$
$$J = \nabla_b J^b = T^{ba}(\nabla_b X_a)$$
$$T^{ab} = T^{ba}$$
$$(\nabla_a X_b) + (\nabla_b X_a) = 0$$

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[4] $= X^c\nabla_c T_{ab} + T_{bc}\nabla_a X^c + T_{ac}\nabla_b X^c + \Gamma^d{}_{ac}X^c T_{bd} + \Gamma^d{}_{bc}X^c T_{ad} - \Gamma^c{}_{da}T_{bc}X^d - T_{ac}\Gamma^c{}_{db}X^d =$

[5] $= X^c\nabla_c T_{ab} + T_{bc}\nabla_a X^c + T_{ac}\nabla_b X^c + \Gamma^c{}_{ad}X^d T_{bc} + \Gamma^c{}_{bd}X^d T_{ac} - \Gamma^c{}_{da}T_{bc}X^d - T_{ac}\Gamma^c{}_{db}X^d =$

[6] This is a straightforward (but long) proof: Se e.g. (McMahon, 2006, s. 133)

[7] This is found by contracting the former equation with δ^c_a: $\delta^c_a(\nabla_b\nabla_c X^a - \nabla_c\nabla_b X^a) = \delta^c_a(R^a{}_{bcd}X^d) \Rightarrow \nabla_b\nabla_a X^a - \nabla_a\nabla_b X^a = R_{bd}X^d$

[8] $X^a\nabla_a R = X^a\nabla_a R^b{}_b = X^a\left(\partial_a R^b{}_b + \Gamma^b{}_{ad}T^d{}_b - \Gamma^d{}_{ba}T^b{}_d\right) = X^a\left(\partial_a R + \Gamma^d{}_{ba}T^b{}_d - \Gamma^d{}_{ba}T^b{}_d\right) = X^a\partial_a R = L_X R$

$$\Rightarrow \qquad J = \frac{1}{2}\nabla_a J^a + \frac{1}{2}\nabla_b J^b = \frac{1}{2}\left(T^{ab}(\nabla_a X_b) + T^{ba}(\nabla_b X_a)\right)$$
$$= \frac{1}{2}T^{ab}\left((\nabla_a X_b) + (\nabla_b X_a)\right) = 0$$

5.3 [k]Lie derivative of a vector – an example

Let

$$w^a = \begin{pmatrix} r \\ \sin\theta \\ \sin\theta\cos\phi \end{pmatrix}$$
$$v^a = \begin{pmatrix} r \\ r^2\cos\theta \\ \sin\phi \end{pmatrix}$$

The Lie derivative

$$u^a = L_v w^a = v^b\partial_b w^a - w^b\partial_b v^a$$
$$\Rightarrow \quad u^r = v^b\partial_b w^r - w^b\partial_b v^r = v^r\partial_r w^r - w^r\partial_r v^r + v^\theta\partial_\theta w^r - w^\theta\partial_\theta v^r + v^\phi\partial_\phi w^r - w^\phi\partial_\phi v^r$$
$$= v^r\partial_r(r) - w^r\partial_r(r) + v^\theta\partial_\theta(r) - w^\theta\partial_\theta(r) + v^\phi\partial_\phi(r) - w^\phi\partial_\phi(r) = v^r - w^r$$
$$\Rightarrow \quad u^\theta = v^b\partial_b w^\theta - w^b\partial_b v^\theta = v^r\partial_r w^\theta - w^r\partial_r v^\theta + v^\theta\partial_\theta w^\theta - w^\theta\partial_\theta v^\theta + v^\phi\partial_\phi w^\theta - w^\phi\partial_\phi v^\theta$$
$$= {}^{9} - w^r(2r\cos\theta) + v^\theta(\cos\theta) - w^\theta(-r^2\sin\theta)$$
$$= -r(2r\cos\theta) + r^2\cos\theta\,(\cos\theta) - \sin\theta\,(-r^2\sin\theta) = r^2(1 - 2\cos\theta)$$
$$\Rightarrow \quad u^\phi = v^b\partial_b w^\phi - w^b\partial_b v^\phi$$
$$= v^r\partial_r w^\phi - w^r\partial_r v^\phi + v^\theta\partial_\theta w^\phi - w^\theta\partial_\theta v^\phi + v^\phi\partial_\phi w^\phi - w^\phi\partial_\phi v^\phi$$
$$= {}^{10}v^\theta(\cos\theta\cos\phi) - v^\phi(\sin\theta\sin\phi) - w^\phi(\cos\phi)$$
$$= r^2\cos\theta\,(\cos\theta\cos\phi) - \sin\phi\,(\sin\theta\sin\phi) - \sin\theta\cos\phi\,(\cos\phi)$$
$$= r^2\cos^2\theta\cos\phi - \sin\theta$$
$$\Rightarrow \quad u = \begin{pmatrix} u^r \\ u^\theta \\ u^\phi \end{pmatrix} = \begin{pmatrix} 0 \\ r^2(1 - 2\cos\theta) \\ r^2\cos^2\theta\cos\phi - \sin\theta \end{pmatrix}$$

