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Space-time		Line-element	Chapter
Three dimensional flat space-time	ds^2	$= -(cdt)^2 + dx^2 + dy^2$	3
Two-dimensional flat space-time	ds^2	$= -X^2 dT^2 + dX^2$	2,3,4
Two-dimensional flat space-time	ds^2	$= -dt^2 + dx^2$	3

3 Four vectors and four velocity

3.1 Four vectors

3.1.1 [a]The sum of two four vectors

If

$$\begin{aligned} a &= (a^0, \quad a^1, \quad a^2, \quad a^3) = (-2, \quad 0, \quad 0, \quad 1) \\ b &= (b^0, \quad b^1, \quad b^2, \quad b^3) = (5, \quad 0, \quad 3, \quad 4) \\ a - 5b &= (-2, \quad 0, \quad 0, \quad 1) - 5 \cdot (5, \quad 0, \quad 3, \quad 4) = (-27, \quad 0, \quad -15, \quad -19) \end{aligned}$$

3.1.2 [b] [c]The product of two four vectors

If

$$
\begin{aligned}
a &= (a^0,\quad a^1,\quad a^2,\quad a^3) = (-2,\quad 0,\quad 0,\quad 1) \\
b &= (b^0,\quad b^1,\quad b^2,\quad b^3) = (5,\quad 0,\quad 3,\quad 4) \\
\eta_{ab} &= \begin{Bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{Bmatrix} \\
a \cdot b &= -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3 = 10 + 0 + 0 + 4 = 14
\end{aligned}
$$

If

$$
\begin{aligned}
v^a &= (2,\quad 1,\quad 1,\quad -1) \\
w^a &= (-1,\quad 3,\quad 0,\quad 1) \\
\eta_{ab} &= \begin{Bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{Bmatrix} \\
v \cdot w &= v_a w^a = \eta_{ab} v^a w^a = -2 - 3 - 0 + 1 = -4
\end{aligned}
$$

3.1.3 [d] Timelike, spacelike or null-vector

If

$$
\begin{aligned}
a &= (a^0,\quad a^1,\quad a^2,\quad a^3) = (-2,\quad 0,\quad 0,\quad 1) \\
b &= (b^0,\quad b^1,\quad b^2,\quad b^3) = (5,\quad 0,\quad 3,\quad 4) \\
\eta_{ab} &= \begin{Bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{Bmatrix} \\
a \cdot a &= -(a^0)^2 + (a^1)^2 + (a^2)^2 + (a^3)^2 = -4 + 0 + 0 + 1 = -3 < 0\ i.e. timelike \\
b \cdot b &= -(b^0)^2 + (b^1)^2 + (b^2)^2 + (b^3)^2 = -25 + 0 + 9 + 16 = 0\ i.e. null\ vector
\end{aligned}
$$

3.1.4 [e]The separation between two events in flat spacetime

The separation $(\Delta s)^2$ between two events in flat space is described by a four-vector

$$
\begin{aligned}
E_1 &= (-1,\quad 3,\quad 2,\quad 4) \\
E_2 &= (4,\quad 0,\quad -1,\quad 1) \\
\eta_{ab} &= \begin{Bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{Bmatrix} \\
(\Delta s)^2 &= (E_2^0 - E_1^0)^2 - (E_2^1 - E_1^1)^2 - (E_2^2 - E_1^2)^2 - (E_2^3 - E_1^3)^2 = 25 - 9 - 9 - 9 = -2 > 0
\end{aligned}
$$

Which is timelike.

3.1.5 [f]Lorentz boost and the conservation of orthogonality

In the inertial system $(t,\quad x,\quad y,\quad z)$ we have the two orthogonal four vectors

$$
\begin{aligned}
a &= (a^t,\quad a^x,\quad a^y,\quad a^z) = (1,\quad 0,\quad 0,\quad 0) \\
b &= (b^t,\quad b^x,\quad b^y,\quad b^z) = (0,\quad 1,\quad 0,\quad 0) \\
\eta_{ab} &= \begin{Bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{Bmatrix} \\
a \cdot b &= -a^t b^t + a^x b^x + a^y b^y + a^z b^z = -1 + 1 = 0
\end{aligned}
$$

The inertial system $(t',\quad x',\quad y',\quad z')$ is related to the system $(t,\quad x,\quad y,\quad z)$ by a uniform velocity v along the x-axis. A four vector is transformed by the Lorenz boost

$$
a^{t'} = \gamma(a^t - v a^x)
$$

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$$
\begin{aligned}
a^{x'} &= \gamma(a^x - v a^t) \\
a^{y'} &= a^y \\
a^{z'} &= a^z \\
\Rightarrow \qquad a' &= (a^{t'},\quad a^{x'},\quad a^{y'},\quad a^{z'}) = (\gamma,\quad -\gamma v,\quad 0,\quad 0) \\
b' &= (b^{t'},\quad b^{x'},\quad b^{y'},\quad b^{z'}) = (-\gamma v,\quad \gamma,\quad 0,\quad 0) \\
a' \cdot b' &= -a^{t'} b^{t'} + a^{x'} b^{x'} + a^{y'} b^{y'} + a^{z'} b^{z'} = \gamma^2 v - \gamma^2 v = 0
\end{aligned}
$$

And we can conclude, that the orthogonality is conserved.

