Content

Space-time		Line-element	Chap-ter
Alcubierre drive warp drive	ds^2	$= -dt^2 + [dx - V_s(t)f(r_s)dt]^2 + dy^2 + dz^2$	4
Classically anti-de-Sitter space-time	ds^2	$= -\cosh^2(r)\,dt^2 + dr^2 + \sinh^2(r)\,d\theta^2 + \sinh^2(r)\sin^2\theta\,d\phi^2$	4
Cylindrical coordinates	ds^2	$= dr^2 + r^2 d\phi^2 + dz^2$	4

http://physicssusan.mono.net/ logik.susan@gmail.com

Einstein cylinder	ds^2	$= -dt^2 + (a_0)^2\big(d\theta^2 + \sin^2\theta\,(d\phi^2 + \sin^2\phi\, d\psi^2)\big)$	9 (4), 13
Hyperbolsk plane – Poincaré Half plane	ds^2	$= \frac{1}{y^2}(dx^2 + dy^2)$	4,7
Rindler metric	ds^2	$= -X^2dT^2 + dX^2$	2,3,4,7
Three-dimensional flat space-time in polar coordinates	ds^2	$= -dt^2 + dr^2 + r^2d\phi^2$	4
Two-dimensional flat space	dS^2	$= dx^2 + dy^2$	4
Two-dimensional flat space-time	ds^2	$= -dt^2 + dx^2$	3,4
Worm hole geometry	ds^2	$= -dt^2 + dr^2 + (b^2 + r^2)(d\theta^2 + \sin^2\theta\, d\phi^2)$	4

4 Christoffel Symbols, Geodesic Equations and Killing Vectors

4.1 Christoffel symbols.

4.1.1 [a]Definitions

The Christoffel symbols of first kind[1]

$$\Gamma_{abc} = \frac{1}{2}(\partial_a g_{bc} + \partial_b g_{ca} - \partial_c g_{ab})$$

The Christoffel symbols of second kind[2]

$$\Gamma^a{}_{bc} = \frac{1}{2}g^{ad}(\partial_c g_{db} + \partial_b g_{dc} - \partial_d g_{bc})$$

The connection between the Christoffel symbols of the first and the second kind[3]

$$\Gamma^a{}_{bc} = g^{ad}\Gamma_{bcd}$$

4.1.1.1 *Important remarks on the notation of the Christoffel symbols.*

We find that there are a variety of notations when introducing the Christoffel symbols, and we will state some of them here.

[b]Misner, Thorne, Wheeler

The Christoffel symbols of first kind[4]

$$\Gamma_{abc} = \frac{1}{2}(\partial_c g_{ba} + \partial_b g_{ac} - \partial_a g_{bc})$$

The Christoffel symbols of second kind[5] [6]

$$\Gamma^a{}_{bc} = g^{ad}\Gamma_{dbc}$$

[c]D'Inverno

The Christoffel symbols of first kind[7]

$$\{ab,c\} = \frac{1}{2}(\partial_a g_{bc} + \partial_b g_{ca} - \partial_c g_{ab})$$

The Christoffel symbols of second kind[8]

$$\left\{ {a \atop bc} \right\} = g^{ad}\{bc,d\}$$

[1] Symmetric in the first two indices: $\Gamma_{abc} = \Gamma_{bac}$
[2] Symmetric in the last two indices: $\Gamma^a{}_{bc} = \Gamma^a{}_{cb}$
[3] Notice: The third subscript is raised
[4] Symmetric in the last two indices: $\Gamma_{abc} = \Gamma_{acb}$
[5] Symmetric in the last two indices: $\Gamma^a{}_{bc} = \Gamma^a{}_{cb}$
[6] Notice: The first subscript is raised
[7] Symmetric in ab
[8] Symmetric in bc

4.1.2 [d]Properties

$$
\begin{aligned}
\Gamma_{abc} &= \frac{1}{2}(\partial_a g_{bc} + \partial_b g_{ca} - \partial_c g_{ab}) = \frac{1}{2}(\partial_b g_{ac} + \partial_a g_{cb} - \partial_c g_{ba}) = \Gamma_{bac} \\
\Gamma_{aaa} &= \frac{1}{2}(\partial_a g_{aa} + \partial_a g_{aa} - \partial_a g_b) = \frac{1}{2}(\partial_a g_{aa}) \\
\Gamma_{aab} &= \frac{1}{2}(\partial_a g_{ab} + \partial_a g_{ba} - \partial_b g_{aa}) = \frac{1}{2}(2\partial_a g_{ab} - \partial_b g_{aa}) \\
\Gamma_{abb} &= \Gamma_{bab} = \frac{1}{2}(\partial_a g_{bb} + \partial_b g_{ba} - \partial_b g_{ab}) = \frac{1}{2}(\partial_a g_{bb}) \\
\Gamma_{abc} + \Gamma_{acb} &= \frac{1}{2}(\partial_a g_{bc} + \partial_b g_{ca} - \partial_c g_{ab}) + \frac{1}{2}(\partial_a g_{bc} + \partial_c g_{ba} - \partial_b g_{ac}) = \partial_a g_{bc} \\
\Gamma^a{}_{bc} &= \frac{1}{2}g^{ad}(\partial_c g_{db} + \partial_b g_{dc} - \partial_d g_{bc}) = \frac{1}{2}g^{ad}(\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{cb}) \\
&= \Gamma^a{}_{cb} \\
\Gamma^a{}_{aa} &= {}^{9}g^{ad}\Gamma_{daa} \\
\Gamma^a{}_{bb} &= g^{ad}\Gamma_{bbd} \\
\Gamma^a{}_{ab} &= \Gamma^a{}_{ba} = {}^{10}g^{ad}\Gamma_{abd}
\end{aligned}
$$

4.1.3 The Christoffel Symbols of a diagonal metric in Three Dimensions

The line element

$$ds^2 = g_{xx}dx^2 + g_{yy}dy^2 + g_{zz}dz^2$$

The metric tensor and its inverse

$$g_{ab} = \begin{Bmatrix} g_{xx} & & \\ & g_{yy} & \\ & & g_{zz} \end{Bmatrix}$$

$$g^{ab} = \begin{Bmatrix} \frac{1}{g_{xx}} & & \\ & \frac{1}{g_{yy}} & \\ & & \frac{1}{g_{zz}} \end{Bmatrix}$$

<u>The Christoffel symbols of first kind</u>

$$
\begin{aligned}
\Gamma_{abc} &= \frac{1}{2}(\partial_a g_{bc} + \partial_b g_{ca} - \partial_c g_{ab}) \\
\Rightarrow \Gamma_{xyz} &= \Gamma_{xzy} = \Gamma_{yzx} = 0 \\
\Gamma_{aaa} &= \frac{1}{2}\partial_a g_{aa} \\
\Rightarrow \Gamma_{xxx} &= \frac{1}{2}\partial_x g_{xx} \\
\Gamma_{yyy} &= \frac{1}{2}\partial_y g_{yy} \\
\Gamma_{zzz} &= \frac{1}{2}\partial_z g_{zz} \\
\Gamma_{aab} &= \frac{1}{2}(2\partial_a g_{ab} - \partial_b g_{aa}) = -\frac{1}{2}\partial_b g_{aa} \\
\Rightarrow \Gamma_{xxy} &= -\frac{1}{2}\partial_y g_{xx}
\end{aligned}
$$

<u>Christoffel symbols of the second kind</u>

$$
\begin{aligned}
\Gamma^a{}_{bc} &= g^{ad}\Gamma_{bcd} \\
\Rightarrow \Gamma^z{}_{xy} &= \Gamma^y{}_{xz} = \Gamma^x{}_{yz} = 0 \\
\Gamma^a{}_{aa} &= {}^{11}g^{ad}\Gamma_{aad} \\
\Rightarrow \Gamma^x{}_{xx} &= g^{xx}\Gamma_{xxx} \\
\Rightarrow \Gamma^y{}_{yy} &= g^{yy}\Gamma_{yyy} \\
\Rightarrow \Gamma^z{}_{zz} &= g^{zz}\Gamma_{zzz} \\
\Gamma^b{}_{aa} &= g^{bd}\Gamma_{aad} \\
\Rightarrow \Gamma^y{}_{xx} &= g^{yy}\Gamma_{xxy}
\end{aligned}
$$