5.4 [11]The 2-sphere with radius a

The line element:

$$dS^2 = a^2 d\theta^2 + a^2\sin^2\theta\, d\phi^2$$

The metric tensor and its inverse:

$$g_{ab} = \begin{Bmatrix} a^2 & \\ & a^2\sin^2\theta \end{Bmatrix}$$

5.4.1 [l]The Christoffel symbols of the 2-sphere with radius a

The Christoffel symbols

$$\Gamma_{abc} = \frac{1}{2}(\partial_a g_{bc} + \partial_b g_{ca} - \partial_c g_{ab}) \qquad \Gamma^a{}_{bc} = g^{ad}\Gamma_{bcd}$$

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[9] $= v^r\partial_r(\sin\theta) - w^r\partial_r(r^2\cos\theta) + v^\theta\partial_\theta(\sin\theta) - w^\theta\partial_\theta(r^2\cos\theta) + v^\phi\partial_\phi(\sin\theta) - w^\phi\partial_\phi r^2\cos\theta =$

[10] $= v^r\partial_r(\sin\theta\cos\phi) - w^r\partial_r(\sin\phi) + v^\theta\partial_\theta(\sin\theta\cos\phi) - w^\theta\partial_\theta(\sin\phi) + v^\phi\partial_\phi(\sin\theta\cos\phi) - w^\phi\partial_\phi(\sin\phi) =$

[11] This is a special case of the three-dimensional space in polar coordinates: $ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$ with constant radius $r = a$

$$\Gamma_{\phi\phi\theta} = -\frac{1}{2}\partial_\theta g_{\phi\phi} = {}^{12} - a^2\cos\theta\sin\theta \quad \Rightarrow \quad \Gamma^\theta{}_{\phi\phi} = g^{\theta\theta}\Gamma_{\phi\phi\theta} = {}^{13} - \cos\theta\sin\theta$$

$$\Gamma_{\theta\phi\phi} = \Gamma_{\phi\theta\phi} = \frac{1}{2}\partial_\theta g_{\phi\phi} = {}^{14}a^2\cos\theta\sin\theta \quad \Rightarrow \quad \Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta} = g^{\phi\phi}\Gamma_{\theta\phi\phi} = {}^{15}\cot\theta$$

5.4.2 [m]Show that the great circle is a solution of the geodesic equation of a two-dimensional sphere.

We use the Euler-Lagrange method.

$$0 = \frac{d}{dS}\left(\frac{\partial F}{\partial \dot{x}^a}\right) - \frac{\partial F}{\partial x^a}$$

$$F = \frac{1}{2}g_{ab}\dot{x}^a\dot{x}^b$$

The line element

$$dS^2 = a^2 d\theta^2 + a^2\sin^2\theta\, d\phi^2$$

$$\Rightarrow \quad F = \frac{1}{2}a^2\dot{\theta}^2 + \frac{1}{2}a^2\sin^2\theta\,\dot{\phi}^2$$