3.2 Four-velocity, Four-impulse, Four-force: Definitions and useful properties

The four-velocity

$$
\begin{aligned}
u &= \left(\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right) = (u^t, u^x, u^y, u^z) = (\gamma, \gamma\vec{V}) = \left(\frac{1}{\sqrt{1-(\vec{V})^2}}, \frac{\vec{V}}{\sqrt{1-(\vec{V})^2}}\right) \\
u^t &= \frac{1}{\sqrt{1-(\vec{V})^2}} \\
\vec{u} &= \frac{\vec{V}}{\sqrt{1-(\vec{V})^2}}
\end{aligned}
$$

The three-velocity:

$$
\vec{V} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)
$$

The γ –factor

$$
\begin{aligned}
\gamma &= \frac{1}{\sqrt{1-(\vec{V})^2}} \\
\dot{\gamma} &= \frac{d}{dt}\left(\frac{1}{\sqrt{1-(\vec{V})^2}}\right) = -2\vec{V}\frac{d\vec{V}}{dt}\left(-\frac{1}{2}\right)\frac{1}{(1-\vec{V}^2)^{\frac{3}{2}}} = \gamma^3 \vec{V} \cdot \vec{A}
\end{aligned}
$$

The four-impulse (also named the energy-momentum vector):

$$
\begin{aligned}
p &= mu = (p^t, \vec{p}) = m\left(\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right) = (m\gamma, m\gamma\vec{V}) \\
p^t &= m\frac{dt}{d\tau} = \frac{m}{\sqrt{1-(\vec{V})^2}} = {}^{1}E \\
\vec{p} &= \frac{m\vec{V}}{\sqrt{1-(\vec{V})^2}}
\end{aligned}
$$

The four-force:

$$
\begin{aligned}
f &= \frac{dp}{d\tau} = (f^t, \vec{f}) \\
f^t &= \gamma\vec{F} \cdot \vec{V} \\
\vec{f} &= \frac{d\vec{p}}{d\tau} = \frac{d\vec{p}}{dt}\frac{dt}{d\tau} = \gamma\vec{F}
\end{aligned}
$$

---

[1] Notice: For small velocities $\vec{V} \ll 1$: $p^t = m\left(1 + \frac{1}{2}(\vec{V})^2\right) = m + \frac{1}{2}m(\vec{V})^2 + \cdots$ which corresponds to the particle rest mass plus the particle kinetic energy, hence p^t is interpreted as the particle energy.

<u>The three-force</u>

$$\vec{F} \equiv \frac{d\vec{p}}{dt}$$

[g]<u>The four-acceleration</u>

$$\begin{aligned}
a &= \frac{du}{d\tau} = (a^t, \vec{a}) \\
a^t &= \frac{du^t}{d\tau} = \frac{dt}{d\tau}\frac{du^t}{dt} = \gamma\dot{\gamma} = \gamma^4 \vec{V}\cdot\vec{A} \\
\vec{a} &= \frac{dt}{d\tau}\frac{d\vec{u}}{dt} = \gamma\frac{d}{dt}(\gamma\vec{V}) = \gamma^2\frac{d\vec{V}}{dt} + \gamma\dot{\gamma}\vec{V} = \gamma^2\vec{A} + \gamma(\gamma^3\vec{V}\cdot\vec{A})\vec{V} = \gamma^2\vec{A}\left(1+\gamma^2\left(\vec{V}\right)^2\right) \\
&= \gamma^2\vec{A}\left(1+\frac{\left(\vec{V}\right)^2}{1-\left(\vec{V}\right)^2}\right) = \gamma^2\vec{A}\left(\frac{1}{1-\left(\vec{V}\right)^2}\right) = \gamma^4\vec{A}
\end{aligned}$$

<u>The three-acceleration</u>

$$\vec{A} = \frac{d\vec{V}}{dt}$$

<u>Notice the following:</u>

$$\begin{aligned}
u\cdot u &= \text{[h]}u_\alpha u^\alpha = \eta_{\alpha\beta}u^\beta u^\alpha = -\gamma^2 + \gamma^2\left(\vec{V}\right)^2 = -\gamma^2\left(1-\left(\vec{V}\right)^2\right) = \text{[2]} - 1 \\
p\cdot p &= m^2 u\cdot u = -m^2 \\
p\cdot p &= \eta_{\alpha\beta}p^\beta p^\alpha = -(p^t)^2 + (\vec{p})^2 = -E^2 + (\vec{p})^2 = -m^2 \\
\Rightarrow \quad E &= \text{[i]}\sqrt{m^2 + (\vec{p})^2} \\
m\frac{d(u\cdot u)}{d\tau} &= 0 \\
\Rightarrow \quad f\cdot u &= \text{[j]}\frac{dp}{d\tau}\cdot u = m\frac{du}{d\tau}\cdot u = \frac{1}{2}m\frac{d(u\cdot u)}{d\tau} = 0 \\
f\cdot u &= \eta_{\alpha\beta}f^\beta u^\alpha = -f^t u^t + \vec{f}\cdot\vec{u} = -f^t\gamma + \gamma\vec{F}\cdot\gamma\vec{V} = 0 \\
\Rightarrow \quad f^t &= \text{[k]}\gamma\vec{F}\cdot\vec{V} \\
f^t &= \frac{dE}{d\tau} = \frac{dE}{dt}\frac{dt}{d\tau} = \gamma\frac{dE}{dt} = \gamma\vec{F}\cdot\vec{V} \\
\Rightarrow \quad \frac{dE}{dt} &= \vec{F}\cdot\vec{V} \\
a\cdot u &= -a^t u^t + \vec{a}\cdot\vec{u} = -\gamma^4\vec{V}\cdot\vec{A}\gamma + \gamma^4\vec{A}\cdot\gamma\vec{V} = 0
\end{aligned}$$

3.3 Four-velocity, Four-impulse, Four-force and world-lines: Examples

World-lines describes the movement of a particle in a space-time with a certain physical condition imposed on it. So in order to find a particle world-line we have to establish two things: 1) The space-time, which is a sort of background, constraint or grid that the particle can move in. 2) The physical condition, e.g. a velocity, acceleration or trajectory. The tricky part in these calculations and finding the world-line[3] is to translate the physical condition into the correct space-time language and to keep track of the coordinates. Often the physical condition is given as a three vector and we have to translate it into a four-vector as the examples below will show.