[9] Only sum over d
[10] Only sum over d
[11] Only sum over d

$$
\begin{aligned}
\Gamma_{xxz} &= -\frac{1}{2}\partial_z g_{xx} &&\Rightarrow & \Gamma^z{}_{xx} &= g^{zz}\Gamma_{xxz}\\
\Gamma_{yyx} &= -\frac{1}{2}\partial_x g_{yy} &&\Rightarrow & \Gamma^x{}_{yy} &= g^{xx}\Gamma_{yyx}\\
\Gamma_{yyz} &= -\frac{1}{2}\partial_z g_{yy} &&\Rightarrow & \Gamma^z{}_{yy} &= g^{zz}\Gamma_{yyz}\\
\Gamma_{zzx} &= -\frac{1}{2}\partial_x g_{zz} &&\Rightarrow & \Gamma^x{}_{zz} &= g^{xx}\Gamma_{zzx}\\
\Gamma_{zzy} &= -\frac{1}{2}\partial_y g_{zz} &&\Rightarrow & \Gamma^y{}_{zz} &= g^{yy}\Gamma_{zzy}
\end{aligned}
$$

$$\Gamma_{abb} = \Gamma_{bab} = \frac{1}{2}\partial_a g_{bb} \qquad \Gamma^b{}_{ab} = \Gamma^b{}_{ba} = {}^{[12]}g^{bd}\Gamma_{abd}$$

$$
\begin{aligned}
\Rightarrow \Gamma_{xyy} &= \Gamma_{yxy} = \frac{1}{2}\partial_x g_{yy} &&\Rightarrow & \Gamma^y{}_{xy} &= \Gamma^y{}_{yx} = g^{yy}\Gamma_{xyy}\\
\Gamma_{xzz} &= \Gamma_{zxz} = \frac{1}{2}\partial_x g_{zz} &&\Rightarrow & \Gamma^z{}_{xz} &= \Gamma^z{}_{zx} = g^{zz}\Gamma_{xzz}\\
\Gamma_{yxx} &= \Gamma_{xyx} = \frac{1}{2}\partial_y g_{xx} &&\Rightarrow & \Gamma^x{}_{yx} &= \Gamma^x{}_{xy} = g^{xx}\Gamma_{yxx}\\
\Gamma_{yzz} &= \Gamma_{zyz} = \frac{1}{2}\partial_y g_{zz} &&\Rightarrow & \Gamma^z{}_{yz} &= \Gamma^z{}_{zy} = g^{zz}\Gamma_{yzz}\\
\Gamma_{zxx} &= \Gamma_{xzx} = \frac{1}{2}\partial_z g_{xx} &&\Rightarrow & \Gamma^x{}_{zx} &= \Gamma^x{}_{xz} = g^{xx}\Gamma_{zxx}\\
\Gamma_{zyy} &= \Gamma_{yzy} = \frac{1}{2}\partial_z g_{yy} &&\Rightarrow & \Gamma^y{}_{zy} &= \Gamma^y{}_{yz} = g^{yy}\Gamma_{zyy}
\end{aligned}
$$

4.2 [e]Cylindrical coordinates.

The line element:

$$ds^2 = dr^2 + r^2 d\phi^2 + dz^2$$

The metric tensor and its inverse:

$$g_{ab} = \begin{Bmatrix} 1 & & \\ & r^2 & \\ & & 1 \end{Bmatrix}$$

$$g^{ab} = \begin{Bmatrix} 1 & & \\ & \frac{1}{r^2} & \\ & & 1 \end{Bmatrix}$$

4.2.1 The non-zero Christoffel symbols

<u>The Christoffel symbols of first kind</u> — <u>Christoffel symbols of the second kind</u>

$$
\begin{aligned}
\Gamma_{abc} &= \frac{1}{2}(\partial_a g_{bc} + \partial_b g_{ca} - \partial_c g_{ab}) && & \Gamma^a{}_{bc} &= g^{ad}\Gamma_{bcd}\\
\Gamma_{\phi\phi r} &= -\frac{1}{2}\partial_r g_{\phi\phi} = -\frac{1}{2}\partial_r(r^2) = -r &&\Rightarrow & \Gamma^r{}_{\phi\phi} &= g^{rr}\Gamma_{\phi\phi r} = -r\\
\Gamma_{r\phi\phi} &= \Gamma_{r\phi\phi} = \frac{1}{2}\partial_r g_{\phi\phi} = \frac{1}{2}\partial_r(r^2) = r &&\Rightarrow & \Gamma^\phi{}_{r\phi} &= \Gamma^\phi{}_{\phi r} = g^{\phi\phi}\Gamma_{r\phi\phi} = \frac{1}{r}
\end{aligned}
$$

4.2.2 The geodesic equation for cylindrical coordinates

[13]<u>The geodesics equation:</u>

$$\frac{d^2x^a}{ds^2} + \Gamma^a{}_{bc}\frac{dx^b}{ds}\frac{dx^c}{ds} = 0$$

---

[12] Only sum over d

[13] In this case we know the Christoffel symbols and want to find the geodesic equations

$\underline{x^a = r:}$ $$\frac{d^2r}{ds^2} + \Gamma^r{}_{bc}\frac{dx^b}{ds}\frac{dx^c}{ds} = 0$$

$\Rightarrow$ $$\frac{d^2r}{ds^2} - r\left(\frac{d\phi}{ds}\right)^2 = 0$$

$\underline{x^a = \phi:}$ $$\frac{d^2\phi}{ds^2} + \Gamma^\phi{}_{bc}\frac{dx^b}{ds}\frac{dx^c}{ds} = 0$$

$\Rightarrow$ $$\frac{d^2\phi}{ds^2} + \Gamma^\phi{}_{r\phi}\frac{dr}{ds}\frac{d\phi}{ds} + \Gamma^\phi{}_{\phi r}\frac{d\phi}{ds}\frac{dr}{ds} = 0$$

$\Rightarrow$ $$\frac{d^2\phi}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\phi}{ds} = 0$$

$\underline{x^a = z:}$ $$\frac{d^2z}{ds^2} + \Gamma^z{}_{bc}\frac{dx^b}{ds}\frac{dx^c}{ds} = 0$$

$\Rightarrow$ $$\frac{d^2z}{ds^2} = 0$$

4.2.3 [14]The Christoffel symbols from the geodesic equations

We have

$$K = \frac{1}{2}g_{ab}\dot{x}^a\dot{x}^b = \frac{1}{2}(\dot{r})^2 + \frac{1}{2}r^2(\dot{\phi})^2 + \frac{1}{2}(\dot{z})^2$$

Now we need

$$\frac{\partial K}{\partial x^a} = \frac{d}{ds}\left(\frac{\partial K}{\partial \dot{x}^a}\right)$$

$\underline{x^a = r:}$ $$\frac{\partial K}{\partial r} = \frac{d}{ds}\left(\frac{\partial K}{\partial \dot{r}}\right)$$

$\Rightarrow$ $$r\dot{\phi}^2 = \frac{d}{ds}(\dot{r}) = \ddot{r}$$

$\Rightarrow$ $$0 = \ddot{r} - r\dot{\phi}^2$$

$\underline{x^a = \phi:}$ $$\frac{\partial K}{\partial \phi} = \frac{d}{ds}\left(\frac{\partial K}{\partial \dot{\phi}}\right)$$

$$0 = \frac{d}{ds}(r^2\dot{\phi}) = 2r\dot{r}\dot{\phi} + r^2\ddot{\phi}$$

$\Rightarrow$ $$0 = \ddot{\phi} + \frac{1}{r}\dot{r}\dot{\phi} + \frac{1}{r}\dot{\phi}\dot{r}$$

$\underline{x^a = z:}$ $$\frac{\partial K}{\partial z} = \frac{d}{ds}\left(\frac{\partial K}{\partial \dot{z}}\right)$$

$\Rightarrow$ $$0 = \frac{d}{ds}(\dot{z}) = \ddot{z}$$

Collecting the results

$$0 = \ddot{r} - r\dot{\phi}^2$$
$$0 = \ddot{\phi} + \frac{1}{r}\dot{r}\dot{\phi} + \frac{1}{r}\dot{\phi}\dot{r}$$
$$0 = \ddot{z}$$

We can now find the Christoffel symbols from the geodesic equation:

$$\Gamma^r{}_{\phi\phi} = -r$$
$$\Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} = \frac{1}{r}$$

[14] In this case we know the geodesic equations and want to find the Christoffel symbols

4.3 The plane in Cartesian coordinates

4.3.1 [f]Solve the geodesics equations of the plane in Cartesian coordinates.