$\underline{x^a = \theta:}$

$$\frac{\partial F}{\partial \theta} = a^2\sin\theta\cos\theta\,\dot{\phi}^2$$

$$\frac{\partial F}{\partial \dot{\theta}} = a^2\dot{\theta}$$

$$\frac{d}{dS}\left(\frac{\partial F}{\partial \dot{\theta}}\right) = a^2\ddot{\theta}$$

$$\Rightarrow \quad \ddot{\theta} = \sin\theta\cos\theta\,\dot{\phi}^2$$

$\underline{x^a = \phi:}$

$$\frac{\partial F}{\partial \phi} = 0$$

$$\frac{\partial F}{\partial \dot{\phi}} = a^2\sin^2\theta\,\dot{\phi}$$

$$\frac{d}{dS}\left(\frac{\partial F}{\partial \dot{\phi}}\right) = \frac{d}{dS}\left(a^2\sin^2\theta\,\dot{\phi}\right) = 2a^2\sin\theta\cos\theta\,\dot{\theta}\dot{\phi} + a^2\sin^2\theta\,\ddot{\phi}$$

$$\Rightarrow \quad \ddot{\phi} = -2\cot\theta\,\dot{\theta}\dot{\phi}$$

Collecting the results

$$\ddot{\theta} = \sin\theta\cos\theta\,\dot{\phi}^2 \qquad (5.1.)$$

$$\ddot{\phi} = -2\cot\theta\,\dot{\theta}\dot{\phi} \qquad (5.2.)$$

The non-zero Christoffel symbols are

$$\Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta} = \cot\theta$$

To show that the great circle is a solution to the geodesic equations we choose to work in the plane $\theta = \frac{\pi}{2}$, so the spherical polar coordinates are

$$x = a\sin\theta\cos\phi = a\cos\phi$$

$$y = a\sin\theta\sin\phi = a\sin\phi$$

$$z = a\cos\theta = 0$$

$$\Rightarrow \quad \dot{x} = -a\dot{\phi}\sin\phi$$

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[12] $= -\frac{1}{2}\partial_\theta(a^2\sin^2\theta) =$

[13] $= \frac{1}{a^2}(-a^2\cos\theta\sin\theta) =$

[14] $= \frac{1}{2}\partial_\theta(a^2\sin^2\theta) =$

[15] $= \frac{1}{a^2\sin^2\theta}a^2\cos\theta\sin\theta =$

$$\dot{y} = a\dot{\phi}\cos\phi$$
$$\Rightarrow \quad \dot{x}^2 + \dot{y}^2 = a^2\dot{\phi}^2$$

The right hand side we can find from the second geodesic eq. (5.2.), which is reduced to $\ddot{\phi} = 0 \Rightarrow \dot{\phi} = constant$. Next we use the law of conservation of the four velocity in Cartesian coordinates to show that the left hand side is a constant as well.

$$\bar{u}\cdot\bar{u} = g_{ab}\frac{dx^a}{dS}\frac{dx^b}{dS} = {}^{16}\dot{x}^2 + \dot{y}^2 = 1$$

And we can conclude that the great circle is a solution.

5.4.3 [n]Killing vectors in the two-dimensional flat space (the 2-sphere):

The line-element

$$dS^2 = a^2 d\theta^2 + a^2\sin^2\theta\, d\phi^2$$

The Christoffel symbols found earlier

$$\Gamma^{\theta}{}_{\phi\phi} = -\sin\theta\cos\theta$$
$$\Gamma^{\phi}{}_{\theta\phi} = \Gamma^{\phi}{}_{\phi\theta} = \cot\theta$$