---

[2] Notice: In the case of positive signature we would have: $= \eta_{\alpha\beta}u^\beta u_\alpha = \gamma^2 - \gamma^2\left(\vec{V}\right)^2 = \gamma^2\left(1-\left(\vec{V}\right)^2\right) = 1$

[3] Also notice, that if we have no physical conditions imposed on the particle, the particle is moving freely and the world-line becomes a geodesic, which we will look at more thoroughly in later chapters.

3.3.1 Parametrizing and four vectors of a free particle with constant velocity.

A particle is moving along the x-axis with constant three velocity: $\vec{V} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = (V_x, 0, 0)$. We want to know a) the particle world-line expressed parametrically as a function of τ: $\big(t(\tau), x(\tau)\big)$ and b) the particle four-velocity: $u = \left(\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right) = (u^t, u^x, 0, 0)$

The particle is moving along the x-axis in a space-time described by

$$\begin{aligned} ds^2 &= -dt^2 + dx^2 \\ g_{ab} &= \eta_{ab} = \begin{Bmatrix} -1 & \\ & 1 \end{Bmatrix} \\ \Rightarrow \quad d\tau^2 &= \text{[4]}\, dt^2 - dx^2 \end{aligned}$$

<u>The world-line:</u>

$$\left(\frac{d\tau}{dt}\right)^2 = 1 - \left(\frac{dx}{dt}\right)^2 = 1 - (V_x)^2 \tag{3.1.}$$

$$\begin{aligned} \Rightarrow \quad d\tau &= \sqrt{1-(V_x)^2}\,dt \\ \Rightarrow \quad \tau - \tau_0 &= \sqrt{1-(V_x)^2}\,t \\ \Rightarrow \quad t(\tau) &= \text{[5]}\,\frac{\tau}{\sqrt{1-(V_x)^2}} \\ \frac{dx}{dt} &= V_x \\ \Rightarrow \quad dx &= V_x dt \\ \Rightarrow \quad x - x_0 &= V_x t \\ \Rightarrow \quad x(\tau) &= V_x t(\tau) = \text{[6]}\,\frac{\tau V_x}{\sqrt{1-(V_x)^2}} \end{aligned}$$

<u>The four-velocity:</u> We use (3.1.) to find the four-velocity $u = \left(\frac{dt}{d\tau}, \quad \frac{dx}{d\tau}\right)$

$$\begin{aligned} \left(\frac{d\tau}{dt}\right)^2 &= 1 - (V_x)^2 \\ \Rightarrow \quad \frac{dt}{d\tau} &= \frac{1}{\sqrt{1-(V_x)^2}} = u^t \\ \frac{dx}{d\tau} &= \frac{dx}{dt}\frac{dt}{d\tau} = V_x\frac{dt}{d\tau} = \frac{V_x}{\sqrt{1-(V_x)^2}} = u^x \\ \Rightarrow \quad u &= \text{[7]}\left(\frac{1}{\sqrt{1-(V_x)^2}}, \frac{V_x}{\sqrt{1-(V_x)^2}}\right) \end{aligned}$$

<u>Collecting the results:</u>

<u>The world-line[8]</u>

$$\begin{aligned} t(\tau) &= \frac{\tau}{\sqrt{1-(V_x)^2}} \\ x(\tau) &= \frac{\tau V_x}{\sqrt{1-(V_x)^2}} \end{aligned}$$

<u>The four-velocity[9]</u>

---

[4] Negative time-signature i.e. $d\tau^2 = -ds^2$. See chapter 2

[5] $t(0) = 0 \Rightarrow \tau_0 = 0$

[6] $x(0) = 0 \Rightarrow x_0 = 0$

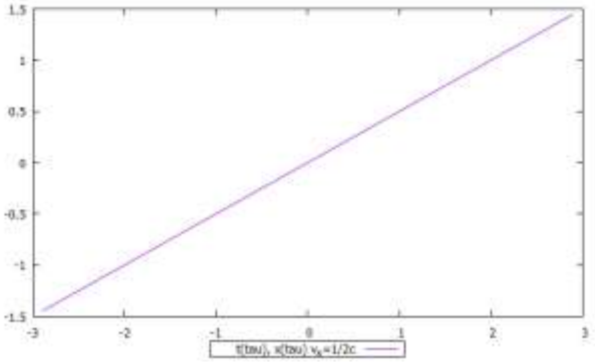
[7] $= \left(\frac{dt}{d\tau}, \frac{dt}{d\tau}V_x\right) = (\gamma, \gamma\vec{V})$

[8] gnuplot> g=1/sqrt(1-(1/2)^2)
gnuplot> plot g*t,g*t/2 title "t(tau), x(tau) v_x=1/2c "

[9] Checking the conservation of four-velocity: $-(u^t)^2 + (u^x)^2 = -\left(\frac{1}{\sqrt{1-(V_x)^2}}\right)^2 + \left(\frac{V_x}{\sqrt{1-(V_x)^2}}\right)^2 = -1$

$$u^t = \frac{1}{\sqrt{1-(V_x)^2}}$$

$$u^x = \frac{V_x}{\sqrt{1-(V_x)^2}}$$



3.3.2 m Four vectors of a free particle with constant velocity.