We use the Euler-Lagrange method.

$$0 = \frac{d}{dS}\left(\frac{\partial F}{\partial \dot{x}^a}\right) - \frac{\partial F}{\partial x^a}$$
$$F = \frac{1}{2} g_{ab}\dot{x}^a \dot{x}^b$$

The line element

$$dS^2 = dx^2 + dy^2$$
$$F = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2$$

<u>$x^a = x$:</u>
$$\frac{\partial F}{\partial x} = 0$$
$$\frac{\partial F}{\partial \dot{x}} = \dot{x}$$
$$\Rightarrow \quad \ddot{x} = 0$$

<u>$x^a = y$:</u>
$$\frac{\partial F}{\partial y} = 0$$
$$\frac{\partial F}{\partial \dot{y}} = \dot{y}$$
$$\Rightarrow \quad \ddot{y} = 0$$

Collecting the results

$$\ddot{x} = 0$$
$$\ddot{y} = 0$$

The solution is obviously a straight line:

$$\ddot{x} = \frac{d^2x}{dS^2} = 0$$
$$\Rightarrow \quad \dot{x} = \frac{dx}{dS} = k_0$$
$$\Rightarrow \quad x = k_0 S + k_1$$
$$\ddot{y} = \frac{d^2y}{dS^2} = 0$$
$$\Rightarrow \quad \dot{y} = \frac{dy}{dS} = c_0$$
$$\Rightarrow \quad y = c_0 S + c_1 = c_0 \frac{(x - k_1)}{k_0} + c_1 = \frac{c_0}{k_0} x + c_1 - c_0 \frac{k_1}{k_0} = K_0 x + K_1$$

4.4 The Hyperbolic Plane

4.4.1 [g]Geodesics in the Hyperbolic Plane

<u>The line element:</u>

$$dS^2 = y^{-2}(dx^2 + dy^2) \qquad y \geq 0$$

To find the geodesics we need a few integrals which we can solve:

<u>a. Killing vector:</u>

Because the metric is independent of x a Killing vector is

$$\boldsymbol{\xi} = (\xi^x, \xi^y) = (1,0)$$

$\boldsymbol{\xi} \cdot \boldsymbol{u}$ is a conserved quantity along a geodesic, where

$$\boldsymbol{u} = (u^x, u^y) = \left(\frac{dx}{dS}, \frac{dy}{dS}\right)$$
$$\Rightarrow \quad \boldsymbol{\xi} \cdot \boldsymbol{u} = \xi_a u^a = g_{ab}\xi^b u^a = g_{ax}\xi^x u^a = g_{xx}\xi^x u^x = y^{-2} u^x = K_1$$

$$\Rightarrow \quad \frac{dx}{dS} = y^2 K_1 \tag{4.1.}$$

b. The line-element

$$dS^2 = y^{-2}(dx^2 + dy^2)$$

$$\Rightarrow \quad 1 = y^{-2}\left(\left(\frac{dx}{dS}\right)^2 + \left(\frac{dy}{dS}\right)^2\right) \tag{4.2.}$$

Substituting eq. (4.1.) into eq. (4.2.)

$$\Rightarrow \quad 1 = y^{-2}\left((y^2K_1)^2 + \left(\frac{dy}{dS}\right)^2\right)$$

$$\Rightarrow \quad \left(\frac{dy}{dS}\right)^2 = y^2K_1^2\left(\frac{1}{K_1^2} - y^2\right)$$

$$\Rightarrow \quad \frac{dy}{dS} = \pm yK_1\sqrt{\frac{1}{K_1^2} - y^2} \tag{4.3.}$$

Combining eq. (4.1.) and eq. (4.3.)

$$\Rightarrow \quad \frac{dx}{dy} = \pm\frac{y}{\sqrt{\frac{1}{K_1^2} - y^2}}$$

$$\Rightarrow \quad dx = \pm\frac{ydy}{\sqrt{\frac{1}{K_1^2} - y^2}}$$

$$\Rightarrow \quad x - x_0 = \pm\int\frac{ydy}{\sqrt{\frac{1}{K_1^2} - y^2}} =^{15} \pm\sqrt{\frac{1}{K_1^2} - y^2}$$

$$\Rightarrow \quad (x - x_0)^2 + y^2 = \frac{1}{K_1^2} \qquad y \geq 0$$

If $y = 0$ the geodesics are the vertical lines $x = x_0 \pm \frac{1}{K_1}$. If $y > 0$ the geodesics are semicircles centered on the x-axis in $(x_0, 0)$ with radius $\frac{1}{K_1}$.

<u>x and y as a function of S:</u>

$$\frac{dy}{dS} = \pm yK_1\sqrt{\frac{1}{K_1^2} - y^2}$$

$$dS = \pm\frac{dy}{yK_1\sqrt{\frac{1}{K_1^2} - y^2}}$$

[15] (Spiegel, 1990) (14.238) $\int\frac{rdr}{\sqrt{a^2-r^2}} = -\sqrt{a^2 - r^2}$

$$\Rightarrow \quad S - S_0 = \pm \int \frac{dy}{yK_1\sqrt{\frac{1}{K_1^2} - y^2}} =^{16} \pm \ln\left(\frac{\frac{1}{K_1} + \sqrt{\frac{1}{K_1^2} - y^2}}{y}\right)$$

$$\Rightarrow \quad \exp(\pm(S - S_0)) = \frac{\frac{1}{K_1} + \sqrt{\frac{1}{K_1^2} - y^2}}{y}$$

$$\Rightarrow \quad 0 = \left(y\exp(\pm(S - S_0)) - \frac{1}{K_1}\right)^2 + y^2 - \frac{1}{K_1^2}$$

$$= y^2\exp(\pm 2(S - S_0)) - \frac{2y}{K_1}\exp(\pm(S - S_0)) + y^2$$

$$= y\left(y\left(1 + \exp(\pm 2(S - S_0))\right) - \frac{2}{K_1}\exp(\pm(S - S_0))\right)$$

$$\Rightarrow \quad y_1 = 0$$

$$y_2 = \frac{2\exp(S - S_0)}{K_1\left(1 + \exp(2(S - S_0))\right)} =^{17} \frac{1}{K_1\cosh(S - S_0)}$$

$$y_3 = \frac{2\exp(-(S - S_0))}{K_1\left(1 + \exp(-2(S - S_0))\right)} =^{18} \frac{1}{K_1\cosh(S - S_0)} = y_2$$

$$\frac{dx}{dS} = y^2 K_1 = \left(\frac{1}{K_1\cosh(S - S_0)}\right)^2 K_1 = \frac{1}{K_1\cosh^2(S - S_0)}$$

$$\Rightarrow \quad x - x_0 = \frac{1}{K_1}\int \frac{1}{\cosh^2(S - S_0)} dS =^{19} \frac{1}{K_1}\tanh(S - S_0) = \frac{1}{K_1}\tanh(S - S_0)$$

Rescaling and collecting the results[20]

$$y = \frac{1}{K_1\cosh(S)}$$

$$x = \frac{1}{K_1}\tanh(S) = y\sinh(S)$$

4.5 The Geodesic of two-dimensional Minkowski space-time

The line element:

$$ds^2 = -dt^2 + dx^2$$

$$\Rightarrow \quad d\tau^2 =^{21} dt^2 - ds^2$$

---

[16] (Spiegel, 1990) (14.241) $\int \frac{dy}{y\sqrt{a^2-y^2}} = -\frac{1}{a}\ln\left(\frac{a+\sqrt{a^2-y^2}}{y}\right)$

[17] $\frac{2e^x}{1+e^{2x}} = \frac{2e^x}{e^x(e^{-x}+e^x)} = \frac{2}{e^{-x}+e^x} = \frac{1}{\cosh x}$

[18] $\frac{2e^{-x}}{1+e^{-2x}} = \frac{2e^{-x}}{e^{-x}(e^x+e^{-x})} = \frac{2}{e^{-x}+e^x} = \frac{1}{\cosh x}$

[19] (Spiegel, 1990) (14.571) $\int \frac{1}{\cosh^2(x)} dx = \tanh x$

[20] Checking: $1 = \frac{1}{y^2}\left(\left(\frac{dx}{dS}\right)^2 + \left(\frac{dy}{dS}\right)^2\right) = K_1^2\cosh^2(S)\left(\left(\frac{d\left(\frac{1}{K_1}\tanh(S)\right)}{dS}\right)^2 + \left(\frac{d\left(\frac{1}{K_1\cosh(S)}\right)}{dS}\right)^2\right) = \cosh^2 S\left(\left(\frac{1}{\cosh^2(S)}\right)^2 + \left(-\frac{1}{\cosh(S)}\tanh(S)\right)^2\right) = \frac{1}{\cosh^2(S)} + \tanh^2(S) = 1$

[21] Negative time-signature i.e. $d\tau^2 = -ds^2$. See chapter 2

$$\Rightarrow \qquad 1 = \left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dx}{d\tau}\right)^2$$