We need: Killings equation and the covariant derivative of a vector

$$0 = {}^{17}\nabla_a X_b + \nabla_b X_a$$
$$\nabla_a X_b = \partial_a X_b - \Gamma^c{}_{ab}X_c$$
$$\underline{a=b=\theta:} \quad \nabla_\theta X_\theta = \partial_\theta X_\theta - \Gamma^c{}_{\theta\theta}X_c = \partial_\theta X_\theta = 0$$
$$\Rightarrow \quad X_\theta = f(\phi)$$
$$\underline{a=b=\phi:} \quad \nabla_\phi X_\phi = \partial_\phi X_\phi - \Gamma^c{}_{\phi\phi}X_c = \partial_\phi X_\phi - \Gamma^\theta{}_{\phi\phi}X_\theta = \partial_\phi X_\phi + \sin\theta\cos\theta\, X_\theta = 0$$
$$\Rightarrow \quad \partial_\phi X_\phi = -\sin\theta\cos\theta\, X_\theta = -\sin\theta\cos\theta\, f(\phi)$$
$$\Rightarrow \quad X_\phi = -\sin\theta\cos\theta\int f(\phi)d\phi + g(\theta)$$
$$\underline{a=\theta, b=\phi:} \quad \nabla_\theta X_\phi + \nabla_\phi X_\theta = \partial_\theta X_\phi - \Gamma^c{}_{\theta\phi}X_c + \partial_\phi X_\theta - \Gamma^c{}_{\phi\theta}X_c$$
$$= \partial_\theta X_\phi + \partial_\phi X_\theta - 2\cot\theta\, X_\phi$$
$$= {}^{18\,19}\int f(\phi)d\phi + \partial_\phi\big(f(\phi)\big) + \partial_\theta\big(g(\theta)\big) - 2\cot\theta\, g(\theta) = 0$$

Separating the variables we get:

$$k_1 = \int f(\phi)d\phi + \partial_\phi\big(f(\phi)\big) = -\partial_\theta\big(g(\theta)\big) + 2\cot\theta\, g(\theta)$$
$$\Rightarrow \quad \partial_\phi\big(f(\phi)\big) = -\int f(\phi)d\phi - k_1$$
$$\Rightarrow \quad \frac{\partial^2 f(\phi)}{\partial\phi^2} = -f(\phi)$$
$$\Rightarrow \quad f(\phi) = {}^{20}A\sin\phi + B\cos\phi$$
$$\Rightarrow \quad \partial_\theta\big(g(\theta)\big) = 2\cot\theta\, g(\theta)$$
$$\Rightarrow \quad g(\theta) = C\sin^2\theta$$

The Killing Vector:

$$\Rightarrow \quad X_\theta = f(\phi) = A\sin\phi + B\cos\phi$$
$$X_\phi = -\sin\theta\cos\theta\int f(\phi)d\phi + g(\theta)$$
$$= -\sin\theta\cos\theta\int(A\sin\phi + B\cos\phi) + C\sin^2\theta$$

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[16] Recall: $dS^2 = dx^2 + dy^2 = a^2d\theta^2 + a^2\sin^2\theta\, d\phi^2$

[17] Do not confuse index a with the parameter a

[18] $= \partial_\theta\left(-\sin\theta\cos\theta\int f(\phi)d\phi + g(\theta)\right) + \partial_\phi\big(f(\phi)\big) - 2\cot\theta\left(-\sin\theta\cos\theta\int f(\phi)d\phi + g(\theta)\right) =$

[19] $= \left(\sin^2\theta - \cos^2\theta\right)\int f(\phi)d\phi + \partial_\theta\big(g(\theta)\big) + \partial_\phi\big(f(\phi)\big) + 2\cos^2\theta\int f(\phi)d\phi - 2\cot\theta\, g(\theta) =$

[20] $k_1 = 0$

$$= -\sin\theta\cos\theta\,(-A\cos\phi + B\sin\phi + D) + C\sin^2\theta$$
$$=^{21} \sin\theta\cos\theta\,(A\cos\phi - B\sin\phi) + C\sin^2\theta$$

The covariant Killing vector:

$$X^\theta = g^{\theta\theta}X_\theta = \frac{1}{a^2}(A\sin\phi + B\cos\phi) = A'\sin\phi + B'\cos\phi$$
$$X^\phi = g^{\phi\phi}X_\phi = \frac{1}{a^2\sin^2\theta}(\sin\theta\cos\theta\,(A\cos\phi - B\sin\phi) + C\sin^2\theta)$$
$$= \cot\theta\,(A'\cos\phi - B'\sin\phi) + C'$$
$$\Rightarrow \quad X = \begin{pmatrix} X^\theta \\ X^\phi \end{pmatrix} = \begin{pmatrix} A'\sin\phi + B'\cos\phi \\ \cot\theta\,(A'\cos\phi - B'\sin\phi) + C' \end{pmatrix}$$