A particle with rest mass m_0 is moving at constant velocity $|\vec{V}| = \frac{c}{\sqrt{2}}$ in a direction 45° to the x-axis. We want to know a) The particle four-velocity: $u = \left(\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right) = (u^t, u^x, u^y, 0)$ and b) the energy-momentum vector

<u>The Three vector</u>

$$\vec{V} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = (V_x, V_y, V_z) = \left(\frac{1}{2}c, \frac{1}{2}c, 0\right)$$

The particle is moving in a space-time described by

$$ds^2 = -(cdt)^2 + dx^2 + dy^2$$

$$g_{ab} = \eta_{ab} = \begin{Bmatrix} -1 & & \\ & 1 & \\ & & 1 \end{Bmatrix}$$

$$\Rightarrow \quad d\tau^2 = (cdt)^2 - dx^2 - dy^2$$

$$\left(\frac{d\tau}{cdt}\right)^2 = 1 - \left(\frac{dx}{cdt}\right)^2 - \left(\frac{dy}{cdt}\right)^2 = 1 - \left(\frac{V_x}{c}\right)^2 - \left(\frac{V_y}{c}\right)^2 = 1 - \left(\frac{\vec{V}}{c}\right)^2 = 1 - \frac{1}{2} = \frac{1}{2}$$

<u>The four-velocity:</u> $u = \left(\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right)$

$$\frac{dt}{d\tau} =^{10} \frac{1}{\sqrt{1-\left(\frac{\vec{V}}{c}\right)^2}} = \frac{1}{\sqrt{\frac{1}{2}}} = \sqrt{2}$$

$$\frac{dx}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau} = V_x \frac{dt}{d\tau} = \frac{V_x}{\sqrt{1-(\vec{V})^2}} = u^x = \frac{1}{2}c\sqrt{2} = \frac{c}{\sqrt{2}}$$

$$\frac{dy}{d\tau} = \frac{dy}{dt}\frac{dt}{d\tau} = V_y \frac{dt}{d\tau} = \frac{V_y}{\sqrt{1-(\vec{V})^2}} = u^y = \frac{1}{2}c\sqrt{2} = \frac{c}{\sqrt{2}}$$

$$\Rightarrow \quad u =^{11} \left(\frac{1}{\sqrt{1-(\vec{V})^2}}, \frac{V_x}{\sqrt{1-(\vec{V})^2}}, \frac{V_y}{\sqrt{1-(\vec{V})^2}}, 0\right) = \left(\sqrt{2}, \frac{c}{\sqrt{2}}, \frac{c}{\sqrt{2}}, 0\right)$$

$$= \sqrt{2}\left(1, \frac{c}{2}, \frac{c}{2}, 0\right)$$

The energy-momentum vector $p = m_0 u$

[10] This is the familiar γ- factor

[11] $= (\gamma, \gamma\vec{V})$

$$
\begin{aligned}
p &= m_0\left(\frac{dt}{d\tau},\frac{dx}{d\tau},\frac{dy}{d\tau},\frac{dz}{d\tau}\right) =^{12} \left(\frac{m_0}{\sqrt{1-\left(\vec{V}\right)^2}},\frac{m_0V_x}{\sqrt{1-\left(\vec{V}\right)^2}},\frac{m_0V_y}{\sqrt{1-\left(\vec{V}\right)^2}},0\right)\\
&= \sqrt{2}m_0\left(1,\frac{c}{2},\frac{c}{2},0\right)
\end{aligned}
$$

The particle energy

$$E = \sqrt{2}m_0$$

The particle impulse

$$\vec{p} = m_0\left(\frac{c}{\sqrt{2}},\frac{c}{\sqrt{2}},0\right) = m_0\gamma\vec{V} \rightarrow m_0\vec{V} \text{ for small velocities}$$

3.4 Non-constant velocities

3.4.1 nParametrizing and four vectors of a free particle with non-constant velocity.

A particle is moving along the x-axis with three velocity: $\vec{V}=\left(\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt}\right)=\left(\frac{kt}{\sqrt{1+k^2t^2}},0,0\right)$. We want to know a) the particle world-line expressed parametrically as a function of τ: $\left(t(\tau),x(\tau)\right)$, b) the particle four-velocity: $u=\left(\frac{dt}{d\tau},\frac{dx}{d\tau},\frac{dy}{d\tau},\frac{dz}{d\tau}\right)=(u^t,u^x,0,0)$, c) the particle four impulse: $p=m_0u$, d) and the particle four force: $f=\frac{dp}{d\tau}=m_0\frac{du}{d\tau}$

We begin by noticing that

$$V_x = \frac{kt}{\sqrt{1+k^2t^2}} \rightarrow 1(=c)\ if\ t\rightarrow\infty$$

The particle is moving along the x-axis in a space-time described by

$$
\begin{aligned}
ds^2 &= -dt^2+dx^2\\
g_{ab} &= \eta_{ab} = \begin{Bmatrix}-1 & \\ & 1\end{Bmatrix}\\
\Rightarrow\quad d\tau^2 &=^{13} dt^2-dx^2
\end{aligned}
$$

<u>The world-line</u>

$$\left(\frac{d\tau}{dt}\right)^2 = 1-\left(\frac{dx}{dt}\right)^2 = 1-\left(\frac{kt}{\sqrt{1+k^2t^2}}\right)^2 = 1-\frac{k^2t^2}{1+k^2t^2} = \frac{1}{1+k^2t^2} \tag{3.2.}$$

$$
\begin{aligned}
\Rightarrow\quad d\tau &= \frac{dt}{\sqrt{1+k^2t^2}}\\
\Rightarrow\quad \tau-\tau_0 &= \int\frac{dt}{\sqrt{1+k^2t^2}} =^{14} \frac{\operatorname{arcsinh}(kt)}{k}\\
\Rightarrow\quad t(\tau) &=^{15} \frac{\sinh(k\tau)}{k}\\
\frac{dx}{dt} &= \frac{kt}{\sqrt{1+k^2t^2}}\\
\Rightarrow\quad dx &= \frac{kt}{\sqrt{1+k^2t^2}}dt\\
\Rightarrow\quad x-x_0 &= \int\frac{kt}{\sqrt{1+k^2t^2}}dt =^{16} \frac{1}{k}\sqrt{1+k^2t^2}
\end{aligned}
$$