Killing vector: Because the metric is independent of t a Killing vector is

$$\boldsymbol{\xi} = (\xi^t, \xi^x) = (1,0)$$

$\boldsymbol{\xi} \cdot \boldsymbol{u}$ is a conserved quantity along a geodesic, where

$$\boldsymbol{u} = (u^t, u^x) = \left(\frac{dt}{d\tau}, \frac{dx}{d\tau}\right)$$

$$\Rightarrow \quad \boldsymbol{\xi} \cdot \boldsymbol{u} = \xi_a u^a = g_{ab}\xi^b u^a = g_{tt}\xi^t u^t + g_{xx}\xi^x u^x = -\frac{dt}{d\tau}$$

$$\Rightarrow \quad \frac{dt}{d\tau} = constant = K$$

Substituting this into the line element we find

$$1 = K^2 - \left(\frac{dx}{d\tau}\right)^2 = K^2 - \left(\frac{dt}{d\tau}\frac{dx}{dt}\right)^2 = K^2 - K^2\left(\frac{dx}{dt}\right)^2$$

$$\Rightarrow \quad \frac{dx}{dt} = \pm\sqrt{\frac{K^2-1}{K^2}} = \pm K'$$

When we solve this we find the familiar geodesics

$$x(t) = \pm K' t$$

Or

$$t(x) = \pm \frac{1}{K'} x$$

We can check whether these are timelike or spacelike and find the expected results

$$ds^2 = -dt^2 + dx^2 = \left(-1 + \left(\frac{dx}{dt}\right)^2\right) dt^2 = (-1 + (K')^2) dt^2$$

$$\begin{cases} < 0\ timelike, if\ K' < 1\ inside\ the\ lightcone \\ = 0\ lightspeed, if\ K' = 1\ on\ the\ lightcone \\ > 0\ spacelike, if\ K' > 1\ outside\ the\ lightcone \end{cases}$$

4.6 The Flat Space-time in two dimensions – [22]Rindler Space-time

4.6.1 [h]The time-like geodesic $\boldsymbol{X(T)}$ of the Flat Space-time metric in two dimensions

To find $X(T)$ we need a few integrals which we can solve:

a.The line element:

$$dS^2 = -X^2 dT^2 + dX^2$$

$$\Rightarrow \quad d\tau^2 = {}^{23}\ X^2 dT^2 - dX^2$$

$$\Rightarrow \quad 1 = X^2 \left(\frac{dT}{d\tau}\right)^2 - \left(\frac{dX}{d\tau}\right)^2 \qquad (4.4.)$$

b. Killing vectors: Because the metric is independent of T a Killing vector is

$$\boldsymbol{\xi} = (\xi^T, \xi^X) = (1,0)$$

$\boldsymbol{\xi} \cdot \boldsymbol{u}$ is a conserved quantity along a geodesic, where

$$\boldsymbol{u} = (u^T, u^X) = \left(\frac{dT}{d\tau}, \frac{dX}{d\tau}\right)$$

$$\Rightarrow \quad \boldsymbol{\xi} \cdot \boldsymbol{u} = \xi_a u^a = g_{ab}\xi^b u^a = g_{TT}\xi^T u^T + g_{XX}\xi^X u^X$$

$$= -X^2 \cdot 1 \cdot \frac{dT}{d\tau} + 1 \cdot 0 \cdot \frac{dX}{d\tau} = -X^2 \frac{dT}{d\tau}$$

[22] See chapter 2

[23] Negative time-signature i.e. $d\tau^2 = -dS^2$. See chapter 2

$$\Rightarrow \quad X^2 \frac{dT}{d\tau} = constant = K \tag{4.5.}$$

Substituting eq. (4.5.) into eq. (4.4.)

$$1 = X^2 \left(\frac{K}{X^2}\right)^2 - \left(\frac{dX}{d\tau}\right)^2 = \frac{K^2}{X^2} - \left(\frac{dX}{d\tau}\right)^2$$

$$\Rightarrow \quad \frac{dX}{d\tau} =^{24} \pm \sqrt{\frac{K^2}{X^2} - 1} \tag{4.6.}$$

<u>If $\frac{dX}{d\tau} > 0$:</u>

Dividing eq. (4.6.) by eq. (4.5.)

$$\Rightarrow \quad \frac{\frac{dX}{d\tau}}{X^2 \frac{dT}{d\tau}} = \frac{\sqrt{\frac{K^2}{X^2} - 1}}{K}$$

$$\Rightarrow \quad \frac{dX}{dT} = \frac{X^2}{K}\sqrt{\frac{K^2}{X^2} - 1} = \frac{X}{K}\sqrt{K^2 - X^2}$$

$$\Rightarrow \quad dT = \frac{K}{X\sqrt{K^2 - X^2}} dX$$

$$\Rightarrow \quad T - T^* = K \int \frac{dX}{X\sqrt{K^2 - X^2}} =^{25\ 26} -\ln\left(\frac{K + \sqrt{K^2 - X^2}}{X}\right)$$

Isolating X

$$\left(\frac{K + \sqrt{K^2 - X^2}}{X}\right) = \exp\big(-(T - T^*)\big)$$

$$\Rightarrow \quad \sqrt{\left(\frac{K}{X}\right)^2 - 1} = \exp\big(-(T - T^*)\big) - \frac{K}{X}$$

$$\Rightarrow \quad \left(\frac{K}{X}\right)^2 - 1 = \left(\exp\big(-(T - T^*)\big) - \frac{K}{X}\right)^2$$

$$= \exp\big(-2(T - T^*)\big) + \left(\frac{K}{X}\right)^2 - \frac{2K}{X}\exp\big(-(T - T^*)\big)$$

$$\Rightarrow \quad \frac{2K}{X} = \frac{\exp\big(-2(T - T^*)\big) + 1}{\exp\big(-(T - T^*)\big)} = \exp\big(-(T - T^*)\big) + \exp(T - T^*)$$

$$= 2\cosh(T - T^*)$$

And we find the geodesics

$$X(T) =^{27} \frac{K}{\cosh(T - T^*)}$$

4.6.2 Are these geodesics space-like or time-like

$$dS^2 = -X^2 dT^2 + dX^2 = \left(-X^2 + \left(\frac{dX}{dT}\right)^2\right) dT^2 = \left(-X^2 + \left(\frac{X}{K}\sqrt{K^2 - X^2}\right)^2\right) dT^2$$

---

[24] $K^2 > X^2$

[25] $\int \frac{dx}{x\sqrt{a^2 - x^2}} = -\frac{1}{a}\ln\left(\frac{a + \sqrt{a^2 - x^2}}{x}\right)$ (Spiegel, 1990) (14.241)

[26] $\left(\frac{K + \sqrt{K^2 - X^2}}{X}\right) > 0 \ if \ K > X$

[27] Notice: If $\frac{dX}{d\tau} < 0$: $X(T) = \frac{K}{\cosh(T^* - T)} = \frac{K}{\cosh(T - T^*)}$

$$= -\frac{X^4}{K^2}dT^2 = -\frac{K^2}{\cosh^4(T-T^*)}dT^2 < 0$$

Which is inside the light-cone and these geodesics are time-like

4.6.3 [i]Is the world-line $X(T) = A\cosh(T)$ time-like or space-like:

$$dS^2 = -X^2dT^2 + dX^2 = \left(-X^2 + \left(\frac{dX}{dT}\right)^2\right)dT^2 = \left(-X^2 + \left(\frac{d(A\cosh T)}{dT}\right)^2\right)dT^2$$
$$= A^2(-\cosh^2 T + \sinh^2 T)dT^2 = -A^2dT^2 < 0$$

Which is inside the light-cone and the world-line is time-like.

4.7 Three-dimensional flat space-time.

4.7.1 [j]Null geodesics in three-dimensional flat space-time.

The line element:

$$ds^2 = -dt^2 + dr^2 + r^2d\phi^2$$
$$d\tau^2 = {}^{28}dt^2 - dr^2 - r^2d\phi^2$$

To find the null-geodesics we need a few integrals which we can solve:

a. Killing vectors:

Because the metric is independent of t a Killing vector is

$$\boldsymbol{\xi} = (\xi^t, \xi^r, \xi^\phi) = (1,0,0)$$

$\boldsymbol{\xi}\cdot\boldsymbol{u}$ is a conserved quantity along a geodesic, where

$$\boldsymbol{u} = (u^t, u^r, u^\phi) = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, \frac{d\phi}{d\tau}\right)$$
$$\Rightarrow \quad \boldsymbol{\xi}\cdot\boldsymbol{u} = \xi_a u^a = g_{ab}\xi^b u^a = g_{at}\xi^t u^a = g_{tt}\xi^t u^t + g_{rt}\xi^t u^r + g_{\phi t}\xi^t u^\phi$$
$$= -u^t$$
$$\Rightarrow \quad \frac{dt}{d\tau} = K_1 \qquad (4.7.)$$

Because the metric is independent of ϕ a Killing vector is

$$\boldsymbol{\zeta} = (\zeta^t, \zeta^r, \zeta^\phi) = (0,0,1)$$
$$\Rightarrow \quad \boldsymbol{\zeta}\cdot\boldsymbol{u} = \zeta_a u^a = g_{ab}\zeta^b u^a = g_{a\phi}\zeta^\phi u^a$$
$$= g_{t\phi}\zeta^\phi u^t + g_{r\phi}\zeta^\phi u^r + g_{\phi\phi}\zeta^\phi u^\phi = r^2u^\phi$$
$$\Rightarrow \quad r^2\frac{d\phi}{d\tau} = K_2 \qquad (4.8.)$$

b. Conservation of the four-velocity for a light ray

$$\boldsymbol{u}\cdot\boldsymbol{u} = u_a u^a = g_{ab}u^b u^a = g_{tt}u^t u^t + g_{rr}u^r u^r + g_{\phi\phi}u^\phi u^\phi$$
$$= \left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dr}{d\tau}\right)^2 - r^2\left(\frac{d\phi}{d\tau}\right)^2 = 0 \qquad (4.9.)$$

Substitute eq. (4.7.) and eq. (4.8.) into eq. (4.9.)

$$\Rightarrow \quad 0 = (K_1)^2 - \left(\frac{dr}{d\tau}\right)^2 - r^2\left(\frac{K_2}{r^2}\right)^2$$
$$\Rightarrow \quad \left(\frac{dr}{d\tau}\right)^2 = (K_1)^2 - \frac{1}{r^2}(K_2)^2$$
$$\Rightarrow \quad \frac{dr}{d\tau} = \sqrt{(K_1)^2 - \frac{1}{r^2}(K_2)^2} \qquad (4.10.)$$

Combining eq. (4.7.) and eq. (4.10.)