Because the metric is independent of ϕ a rotational Killing vector is

$$X =^{22} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

<u>The translational Killing vectors:</u>

The translational Killing vectors we can find from the line-element in Cartesian coordinates

$$dS^2 = dx^2 + dy^2$$

The Christoffel symbols

$$\Gamma^a{}_{bc} = 0$$

The Killing equation:

$$\partial_a X_b = 0$$
$$\Rightarrow \quad X = \begin{pmatrix} X^x \\ X^y \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}$$

Collecting the results:

$$X = \begin{pmatrix} X^\theta \\ X^\phi \end{pmatrix} = \begin{pmatrix} A'\sin\phi + B'\cos\phi \\ \cot\theta\,(A'\cos\phi - B'\sin\phi) + C' \end{pmatrix}$$
$$X = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$X = \begin{pmatrix} X^x \\ X^y \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}$$

<u>Definition of the angular momentum operators:</u>

In the coordinate basis:

$$X = X^a\partial_a = X^\theta\partial_\theta + X^\phi\partial_\phi$$
$$= (A'\sin\phi + B'\cos\phi)\partial_\theta + (\cot\theta\,(A'\cos\phi - B'\sin\phi) + C')\partial_\phi$$
$$= A'\left(\sin\phi\,\partial_\theta + \cot\theta\cos\phi\,\partial_\phi\right) + B'(\cos\phi\,\partial_\theta - \cot\theta\sin\phi) + C'\partial_\phi$$

°Which defines the angular momentum operators

$$L_x = \sin\phi\,\partial_\theta + \cot\theta\cos\phi\,\partial_\phi$$
$$L_y = \cos\phi\,\partial_\theta - \cot\theta\sin\phi$$
$$L_z = \partial_\phi$$

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[21] D has to be zero because: $X_\phi = -\sin\theta\cos\theta\int f(\phi)d\phi + g(\theta) = \sin\theta\cos\theta\,\partial_\phi\big(f(\phi)\big) + g(\theta) = \sin\theta\cos\theta\,\partial_\phi(A\sin\phi + B\cos\phi) + C\sin^2\theta = \sin\theta\cos\theta\,(A\cos\phi - B\sin\phi) + C\sin^2\theta$

[22] This can be found if we choose $A' = B'$ and $C' = 1$ and rotate the coordinate system into a system where $\theta = \frac{\pi}{2}$ and $\phi = -\frac{\pi}{4}$.

5.5 [23]The Plane in polar coordinates

The line element

$$dS^2 = dr^2 + r^2 d\phi^2$$

5.5.1 [p]Geodesics Equations and Christoffel symbols of the plane in polar coordinates

We use the Euler-Lagrange method.

$$0 = \frac{d}{dS}\left(\frac{\partial F}{\partial \dot{x}^a}\right) - \frac{\partial F}{\partial x^a}$$

$$F = \frac{1}{2} g_{ab} \dot{x}^a \dot{x}^b$$

The line element

$$dS^2 = dr^2 + r^2 d\phi^2$$

$$F = \frac{1}{2}\dot{r}^2 + \frac{1}{2} r^2 \dot{\phi}^2$$

$\underline{x^a = r:}$

$$\frac{\partial F}{\partial r} = r\dot{\phi}^2$$

$$\frac{\partial F}{\partial \dot{r}} = \dot{r}$$

$$\Rightarrow \quad \ddot{r} = r\dot{\phi}^2$$

$\underline{x^a = \phi:}$

$$\frac{\partial F}{\partial \phi} = 0$$

$$\frac{\partial F}{\partial \dot{\phi}} = r^2 \dot{\phi}$$

$$\Rightarrow \quad 0 = \frac{d}{dS}(r^2 \dot{\phi}) = 2r\dot{r}\dot{\phi} + r^2 \ddot{\phi}$$