[12] $=\left(m_0\gamma,m_0\gamma\vec{V}\right)$

[13] Negative time-signature i.e. $d\tau^2=-ds^2$. See chapter 2

[14] $\int\frac{dx}{\sqrt{x^2+a^2}}=\operatorname{arcsinh}(\frac{x}{a})$ (Spiegel, 1990) (14.182)

[15] $t(0)=0\Rightarrow\tau_0=0$

[16] $\int\frac{xdx}{\sqrt{x^2+a^2}}=\sqrt{x^2+a^2}$ (Spiegel, 1990) (14.183)

$$\Rightarrow \quad x(\tau) = {}^{17}\frac{1}{k}\sqrt{1+k^2t^2}-\frac{1}{k} = \frac{1}{k}\sqrt{1+k^2\left(\frac{\sinh(k\tau)}{k}\right)^2}-\frac{1}{k} = {}^{18}\frac{\cosh(k\tau)}{k}-\frac{1}{k}$$

The four-velocity: We use eq. (3.2.) to find the four-velocity $u=\left(\frac{dt}{d\tau},\frac{dx}{d\tau},\frac{dy}{d\tau},\frac{dz}{d\tau}\right)$

$$\left(\frac{d\tau}{dt}\right)^2 = \frac{1}{1+k^2t(\tau)^2}$$

$$\Rightarrow \quad \frac{dt}{d\tau} = \sqrt{1+k^2t(\tau)^2} = \sqrt{1+k^2\left(\frac{\sinh(k\tau)}{k}\right)^2} = \cosh(k\tau) = u^t$$

$$\frac{dx}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau} = \frac{kt}{\sqrt{1+k^2t^2}}\sqrt{1+k^2t^2} = kt(\tau) = \sinh(k\tau) = u^x$$

$$\Rightarrow \quad u = (\cosh(k\tau), \sinh(k\tau))$$

°Notice we can rewrite the three-velocity as a function of τ

$$\frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} = \frac{\sinh(k\tau)}{\cosh(k\tau)} = \tanh(k\tau)$$

ᵖThe four-force: $f = \left(\frac{dp^t}{d\tau}, \frac{dp^x}{d\tau}\right) = m_0\left(\frac{du^t}{d\tau}, \frac{du^x}{d\tau}\right)$

$$m_0\frac{du^t}{d\tau} = m_0\frac{d}{d\tau}\cosh(k\tau) = m_0k\sinh(k\tau) = f^t$$

$$m_0\frac{du^x}{d\tau} = m_0\frac{d}{d\tau}\sinh(k\tau) = m_0k\cosh(k\tau) = f^x$$

$$\Rightarrow \quad f = (m_0k\sinh(k\tau), m_0k\cosh(k\tau))$$

Collecting the results:

The world-line[19]

$$t(\tau) = \frac{\sinh(k\tau)}{k}$$

$$x(\tau) = \frac{\cosh(k\tau)}{k} - \frac{1}{k}$$

The four-velocity[20]

$$u^t(\tau) = \cosh(k\tau)$$

$$u^x(\tau) = \sinh(k\tau)$$

The four-impulse

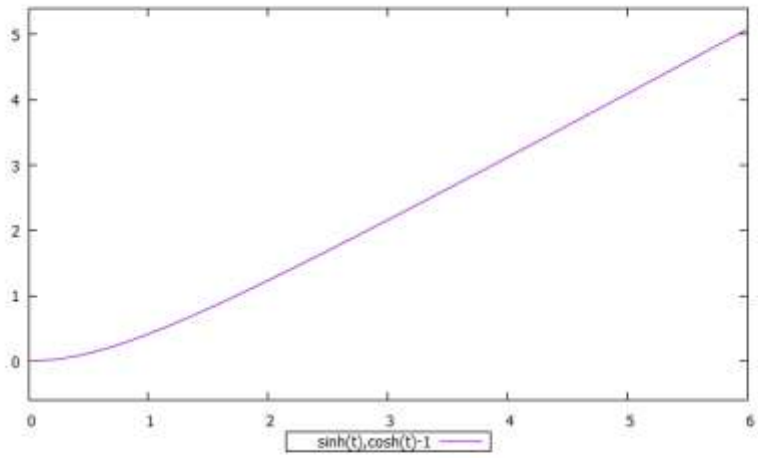
$$p^t(\tau) = m_0\cosh(k\tau)$$

$$p^x(\tau) = m_0\sinh(k\tau)$$

The four-force

$$f^t(\tau) = km_0\sinh(k\tau)$$

$$f^x(\tau) = km_0\cosh(k\tau)$$



[17] $x(0) = 0 \Rightarrow x_0 = -\frac{1}{k}$

[18] $\cosh^2 x - \sinh^2 x = 1$ (Spiegel, 1990) (8.11)

[19] gnuplot> plot sinh(t),cosh(t)-1 k=1"

[20] Checking the conservation of four-velocity: $-(u^t)^2 + (u^x)^2 = -\cosh^2(k\tau) + \sinh^2(k\tau) = -1$

3.4.2 [q]Parametrizing and four vectors of a free particle with constant acceleration.