[28] Negative time-signature i.e. $d\tau^2 = -ds^2$. See chapter 2

$$\Rightarrow \quad \frac{dr}{dt} = \frac{1}{K_1}\sqrt{(K_1)^2 - \frac{1}{r^2}(K_2)^2} = \sqrt{1 - \frac{1}{r^2}\left(\frac{K_2}{K_1}\right)^2}$$

$$\Rightarrow \quad dt = \frac{dr}{\sqrt{1 - \frac{1}{r^2}\left(\frac{K_2}{K_1}\right)^2}} = \frac{rdr}{\sqrt{r^2 - \left(\frac{K_2}{K_1}\right)^2}}$$

$$\Rightarrow \quad t - t_0 = \int \frac{rdr}{\sqrt{r^2 - \left(\frac{K_2}{K_1}\right)^2}} =^{29} \sqrt{r^2 - \left(\frac{K_2}{K_1}\right)^2} \tag{4.11.}$$

Notice we can rewrite this into a hyperboloid.

$$\left(\frac{K_2}{K_1}\right)^2 = r^2 - (t - t_0)^2 \tag{4.12.}$$

Combining eq. (4.7.), eq. (4.8.) and eq. (4.11.)

$$\Rightarrow \quad r^2 \frac{d\phi}{dt} = \frac{K_2}{K_1}$$

$$\Rightarrow \quad \frac{d\phi}{dt} = \frac{K_2}{K_1} \frac{1}{(t - t_0)^2 + \left(\frac{K_2}{K_1}\right)^2}$$

$$\Rightarrow \quad \phi - \phi_0 = \frac{K_2}{K_1} \int \frac{dt}{(t - t_0)^2 + \left(\frac{K_2}{K_1}\right)^2} =^{30} \tan^{-1}\left[\frac{K_1}{K_2}(t - t_0)\right]$$

$$\Rightarrow \quad t - t_0 = \frac{K_2}{K_1}\tan(\phi - \phi_0) \tag{4.13.}$$

Combining eq. (4.11.) and eq. (4.13.)

$$\Rightarrow \quad \sqrt{r^2 - \left(\frac{K_2}{K_1}\right)^2} = \frac{K_2}{K_1}\tan(\phi - \phi_0)$$

$$\Rightarrow \quad r = \pm \frac{K_2}{K_1}\sqrt{\tan^2(\phi - \phi_0) + 1} \tag{4.14.}$$

Collecting the results we find the null-geodesics:

$$\left(\frac{K_2}{K_1}\right)^2 = r^2 - (t - t_0)^2 \tag{4.12.}$$

$$t - t_0 = \frac{K_2}{K_1}\tan(\phi - \phi_0) \tag{4.13.}$$

$$r = \pm \frac{K_2}{K_1}\sqrt{\tan^2(\phi - \phi_0) + 1} \tag{4.14.}$$

Light rays moves on straight lines in curved space. From our point of view the tip of the light cone (t, r) moves along a hyperbolic path eq. (4.14.).

[29] (Spiegel, 1990) (14.210) $\int \frac{rdr}{\sqrt{r^2 - a^2}} = \sqrt{r^2 - a^2}$

[30] (Spiegel, 1990) (14.125) $\int \frac{dt}{t^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{t}{a}$

4.8 The worm-hole geometry

4.8.1 [k]Volume in the Wormhole geometry

The three-dimensional volume on a $t = constant$ slice of the wormhole geometry bounded by two spheres of coordinate radius R on each side of the throat.

The line-element

$$ds^2 = -dt^2 + dr^2 + (b^2 + r^2)(d\theta^2 + \sin^2\theta\, d\phi^2)$$
$$d\tau^2 = {}^{31}dt^2 - dr^2 - (b^2 + r^2)(d\theta^2 + \sin^2\theta\, d\phi^2)$$

The volume

$$V = \int dl^1 \int dl^2 \int dl^3 = \int_{-R}^{R} (b^2 + r^2)dr \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi = 4\pi \int_{-R}^{R} (b^2 + r^2)dr$$
$$= 4\pi \left[rb^2 + \frac{1}{3}r^3 \right]_{-R}^{R} = 4\pi \left(\left(Rb^2 + \frac{1}{3}R^3 \right) - \left((-R)b^2 + \frac{1}{3}(-R)^3 \right) \right)$$
$$= \frac{4\pi}{3} * 2R * (3b^2 + R^2)$$

4.8.2 [l]Geodesic Equations in a Wormhole Geometry

We use the Euler-Lagrange method.

$$0 = \frac{d}{ds}\left(\frac{\partial F}{\partial \dot{x}^a} \right) - \frac{\partial F}{\partial x^a}$$
$$F = \frac{1}{2} g_{ab} \dot{x}^a \dot{x}^b$$

The line element

$$ds^2 = -dt^2 + dr^2 + (b^2 + r^2)(d\theta^2 + \sin^2\theta\, d\phi^2)$$
$$\Rightarrow \quad F = -\frac{1}{2}\dot{t}^2 + \frac{1}{2}\dot{r}^2 + \frac{1}{2}(b^2 + r^2)\dot{\theta}^2 + \frac{1}{2}(b^2 + r^2)\sin^2\theta\, \dot{\phi}^2$$

<u>$x^a = t$:</u>

$$\frac{\partial F}{\partial t} = 0$$
$$\frac{\partial F}{\partial \dot{t}} = -\dot{t}$$
$$\Rightarrow \quad \ddot{t} = 0$$

<u>$x^a = r$:</u>

$$\frac{\partial F}{\partial r} = r\left(\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2 \right)$$
$$\frac{\partial F}{\partial \dot{r}} = \dot{r}$$
$$\Rightarrow \quad \ddot{r} = r\dot{\theta}^2 + r\sin^2\theta\, \dot{\phi}^2$$

<u>$x^a = \theta$:</u>

$$\frac{\partial F}{\partial \theta} = (b^2 + r^2)\sin\theta\cos\theta\, \dot{\phi}^2$$
$$\frac{\partial F}{\partial \dot{\theta}} = (b^2 + r^2)\dot{\theta}$$
$$\frac{d}{ds}\left(\frac{\partial F}{\partial \dot{\theta}} \right) = 2r\dot{r}\dot{\theta} + (b^2 + r^2)\ddot{\theta}$$
$$\Rightarrow \quad \ddot{\theta} = \sin\theta\cos\theta\, \dot{\phi}^2 - \frac{2r}{(b^2 + r^2)}\dot{r}\dot{\theta}$$

<u>$x^a = \phi$:</u>

$$\frac{\partial F}{\partial \phi} = 0$$
$$\frac{\partial F}{\partial \dot{\phi}} = (b^2 + r^2)\sin^2\theta\, \dot{\phi}$$

---

[31] Negative time-signature i.e. $d\tau^2 = -ds^2$. See chapter 2

$$\Rightarrow \quad 0 = \frac{d}{ds}\left((b^2+r^2)\sin^2\theta\,\dot\phi\right)$$
$$= 2r\sin^2\theta\,\dot r\dot\phi + 2(b^2+r^2)\sin\theta\cos\theta\,\dot\theta\dot\phi + (b^2+r^2)\sin^2\theta\,\ddot\phi$$
$$\Rightarrow \quad \ddot\phi = -\frac{2r}{(b^2+r^2)}\dot r\dot\phi - 2\cot\theta\,\dot\theta\dot\phi$$

Collecting the results

$$\ddot t = 0$$
$$\ddot r = r\dot\theta^2 + r\sin^2\theta\,\dot\phi^2$$
$$\ddot\theta = \sin\theta\cos\theta\,\dot\phi^2 - \frac{2r}{(b^2+r^2)}\dot r\dot\theta$$
$$\ddot\phi = -\frac{2r}{(b^2+r^2)}\dot r\dot\phi - 2\cot\theta\,\dot\theta\dot\phi$$