$$\Rightarrow \quad \ddot{\phi} = -\frac{2}{r}\dot{r}\dot{\phi}$$

Collecting the results

$$\ddot{r} = r\dot{\phi}^2$$

$$\ddot{\phi} = -\frac{2}{r}\dot{r}\dot{\phi}$$

[q]The non-zero Christoffel symbols are

$$\Gamma^r{}_{\phi\phi} = -r$$

$$\Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} = \frac{1}{r}$$

5.5.2 [r] [24]Christoffel symbols calculated directly from the metric tensor

<u>The Christoffel symbols of first kind</u> <u>Christoffel symbols of the second kind</u>

$$\Gamma_{abc} = \frac{1}{2}(\partial_a g_{bc} + \partial_b g_{ca} - \partial_c g_{ab}) \qquad \Gamma^a{}_{bc} = g^{ad}\Gamma_{bcd}$$

$$\Gamma_{\phi\phi r} = -\frac{1}{2}\partial_r g_{\phi\phi} = -\frac{1}{2}\partial_r(r^2) = -r \quad \Rightarrow \quad \Gamma^r{}_{\phi\phi} = g^{rr}\Gamma_{\phi\phi r} = -r$$

$$\Gamma_{r\phi\phi} = \Gamma_{r\phi\phi} = \frac{1}{2}\partial_r g_{\phi\phi} = \frac{1}{2}\partial_r(r^2) = r \quad \Rightarrow \quad \Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} = g^{\phi\phi}\Gamma_{r\phi\phi} = \frac{1}{r}$$

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[23] This is a special case of the three-dimensional space in polar coordinates: $ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$ in the equatorial plane where $\theta = \frac{\pi}{2}$

[24] This is a copy of the calculation done in cylindrical coordinates: $ds^2 = dr^2 + r^2 d\phi^2 + dz^2$. We can do that because the metric tensor does not depend on the z-coordinate.

5.5.3 [s]Geodesics in the Plane Using Polar Coordinates.

We know that the geodesics in the plane are straight lines. We can show this by using the line element, the Killing vector ξ and the conserved quantity $l = \xi \cdot u = const$.

The line element

$$dS^2 = dr^2 + r^2 d\phi^2$$

$$\Rightarrow \quad 1 = \left(\frac{dr}{dS}\right)^2 + r^2\left(\frac{d\phi}{dS}\right)^2$$

The metric is independent of ϕ so we have a Killing vector

$$\xi = (0,1)$$

And a conserved quantity

$$l = \xi \cdot u = \xi_a u^a = g_{ab}\xi^b u^a = g_{rr}\xi^r u^r + g_{\phi\phi}\xi^\phi u^\phi = 1 \cdot 0 \cdot \frac{dr}{dS} + r^2 \cdot 1 \cdot \frac{d\phi}{dS}$$

$$= {}^{25} r^2 \frac{d\phi}{dS}$$

This we can substitute in the line element and integrate

$$\Rightarrow \quad 1 = \left(\frac{dr}{dS}\right)^2 + r^2\left(\frac{d\phi}{dS}\right)^2$$

$$\left(\frac{dr}{dS}\right)^2 = 1 - r^2\left(\frac{d\phi}{dS}\right)^2 = 1 - r^2\left(\frac{l}{r^2}\right)^2 = 1 - \left(\frac{l}{r}\right)^2$$

$$\Rightarrow \quad \frac{dr}{dS} = \left(1 - \left(\frac{l}{r}\right)^2\right)^{\frac{1}{2}}$$

We want to find ϕ as a function of r

$$\frac{d\phi}{dr} = \frac{d\phi}{dS}\frac{dS}{dr} = \frac{l}{r^2\left(1 - \left(\frac{l}{r}\right)^2\right)^{\frac{1}{2}}} = \frac{l}{r(r^2 - l^2)^{\frac{1}{2}}} = \frac{l}{r(r^2 - l^2)^{\frac{1}{2}}}$$

$$\Rightarrow \quad d\phi = l\int \frac{dr}{r(r^2 - l^2)^{\frac{1}{2}}}$$

$$\Rightarrow \quad \phi = {}^{26} l \frac{2}{2\sqrt{l^2}} \cos^{-1}\sqrt{\frac{l^2}{r^2}} + \phi^* = \cos^{-1}\left(\frac{l}{r}\right) + \phi^*$$

$$\Rightarrow \quad l = r\cos(\phi - \phi^*) = r\cos\phi\cos\phi^* + r\sin\phi\sin\phi^* = x\cos\phi^* + y\sin\phi^*$$

$$\Rightarrow \quad y = -x\cot\phi^* + \frac{l}{\sin\phi^*} = ax + b$$

And the geodesic is a straight line as expected.