A particle is moving along the x-axis with an acceleration, when measured in the particle rest-frame is always constant g i.e. $\vec{F} = (mg, 0, 0)$. We want to know a) the particle world-line expressed parametrically as a function of τ: $\big(t(\tau), x(\tau)\big)$ and b) the particle four-velocity: $u = \left(\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right) = (u^t, u^x, 0, 0)$

The particle is moving along the x-axis in a space-time described by

$$
\begin{aligned}
ds^2 &= -dt^2 + dx^2 \\
g_{ab} &= \eta_{ab} = \begin{Bmatrix} -1 & \\ & 1 \end{Bmatrix} \\
\Rightarrow \quad d\tau^2 &= {}^{21}dt^2 - dx^2 \\
\Rightarrow \quad \left(\frac{d\tau}{dt}\right)^2 &= 1 - \left(\frac{dx}{dt}\right)^2
\end{aligned}
$$

<u>The world-line</u>

From the space-time we know

$$
\begin{aligned}
\left(\frac{d\tau}{dt}\right)^2 &= 1 - \left(\frac{dx}{dt}\right)^2 = 1 - \left(\frac{d\tau}{dt}\frac{dx}{d\tau}\right)^2 = 1 - \left(\frac{d\tau}{dt}\right)^2\left(\frac{dx}{d\tau}\right)^2 \\
\Rightarrow \quad \frac{d\tau}{dt} &= \frac{1}{\sqrt{1 + \left(\frac{dx}{d\tau}\right)^2}}
\end{aligned}
$$

The physical condition

$$
\vec{F} = (mg, 0, 0)
$$

The four force

$$
\begin{aligned}
\Rightarrow \quad f^x &= \frac{dt}{d\tau}F_x = \frac{dt}{d\tau}mg = \frac{du^x}{d\tau} = \frac{dt}{d\tau}\frac{du^x}{dt} \\
\Rightarrow \quad \frac{du^x}{dt} &= mg \\
\Rightarrow \quad \frac{d^2x}{d\tau^2} &= \frac{d}{d\tau}\left(\frac{dx}{d\tau}\right) = \frac{dt}{d\tau}\frac{d}{dt}\left(\frac{dx}{d\tau}\right) = \frac{dt}{d\tau}\frac{du^x}{dt} = \frac{dt}{d\tau}mg = mg\sqrt{1 + \left(\frac{dx}{d\tau}\right)^2}
\end{aligned}
$$

This we can solve in order to find $u^x(\tau) = \frac{dx}{d\tau}$

$$
\begin{aligned}
\Rightarrow \quad \frac{du^x}{d\tau} &= mg\sqrt{1 + (u^x)^2} \\
\Rightarrow \quad \tau - \tau_0 &= \frac{1}{mg}\int \frac{d(u^x)}{\sqrt{1 + (u^x)^2}} = \frac{{}^{22}\operatorname{arcsinh}\big(u^x(\tau)\big)}{mg} \\
\Rightarrow \quad u^x(\tau) &= \sinh mg(\tau - \tau_0) \\
\Rightarrow \quad \frac{dx}{d\tau} &= {}^{23}\sinh(mg\tau) \\
\Rightarrow \quad x - x_0 &= \int \sinh(mg\tau)\, d\tau = \cosh(mg\tau) \\
\Rightarrow \quad x(\tau) &= {}^{24}\cosh(mg\tau) - 1 \\
\frac{d\tau}{dt} &= \frac{1}{\sqrt{1 + \left(\frac{dx}{d\tau}\right)^2}} = \frac{1}{\sqrt{1 + (\sinh(mg\tau))^2}} = \frac{1}{\cosh(mg\tau)}
\end{aligned}
$$

---

[21] Negative time-signature i.e. $d\tau^2 = -ds^2$. See chapter 2

[22] $\int \frac{dx}{\sqrt{x^2+a^2}} = \operatorname{arcsinh}\frac{x}{a}$ (Spiegel, 1990) (14.182)

[23] Assuming that $u^x(0) = 0$

[24] Assuming that $x(0) = 0$

$$\Rightarrow \qquad t - t_0 = \int \cosh(mg\tau)\, d\tau = \sinh(mg\tau)$$

$$\Rightarrow \qquad t(\tau) \overset{25}{=} \sinh(mg\tau)$$

Collecting the results:

The world-line[26]

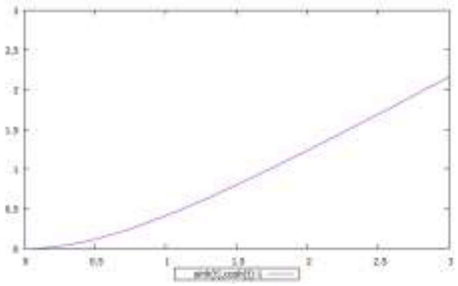
$$t(\tau) = \sinh(mg\tau)$$
$$x(\tau) = \cosh(mg\tau) - 1$$

The four-velocity[27]

$$u^t(\tau) = \cosh(mg\tau)$$
$$u^x(\tau) = \sinh(mg\tau)$$

The four-force

$$f^x(\tau) = mg\cosh(mg\tau)$$



3.4.3 Charged particle in a magnetic field.

A particle with charge q and rest mass m_0 is moving in a circular orbit with radius r in a uniform magnetic field $\vec{B}$ with total energy $E = \sqrt{m_0^2 + (\vec{p})^2}$ and three-force $\vec{F} = q(\vec{V} \times \vec{B})$. We want to find 1) The radius of the orbit and 2) The four-force.

$\vec{V}$, the velocity, is a tangent vector to the circular orbit and $\vec{B}$, the magnetic field is perpendicular to the circular orbit and the three-force is radial (pointing inwards).