[m]The non-zero Christoffel symbols are

$$\Gamma^r{}_{\theta\theta} = -r \qquad \Gamma^r{}_{\phi\phi} = -r\sin^2\theta$$
$$\Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta \qquad \Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = \frac{r}{(b^2+r^2)}$$
$$\Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} = \frac{r}{(b^2+r^2)} \qquad \Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta} = \cot\theta$$

4.8.3 [n]The travel time through a wormhole

Use the geodesic equations to calculate the proper travel time of an astronaut travelling through a worm-hole throat along the coordinate radius r from $r = R$ to $r = -R$. The initial radial four-velocity is $u^r \equiv U$, and because of spherically symmetry $u^\theta = u^\phi = 0$

The four-velocity is

$$u^a = (u^t, u^r, u^\theta, u^\phi) = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, \frac{d\theta}{d\tau}, \frac{d\phi}{d\tau}\right) = {}^{32}\left(\sqrt{1+U^2}, U, 0, 0\right)$$

we will only look at u^r. We use the geodesic equation

$$\ddot r = r\dot\theta^2 + r\sin^2\theta\,\dot\phi^2$$

which we can rewrite

$$\frac{d^2r}{d\tau^2} = r\dot\theta^2 + r\sin^2\theta\,\dot\phi^2 = 0$$
$$\Rightarrow \quad \frac{d^2r}{d\tau^2} = \frac{d}{d\tau}\left(\frac{dr}{d\tau}\right) = \frac{du^r}{d\tau} = 0$$

which implies that $u^r = U$ is a constant along the astronauts world-line. So we can solve

$$u^r = \frac{dr}{d\tau} = U$$
$$\Rightarrow \quad d\tau = \frac{1}{U}dr$$
$$\Rightarrow \quad \Delta\tau = \int_{-R}^{R}\frac{1}{U}dr = \frac{1}{U}[r]_{-R}^{R} = \frac{1}{U}\left(R-(-R)\right) = \frac{2R}{U}$$

So the travel time through the wormhole $\Delta\tau = \frac{2R}{U}$ is very much alike the usual time/speed calculation: distance=time*velocity except with the velocity replaced by the four velocity.

4.8.3.1 *Is the trajectory time-like or space-like?*

The line-element[33]

[32] u^t is found from the fact that the $u^a u_a = -1$ is a conserved quantity: $u^a u_a = u^a \eta_{ij} u^a = -(u^t)^2 + (u^r)^2 + (u^\theta)^2 + (u^\phi)^2 = -(u^t)^2 + U^2 = -1 \Rightarrow u^t = \sqrt{1+U^2}$, where $1 + U^2 > 0$

[33] If we instead used the line-element $ds^2 = -dt^2 + dr^2 + (b^2+r^2)(d\theta^2 + \sin^2\theta\,d\phi^2)$ we would find $dt^2\left(\frac{1}{1+U^2}\right) < 0$, which is time-like.

$$d\tau^2 = dt^2 - dr^2 - (b^2 + r^2)(d\theta^2 + \sin^2\theta\, d\phi^2)$$
$$= dt^2(1 - \left(\frac{dr}{dt}\right)^2 - (b^2 + r^2)\left(\left(\frac{d\theta}{dt}\right)^2 + \sin^2\theta\left(\frac{d\phi}{dt}\right)^2\right)$$
$$= {}^{34}dt^2\left(1 - \left(\frac{dr}{d\tau}\frac{d\tau}{dt}\right)^2 - (b^2 + r^2)((0)^2 + \sin^2\theta\,(0)^2)\right) = dt^2\left(1 - \left(U \cdot \frac{1}{\sqrt{1+U^2}}\right)^2\right)$$
$$= dt^2\left(\frac{1 + U^2 - U^2}{1 + U^2}\right) = dt^2\left(\frac{1}{1+U^2}\right) > 0$$

i.e. the trajectory is time-like.

4.9 °Warp space-time – The Alcubierre Drive

The line-element

$$ds^2 = -dt^2 + [dx - V_s(t)f(r_s)dt]^2 + dy^2 + dz^2$$
$$\Rightarrow \quad d\tau^2 = {}^{35}dt^2 - [dx - V_s(t)f(r_s)dt]^2 - dy^2 - dz^2$$
$$= (1 - V_s(t)^2 f(r_s)^2)dt^2 + 2V_s(t)f(r_s)dxdt - dx^2 - dy^2 - dz^2$$

Notice: The line-element is dependent on the velocity of the spaceship $V_s(t)$. If the velocity is zero the line-element reduces to flat Minkowsky space-time.

The metric

$$g_{ab} = \begin{Bmatrix} -1 + V_s(t)^2 f(r_s)^2 & -V_s(t)f(r_s) & 0 & 0 \\ -V_s(t)f(r_s) & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Bmatrix}$$

Where $V_s(t) \equiv \frac{dx_s(t)}{dt}$, $r_s^2 \equiv \left[\left(x - x_s(t)\right)^2 + y^2 + z^2\right]$ and $f(r_s)$ is a smooth positive [36]function that satisfies $f(0) = 1$ and decreases away from the origin to vanish for $r_s \gg R$ for some R.

The trajectory

A spaceship travels along a curve

$$(t, x, y, z) = (t, x_s(t), 0,0)$$

With the four-velocity

$$u = (u^t, u^x, u^y, u^z) = \left(\frac{dt}{d\tau}, \frac{dx_s(t)}{d\tau}, 0,0\right)$$

Manipulating the line element we get

$$d\tau^2 = dt^2 - [dx - V_s(t)f(r_s)dt]^2 - dy^2 - dz^2$$
$$= \left(1 - \left[\frac{dx}{dt} - V_s(t)f(r_s)\right]^2 - \left(\frac{dy}{dt}\right)^2 - \left(\frac{dz}{dt}\right)^2\right)dt^2$$
$$= \left(1 - \left[\frac{dx_s(t)}{dt} - V_s(t)f(r_s)\right]^2\right)dt^2$$
$$= (1 - [1 - f(r_s)]^2 V_s(t)^2)dt^2 > 0$$

Which means the trajectory is time-like and at every point along the curve

$$(t, x, y, z) = (t, x_s(t), 0,0)$$

the four-velocity of the spaceship lies inside the light cone if $V_s(t)^2 < 1$, i.e. smaller than the velocity of light.

[34] Because all the differentials with respect to θ and ϕ are zero, we can use the chain rule in this simple manner.

[35] Negative time-signature i.e. $d\tau^2 = -ds^2$. See chapter 2

[36] $f(r_s) = \frac{\tanh(\sigma(r_s+R)) - \tanh(\sigma(r_s-R))}{2\tanh(\sigma R)}$ with arbitrary parameters $R > 0$ and $\sigma > 0$

4.9.1 Ship time and coordinate time.

Imagine a spaceship traveling between two space-stations. What is the spaceship proper time $\Delta\tau$ compared to the coordinate time $t = \Delta T$.

The line element

$$d\tau^2 = dt^2 - [dx - V_s(t)f(r_s)dt]^2 - dy^2 - dz^2$$

$$\Rightarrow \quad \left(\frac{d\tau}{dt}\right)^2 = 1 - \left[\frac{dx}{dt} - V_s(t)f(r_s)\right]^2 - \left(\frac{dy}{dt}\right)^2 - \left(\frac{dz}{dt}\right)^2$$

The ship moves on a curve in the (x, y)-plane with the coordinates $(t, x_s(t), 0,0)$

$$\Rightarrow \quad \left(\frac{d\tau}{dt}\right)^2 = 1 - \left[\frac{dx_s(t)}{dt} - V_s(t)f(r_s)\right]^2 = 1 - [V_s(t) - V_s(t)f(r_s)]^2$$

$$= 1 - V_s(t)^2[1 - f(r_s)]^2$$

$$d\tau = \sqrt{1 - V_s(t)^2[1 - f(r_s)]^2}dt$$

If we assume the spaceship has a constant velocity $V_s(t)^2 = V_s(0)^2$

$$\Rightarrow \quad \Delta\tau = \sqrt{1 - V_s(0)^2[1 - f(r_s)]^2}\Delta T$$

In detail

$$\Delta T \to \begin{cases} \Delta\tau \quad if\ r_s \to 0\,, f(0) = 1 & (i) \\ \dfrac{\Delta\tau}{\sqrt{1 - V_s(0)^2[1 - f(r_s)]^2}} \gg \Delta\tau \quad if\ 0 < r_s < R & (ii) \\ \dfrac{\Delta\tau}{\sqrt{1 - V_s(0)^2}} > \Delta\tau \quad if\ r_s \gg R, f(r_s) = 0 & (iii) \end{cases}$$

where (ii) corresponds to the warp period and (iii) to the Minkowski space-time

On the light cone, $d\tau^2 = 0$:

The line-element is depending on the velocity of the space ship $V_s(t)$. This has a rather peculiar effect.

$$0 = -dt^2 + [dx - V_s(t)f(r_s)dt]^2 = \left(-1 + \left[\frac{dx}{dt} - V_s(t)f(r_s)\right]^2\right)dt^2$$

$$\Rightarrow \quad \frac{dx}{dt} = \pm 1 + V_s(t)f(r_s)$$

Now, as you can see there are areas where $\frac{dx}{dt}$ is larger the than one. This means that from our point of view, there are areas where the spaceship seems to move with a velocity larger than the speed of light. There is no contradiction here though, because locally the velocity of the spaceship $V_s(t)$ is smaller than the speed of light.