5.5.4 [t]Covariant derivative of a vector in the polar plane – example.

We want to find the covariant derivative $\nabla_a V^a$ of the contravariant vector $V^a = (V^r, V^\phi) = (r^2\cos\phi, -\sin\phi)$ in the polar plane[27].

We have

[25] Notice this is equivalent with the geodesic equation we found before $\frac{d}{ds}\left(r^2\dot{\phi}\right) = 0$

[26] $\int \frac{dx}{x\sqrt{x^n - a^n}} = \frac{2}{n\sqrt{a^n}}\cos^{-1}\sqrt{\frac{a^n}{x^n}}$ (Spiegel, 1990) (14.334)

[27] This is a special case of three-dimensional flat space in polar coordinates: $ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$ with $\theta = \frac{\pi}{2}$

$$\begin{aligned}
dS^2 &= dr^2 + r^2 d\phi^2 \\
\Gamma^r{}_{\phi\phi} &= -r \\
\Gamma^\phi{}_{r\phi} &= \Gamma^\phi{}_{\phi r} = \frac{1}{r} \\
V^r &= r^2 \cos\phi \\
V^\phi &= -\sin\phi \\
\nabla_b V^a &= \partial_b V^a + \Gamma^a{}_{bc} V^c
\end{aligned}$$

Now

$$\begin{aligned}
\nabla_a V^a &= {}^{28}\partial_r V^r + \Gamma^r{}_{rc} V^c + \partial_\phi V^\phi + \Gamma^\phi{}_{\phi c} V^c \\
&= \partial_r V^r + \Gamma^r{}_{rr} V^r + \Gamma^r{}_{r\phi} V^\phi + \partial_\phi V^\phi + \Gamma^\phi{}_{\phi r} V^r + \Gamma^\phi{}_{\phi\phi} V^\phi \\
&= \partial_r (r^2 \cos\phi) + \partial_\phi (-\sin\phi) + \frac{1}{r} r^2 \cos\phi = (3r - 1)\cos\phi
\end{aligned}$$

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[a] (McMahon, 2006, s. 171), (d'Inverno, 1992, p. 74)
[b] (McMahon, 2006, s. 67) , (d'Inverno, 1992, p. 73)
[c] (McMahon, 2006, s. 69)
[d] (McMahon, 2006, s. 78)
[e] (McMahon, 2006, s. 81), (d'Inverno, 1992, p. 72), https://en.wikipedia.org/wiki/Lie_derivative
[f] (McMahon, 2006, s. 168), (d'Inverno, 1992, p. 103)
[g] These calculations were kindly provided to med by Mr. John Fredsted
[h] (McMahon, 2006, s. 177), (d'Inverno, 1992, p. 77)
[i] (McMahon, 2006, s. 168)
[j] (McMahon, 2006, s. 178)
[k] (McMahon, 2006, s. 91)
[l] (McMahon, 2006, s. 74)
[m] (Hartle, 2003, p. 183)
[n] (McMahon, 2006, s. 170)
[o] Compared to (McMahon, 2006, s. 177) these are rotated by $\frac{\pi}{2}$

[28] Notice: This is a special case where $a = b$, hence the sum. If the problem was $\nabla_b V^a$, we would have to specify a and b in order to establish, whether it was $\nabla_r V^r, \nabla_\phi V^\phi, \nabla_r V^\phi$ or $\nabla_\phi V^r$ we wanted to find.

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[p] (Hartle, 2003, p. 171)
[q] (Hartle, 2003, p. 173)
[r] (Carroll, 2004, p. 100)
[s] (Hartle, 2003, p. 177)
[t] (McMahon, 2006, s. 69)