The centripetal acceleration is radial(pointing outwards) is given by

$$\vec{a_c} \overset{28}{=} \frac{d\vec{V}}{dt} = \frac{\left(\vec{V}\right)^2}{r}\frac{\vec{r}}{r}$$

The three-force:

$$\vec{F} = \frac{d\vec{p}}{dt} = m_0 \frac{d}{dt}\left(\frac{m\vec{V}}{\sqrt{1 - \left(\vec{V}\right)^2}}\right) = m_0\gamma\frac{d\vec{V}}{dt} = m_0\gamma\frac{\left(\vec{V}\right)^2}{r}\frac{\vec{r}}{r}$$

The magnetic three-force and the centripetal three-force has the same size.

$$\left|m_0\gamma\frac{\left(\vec{V}\right)^2}{r}\frac{\vec{r}}{r}\right| = \left|q\left(\vec{V} \times \vec{B}\right)\right|$$

$$\Rightarrow \qquad m_0\gamma\frac{\left(\vec{V}\right)^2}{r} = q\left|\vec{V}\right|B$$

$$\Rightarrow \qquad m_0\gamma\left|\vec{V}\right| = rqB$$

$$\Rightarrow \qquad |\vec{p}| = rqB$$

$$\Rightarrow \qquad r = \frac{|\vec{p}|}{qB} = \frac{\sqrt{E^2 - m_0^2}}{qB}$$

The four-force:

$$f^t = \gamma\vec{F}\cdot\vec{V} \overset{29}{=} 0$$

$$\left|\vec{f}\right| = \left|\frac{d\vec{p}}{d\tau}\right| = \gamma\left|\vec{F}\right| = \gamma q\left|\vec{V}\right|B = \frac{qB}{m_0}|\vec{p}| = \frac{qB}{m_0}\sqrt{E^2 - m_0^2}$$

which is a radial component.

---

[25] Assuming that $t(0) = 0$

[26] gnuplot> plot sinh(t),cosh(t)-1 title "t(tau)=sinh(tau),x(tau)=cosh(tau)-1"

[27] Checking the conservation of four-velocity: $-(u^t)^2 + (u^x)^2 = -\cosh^2(g\tau) + \sinh^2(g\tau) = -1$

[28] In this case $|\vec{V}|$is constant, but the particle is stille accelerated because the direction of $\vec{V}$ constantly is changing tangentially along the circular movement.

[29] Because $\vec{F}$ and $\vec{V}$ are perpendicular

Notice: The relativistic factor shows itself in the $\sqrt{E^2 - m_0^2}$ component.

3.5 [s]Observers

So far we have only looked at the phenomena itself, but what happens when an observer in a laboratory becomes involved. Clearly we have to take into account if the laboratory and the observed phenomena is moving with respect to each other.
The observers laboratory is described by a set of four orthonormal vectors varying with the observers proper time: $e_{\hat{0}}$, $e_{\hat{1}}$, $e_{\hat{2}}$ and $e_{\hat{3}}$. These four vectors define a time direction $e_{\hat{0}}$ and three space or spatial directions $e_{\hat{1}}$, $e_{\hat{2}}$ and $e_{\hat{3}}$. It is important to notice that the time-like unit vector equals the observers four-velocity.

$$e_{\hat{0}}(\tau) = u_{obs}(\tau)$$

And

$$\begin{aligned} e_{\hat{0}} \cdot e_{\hat{0}} &= -1 \\ e_{\hat{1}} \cdot e_{\hat{1}} &= e_{\hat{2}} \cdot e_{\hat{2}} = e_{\hat{3}} \cdot e_{\hat{3}} = 1 \end{aligned}$$

If a particle with four-impuls p is observed in a laboratory as described above, the measured components $p^{\hat{\alpha}}$ along each direction are given by

$$\begin{aligned} p^{\hat{0}} &= {}^{[30]} - p \cdot e_{\hat{0}} \\ p^{\hat{1}} &= {}^{[31]} p \cdot e_{\hat{1}} \\ p^{\hat{2}} &= p \cdot e_{\hat{2}} \\ p^{\hat{3}} &= p \cdot e_{\hat{3}} \end{aligned} \tag{3.3.}$$

Now because $p^{\hat{0}}$ equals the measured energy in the laboratory we can rewrite eq. (3.3.)

$$E_{obs} = -p \cdot u_{obs}$$

[t]We can also describe – which I prefer - an observers laboratory in coordinate space defined by its space-time (if the metric is diagonal)

$$ds^2 = g_{00}(dx^0)^2 + g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2$$

The basis:

$$\begin{aligned} (e_{\hat{0}})^\alpha &= \left((-g_{00})^{-\frac{1}{2}}, 0, 0{,}0\right) \\ (e_{\hat{1}})^\alpha &= \left(0, (g_{11})^{-\frac{1}{2}}, 0, 0\right) \\ (e_2)^\alpha &= \left(0, 0, (g_{22})^{-\frac{1}{2}}, 0\right) \\ (e_{\hat{3}})^\alpha &= \left(0, 0, 0, (g_{33})^{-\frac{1}{2}}\right) \end{aligned}$$

Notice:

$$\begin{aligned} e_{\hat{0}} \cdot e_{\hat{0}} &= (e_{\hat{0}})_\alpha \cdot (e_{\hat{0}})^\alpha = g_{\alpha\beta}(e_{\hat{0}})^\beta \cdot (e_{\hat{0}})^\alpha = g_{00}(-g_{00})^{-\frac{1}{2}}(-g_{00})^{-\frac{1}{2}} = -1 \\ e_{\hat{1}} \cdot e_{\hat{1}} &= (e_{\hat{1}})_\alpha \cdot (e_{\hat{1}})^\alpha = g_{\alpha\beta}(e_{\hat{1}})^\beta \cdot (e_{\hat{1}})^\alpha = g_{11}(g_{11})^{-\frac{1}{2}}(g_{11})^{-\frac{1}{2}} = 1 \end{aligned}$$

If a particle with four-impuls p is observed in a laboratory as described above, the measured components $p^{\hat{\alpha}}$ along each direction are given by

$$\begin{aligned} p^{\hat{0}} &= {}^{[32]} - p \cdot e_{\hat{0}} \\ p^{\hat{1}} &= {}^{[33]} p \cdot e_{\hat{1}} \\ p^{\hat{2}} &= p \cdot e_{\hat{2}} \\ p^{\hat{3}} &= p \cdot e_{\hat{3}} \end{aligned}$$