4.10 PClassic Anti-de Sitter Spacetime

4.10.1 Classic Anti-de Sitter Space-time is conformally related to the Einstein cylinder

The line element

$$ds^2 = -\cosh^2(r)\,dt^2 + dr^2 + \sinh^2(r)\,d\theta^2 + \sinh^2(r)\sin^2\theta\,d\phi^2$$

We use the transformation

$$\cosh(r) = \frac{1}{\cos\psi}$$

$$\Rightarrow \quad \sinh^2(r) = \frac{1}{\cos^2\psi} - 1 = \tan^2\psi$$

$$d(\cosh(r)) = d\left(\frac{1}{\cos\psi}\right)$$

$$\sinh(r)\,dr = \frac{\sin\psi}{\cos^2\psi}d\psi$$

$$\Rightarrow \qquad dr^2 = \frac{\sin^2\psi}{\sinh^2(r)\cos^4\psi}d\psi^2 = \frac{1}{\cos^2\psi}d\psi^2$$

$$\Rightarrow \qquad ds^2 = -\frac{1}{\cos^2\psi}dt^2 + \frac{1}{\cos^2\psi}d\psi^2 + \tan^2\psi\, d\theta^2 + \tan^2\psi\sin^2\theta\, d\phi^2$$

$$= \frac{1}{\cos^2\psi}(-dt^2 + d\psi^2 + \sin^2\psi\, d\theta^2 + \sin^2\psi\sin^2\theta\, d\phi^2)$$

$$= \frac{1}{\cos^2\psi}\left(-dt^2 + d\psi^2 + \sin^2\psi\,(d\theta^2 + \sin^2\theta\, d\phi^2)\right)$$

Which is conformally related to the Einstein cylinder

$$ds^2 = -dt^2 + (a_0)^2\left(d\theta^2 + \sin^2\theta\,(d\phi^2 + \sin^2\phi\, d\psi^2)\right)$$

4.10.2 The Path of a light ray – the null geodesics in the Classic Anti de Sitter space-time

To find the null-geodesics we need some integrals which we can solve:

a. Killing vectors:

Because the metric is independent of t a Killing vector is

$$\boldsymbol{\xi} = (\xi^t, \xi^r, \xi^\theta, \xi^\phi) = (1,0,0,0)$$

$\boldsymbol{\xi}\cdot\boldsymbol{u}$ is a conserved quantity along a geodesic, where

$$\boldsymbol{u} = (u^t, u^r, u^\theta, u^\phi) = \left(\frac{dt}{ds}, \frac{dr}{ds}, \frac{d\theta}{ds}, \frac{d\phi}{ds}\right)$$

$$\Rightarrow \qquad \boldsymbol{\xi}\cdot\boldsymbol{u} = \xi_a u^a = g_{ab}\xi^b u^a = g_{at}\xi^t u^a$$

$$= g_{tt}\xi^t u^t + g_{rt}\xi^t u^r + g_{\theta t}\xi^t u^\theta + g_{\phi t}\xi^t u^\phi = -\cosh^2(r)\,\dot{t}$$

$$\Rightarrow \qquad \dot{t} = \frac{K_1}{\cosh^2(r)} \qquad (4.15.)$$

Because the metric is independent of ϕ a Killing vector is

$$\boldsymbol{\zeta} = (\zeta^t, \zeta^r, \zeta^\theta, \zeta^\phi) = (0,0,0,1)$$

$$\Rightarrow \qquad \boldsymbol{\zeta}\cdot\boldsymbol{u} = \zeta_a u^a = g_{ab}\zeta^b u^a = g_{a\phi}\zeta^\phi u^a$$

$$= g_{t\phi}\zeta^\phi u^t + g_{r\phi}\zeta^\phi u^r + +g_{\theta\phi}\zeta^\phi u^\theta + g_{\phi\phi}\zeta^\phi u^\phi$$

$$= \sinh^2(r)\sin^2\theta\, u^\phi$$

$$\Rightarrow \qquad \dot{\phi} = \frac{K_2}{\sinh^2(r)\sin^2\theta} \qquad (4.16.)$$

b. Conservation of the four-velocity for a light ray

$$\boldsymbol{u}\cdot\boldsymbol{u} = u_a u^a = g_{ab}u^b u^a$$

$$= g_{tt}u^t u^t + g_{rr}u^r u^r + g_{\theta\theta}u^\theta u^\theta + g_{\phi\phi}u^\phi u^\phi$$

$$= -\cosh^2(r)\,\dot{t}^2 + \dot{r}^2 + \sinh^2(r)\,\dot{\theta}^2 + \sinh^2(r)\sin^2\theta\,\dot{\phi}^2 = 0$$

$$\Rightarrow \qquad 0 = -\cosh^2(r)\,\dot{t}^2 + \dot{r}^2 + \sinh^2(r)\,\dot{\theta}^2 + \sinh^2(r)\sin^2\theta\,\dot{\phi}^2 \qquad (4.17.)$$

c. The geodesic equations.

We use the Euler-Lagrange method.

$$0 = \frac{d}{ds}\left(\frac{\partial F}{\partial \dot{x}^a}\right) - \frac{\partial F}{\partial x^a}$$

$$F = \frac{1}{2}g_{ab}\dot{x}^a\dot{x}^b$$

$$= -\frac{1}{2}\cosh^2(r)\,\dot{t}^2 + \frac{1}{2}\dot{r}^2 + \frac{1}{2}\sinh^2(r)\,\dot{\theta}^2 + \frac{1}{2}\sinh^2(r)\sin^2\theta\,\dot{\phi}^2$$

$x^a = t$:

$$\frac{\partial F}{\partial t} = 0$$

$$\frac{\partial F}{\partial \dot{t}} = -\cosh^2(r)\,\dot{t}$$

$$\frac{d}{ds}\left(\frac{\partial F}{\partial \dot{t}}\right) = -\cosh^2(r)\,\ddot{t} - 2\sinh(r)\,\dot{t}\dot{r}$$

$$\Rightarrow \qquad 0 = \cosh^2(r)\,\ddot{t} + 2\sinh(r)\,\dot{t}\dot{r} \qquad (4.18.)$$

$\underline{x^a = r:}$
$$\frac{\partial F}{\partial r} = -\sinh(r)\,\dot{t}^2 + \cosh(r)\,\dot{\theta}^2 + \cosh(r)\sin^2\theta\,\dot{\phi}^2$$
$$\frac{\partial F}{\partial \dot{r}} = \dot{r}$$
$$\frac{d}{ds}\left(\frac{\partial F}{\partial \dot{r}}\right) = \ddot{r}$$
$$\Rightarrow \quad 0 = \ddot{r} + \sinh(r)\,\dot{t}^2 - \cosh(r)\,\dot{\theta}^2 - \cosh(r)\sin^2\theta\,\dot{\phi}^2 \qquad (4.19.)$$

$\underline{x^a = \theta:}$
$$\frac{\partial F}{\partial \theta} = \sinh^2(r)\cos\theta\,\dot{\phi}^2$$
$$\frac{\partial F}{\partial \dot{\theta}} = \sinh^2(r)\,\dot{\theta}$$
$$\frac{d}{ds}\left(\frac{\partial F}{\partial \dot{\theta}}\right) = 2\cosh(r)\,\dot{r}\dot{\theta} + \sinh^2(r)\,\ddot{\theta}$$
$$\Rightarrow \quad 0 = 2\cosh(r)\,\dot{r}\dot{\theta} + \sinh^2(r)\,\ddot{\theta} - \sinh^2(r)\cos\theta\,\dot{\phi}^2 \qquad (4.20.)$$

$\underline{x^a = \phi:}$
$$\frac{\partial F}{\partial \phi} = 0$$
$$\frac{\partial F}{\partial \dot{\phi}} = \sinh^2(r)\sin^2\theta\,\dot{\phi}$$
$$\frac{d}{ds}\left(\frac{\partial F}{\partial \dot{\phi}}\right) = 2\cosh(r)\sin^2\theta\,\dot{\phi} + 2\sinh^2(r)\cos\theta\,\dot{\phi} + \sinh^2(r)\sin^2\theta\,\ddot{\phi}$$
$$\Rightarrow \quad 0 = 2\cosh(r)\sin^2\theta\,\dot{\phi} + 2\sinh^2(r)\cos\theta\,\dot{\phi} + \sinh^2(r)\sin^2\theta\,\ddot{\phi} \qquad (4.21.)$$

Collecting the results
$$\dot{t} = \frac{K_1}{\cosh^2(r)} \qquad (4.15.)$$
$$\dot{\phi} = \frac{K_2}{\sinh^2(r)\sin^2\theta} \qquad (4.16.)$$
$$0 = -\cosh^2(r)\,\dot{t}^2 + \dot{r}^2 + \sinh^2(r)\,\dot{\theta}^2 + \sinh^2(r)\sin^2\theta\,\dot{\phi}^2 \qquad (4.17.)$$
$$0 = \cosh^2(r)\,\ddot{t} + 2\sinh(r)\,\dot{t}\dot{r} \qquad (4.18.)$$
$$0 = \ddot{r} + \sinh(r)\,\dot{t}^2 - \cosh(r)\,\dot{\theta}^2 - \cosh(r)\sin^2\theta\,\dot{\phi}^2 \qquad (4.19.)$$
$$0 = 2\cosh(r)\,\dot{r}\dot{\theta} + \sinh^2(r)\,\ddot{\theta} - \sinh^2(r)\cos\theta\,\dot{\phi}^2 \qquad (4.20.)$$
$$0 = 2\cosh(r)\sin^2\theta\,\dot{\phi} + 2\sinh^2(r)\cos\theta\,\dot{\phi} + \sinh^2(r)\sin^2\theta\,\ddot{\phi} \qquad (4.21.)$$

<u>The coordinates t and r:</u>

We need
$$\ddot{t} = -\frac{2\sinh(r)\cosh(r)}{\cosh^4(r)}K_1 \qquad (4.22.)$$
Substituting eq. (4.15.) and eq. (4.22.) into eq. (4.18.)
$$\Rightarrow \quad 0 = \cosh^2(r)\left(-\frac{2\sinh(r)\cosh(r)}{\cosh^4(r)}K_1\right) + 2\sinh(r)\left(\frac{K_1}{\cosh^2(r)}\right)\dot{r}$$
$$\Rightarrow \quad 0 = -\cosh(r) + \dot{r}$$
$$\Rightarrow \quad \dot{r} = \cosh(r) \qquad (4.23.)$$
Dividing eq. (4.15.) with eq. (4.23.)
$$\frac{\dot{t}}{\dot{r}} = \frac{dt}{dr} = \frac{K_1}{\cosh^3(r)}$$
$$\Rightarrow \quad \int_{t_0}^{t} dt = \int_{r_0}^{r} \frac{K_1}{\cosh^3(r)}\,dr$$

$$\Rightarrow \qquad t - t_0 = {}^{37}\left[\frac{\sinh(r)}{2\cosh^2(r)}\right]_{r_0}^{r} + \frac{1}{2}\int_{r_0}^{r}\frac{dr}{\cosh(r)} = {}^{38}\left[\frac{\sinh(r)}{2\cosh^2(r)} + \tan^{-1}(e^r)\right]_{r_0}^{r}$$
$$= \frac{\sinh(r)}{2\cosh^2(r)} + \tan^{-1}(e^r) - \left(\frac{\sinh(r_0)}{2\cosh^2(r_0)} + \tan^{-1}(e^{r_0})\right)$$
$$= \begin{cases} \dfrac{\sinh(r)}{2\cosh^2(r)} + \tan^{-1}(e^r) - K_3 \; if \; r_0 \to 0 \\ \qquad \to K_4 \; if \; r \to \infty \end{cases}$$

Interpreting this means, that no matter how far the light travels in this spacetime from $r = 0$ to $r \to \infty$ this happens within a limited time[39].

As an exercise we will look at the other coordinates as well.

The coordinates r and θ :

We need

$$\ddot{\phi} = -2K_2\left(\frac{2\cosh(r)\sinh(r)}{\sinh^4(r)\sin^2\theta}\dot{r} + \frac{\cos\theta\sin\theta}{\sinh^2(r)\sin^4\theta}\dot{\theta}\right)$$
$$= -2K_2\left(\frac{2\cosh(r)\sinh(r)}{\sinh^4(r)\sin^2\theta}\cosh(r) + \frac{\cos\theta\sin\theta}{\sinh^2(r)\sin^4\theta}\dot{\theta}\right) \qquad (4.24.)$$

Manipulate eq. (4.23.) and substitute eq. (4.16.) and eq. (4.24.)

$$0 = 2\cosh(r)\sin^2\theta\,\dot{\phi} + 2\sinh^2(r)\cos\theta\,\dot{\phi} + \sinh^2(r)\sin^2\theta\,\ddot{\phi}$$
$$= {}^{40}2K_2\left[\frac{\cosh(r)}{\sinh^2(r)} + \frac{\cos\theta}{\sin^2\theta} - \frac{\cosh^2(r)}{\sinh(r)} - \frac{\cos\theta}{\sin\theta}\dot{\theta}\right]$$
$$\Rightarrow \qquad \dot{\theta} = \frac{\cosh(r)\tan\theta}{\sinh^2(r)} + \frac{1}{\sin\theta} - \frac{\cosh^2(r)\tan\theta}{\sinh(r)} \qquad (4.25.)$$

Dividing (IX) with (VIII):

$$\frac{\dot{\theta}}{\dot{r}} = \frac{d\theta}{dr} = \frac{\tan\theta}{\sinh^2(r)} + \frac{1}{\cosh(r)\sin\theta} - \frac{\tan\theta}{\tanh(r)}$$

This illustrates how difficult it is to solve the geodesic equations in GR.

Referencer

Charles Misner, K. T. (2017). *Gravitation.* Princeton University Press.
Choquet-Bruhat, Y. (2015). *Introduction to General Relativity, Black Holes and Cosmology.* Oxford: Oxford University Press.
d'Inverno, R. (1992). *Introducing Einstein's Relativity.* Oxford: Clarendon Press.
Hartle, J. B. (2003). *An Introduction to Einstein's General Relativity.* San Francisco: Pearson Education.
Hartle, J. B. (2003). *Gravity - An introduction to Einstein's General Relativity.* Addison Wesley.
Kay, D. C. (1988). *Tensor Calculus.* McGraw-Hill.
McMahon, D. (2006). *Relativity Demystified.* McGraw-Hill.
Penrose, R. (2004). *The Road to Reality.* New York: Vintage Books.
Spiegel, M. R. (1990). *SCHAUM'S OUTLINE SERIES: Mathematical Handbook of FORMULAS and TABLES.* McGraw-Hill Publishing Company.

---

[37] $\int \frac{dr}{\cosh^n(r)} = \frac{\sinh(r)}{(n-1)\cosh^{n-1}(r)} + \frac{(n-2)}{(n-1)}\int \frac{dr}{\cosh^{n-2}(r)}$ (14.588) (Spiegel, 1990)

[38] $\int \frac{dr}{\cosh(r)} = 2\tan^{-1}(e^r)$ (14.567) (Spiegel, 1990)

[39] Notice: We haven't used eq. (III) so this result also valid for an object with mass.

[40] $= 2\cosh(r)\sin^2\theta\frac{K_2}{\sinh^2(r)\sin^2\theta} + 2\sinh^2(r)\cos\theta\frac{K_2}{\sinh^2(r)\sin^2\theta} + \sinh^2(r)\sin^2\theta\left[-2K_2\left(\frac{2\cosh(r)\sinh(r)}{\sinh^4(r)\sin^2\theta}\cosh(r) + \frac{\cos\theta\sin\theta}{\sinh^2(r)\sin^4\theta}\dot{\theta}\right)\right] =$

[a] (McMahon, 2006, s. 72-73), (Kay, 1988, s. 68-70, 74-75)
[b] (Charles Misner, 2017, s. 314)
[c] (d'Inverno, 1992, p. 83)
[d] (McMahon, 2006, s. 324), (Kay, 1988, s. 68-70, 74-75)
[e] (McMahon, 2006, s. 83)
[f] (Hartle, 2003, p. 183)
[g] (Penrose, 2004, s. 50), (Hartle, Gravity - An introduction to Einstein's General Relativity, 2003, p. 184)
[h] (McMahon, 2006, s. 84), (Hartle, An Introduction to Einstein's General Relativity, 2003, s. 143, 165, 184), (Kay, 1988, s. 126)
[i] (Hartle, 2003, p. 143)
[j] (Hartle, 2003, p. 184)
[k] (Hartle, 2003, p. 166)
[l] (Hartle, 2003, p. 172)
[m] (Hartle, 2003, p. 174)
[n] (Hartle, 2003, p. 175)
[o] https://en.wikipedia.org/wiki/Alcubierre_drive, (Hartle, 2003, pp. 144, 166)
[p] (Choquet-Bruhat, 2015, s. 97)