[30] The components are defined by: $p = p^{\hat{\alpha}} e_{\hat{\alpha}} \Rightarrow p \cdot e_{\hat{0}} = p^{\hat{\alpha}} e_{\hat{\alpha}} \cdot e_{\hat{0}} = p^{\hat{0}} e_{\hat{0}} \cdot e_{\hat{0}} = -p^{\hat{0}}$

[31] The components are defined by: $p = p^{\hat{\alpha}} e_{\hat{\alpha}} \Rightarrow p \cdot e_{\hat{1}} = p^{\hat{\alpha}} e_{\hat{\alpha}} \cdot e_{\hat{1}} = p^{\hat{1}} e_{\hat{1}} \cdot e_{\hat{1}} = p^{\hat{1}}$

[32] The components are defined by: $p = p^{\hat{\alpha}} e_{\hat{\alpha}} \Rightarrow p \cdot e_{\hat{0}} = p^{\hat{\alpha}} e_{\hat{\alpha}} \cdot e_{\hat{0}} = p^{\hat{0}} e_{\hat{0}} \cdot e_{\hat{0}} = -p^{\hat{0}}$

[33] The components are defined by: $p = p^{\hat{\alpha}} e_{\hat{\alpha}} \Rightarrow p \cdot e_{\hat{1}} = p^{\hat{\alpha}} e_{\hat{\alpha}} \cdot e_{\hat{1}} = p^{\hat{1}} e_{\hat{1}} \cdot e_{\hat{1}} = p^{\hat{1}}$

3.5.1 [u]Energy of a stationary particle measured by and observer with constant velocity $\vec{V}$

A stationary particle with rest-mass m_0 has four-momentum $p^a = (m_0, \vec{p}) = (m_0, 0)$ is observed by an observer in a laboratory with constant velocity $\vec{V}$. We want to find the observed energy.
The space-time is the ordinary Minkowsky space so

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

$$\eta_{ab} = \begin{Bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{Bmatrix}$$

The observers four-velocity:

$$u_{obs} = (\gamma, \gamma\vec{V})$$

$$\Rightarrow \quad E_{obs} = -p \cdot u_{obs} = -p^a \cdot u_{obs,a} = -\eta_{ab} p^a \cdot u^b_{obs} = -\eta_{00} p^0 \cdot u^0_{obs} = m_0 \gamma$$

3.5.2 [v]Particle with four-momentum $\boldsymbol{p}$

A particle with four-momentum p and rest-mass m is passing through an observers system with four-velocity u_{obs}. Because $p^2 = -m^2$ is an invariant we know that

$$-m^2 = p^2 = -E_{obs}^2 + (\vec{p}_{obs})^2 = -(-p \cdot u_{obs})^2 + (\vec{p}_{obs})^2$$

The measured impulse in the laboratory

$$\Rightarrow \quad (\vec{p}_{obs})^2 = (p \cdot u_{obs})^2 + p^2 = (p \cdot u_{obs})^2 - m^2$$

3.5.3 [w]An accelerating observer

Imaging an accelerating observer moving in a frame where[34] $u^\alpha_{obs} = (\cosh a\tau, \sinh a\tau, 0, 0)$ passing by a stationary star emitting photons with wave vector $k^\alpha =$ [35]$(\omega_*, \omega_*, 0, 0)$. What is the observed frequency?
The space-time is the ordinary Minkowsky space so

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

$$\eta_{ab} = \begin{Bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{Bmatrix}$$

We have

$$E_{obs} = -p \cdot u_{obs}$$

For photons we know that

$$E = \hbar\omega$$
$$p = \hbar k$$

$$\Rightarrow \quad \omega(\tau) = -k \cdot u_{obs} = -k^\alpha u_{obs,\alpha} = -\eta_{\alpha\beta} k^\alpha u^\beta_{obs} = -\eta_{tt} k^t u^t_{obs} - \eta_{xx} k^x u^x_{obs}$$
$$= k^t u^t_{obs} - k^x u^x_{obs} = \omega_* (\cosh a\tau - \sinh a\tau) = \omega_* \exp(-a\tau)$$

Notice:
1) When the observer is moving towards the star $\tau < 0$

$$\Rightarrow \quad \omega(\tau) > \omega_*$$

The light is blue-shifted
2) When the observer is moving away from the star $\tau > 0$

$$\Rightarrow \quad \omega(\tau) < \omega_*$$

The light is red-shifted
[x]Notice: The orthonormal basis describing the laboratory:

$$e^\alpha_{\hat{0}} = u^\alpha_{obs} = (\cosh(a\tau), \sinh(a\tau), 0, 0)$$
$$e^\alpha_{\hat{1}} = (\sinh(a\tau), \cosh(a\tau), 0, 0)$$
$$e^\alpha_{\hat{2}} = (0, 0, 1, 0)$$

[34] This is the frame for a constant acceleration particle – see former examples.
[35] Notice: $k^2 = k^\alpha k_\alpha = \eta_{\alpha\beta} k^\alpha k^\beta = -(k^t)^2 + (k^x)^2 = -\omega_*^2 + \omega_*^2 = 0$ (photon)

$$e_{\hat{3}}^{\alpha} = (0,0,0,1)$$

The measured components of the wave-vector $k^{\alpha} = (\omega_*, \omega_*, 0, 0)$ along each direction are given by

$$\begin{aligned}
k^{\hat{0}} &= -k \cdot e_{\hat{0}} = -k^{\alpha} \cdot e_{\hat{0},\alpha} = -\eta_{\alpha\beta} k^{\alpha} \cdot e_{\hat{0}}^{\beta} = -\left(-k^{t} \cdot e_{\hat{0}}^{t} + k^{x} \cdot e_{\hat{0}}^{x}\right) \\
&= \omega_*(\cosh(a\tau) - \sinh(a\tau)) \\
k^{\hat{1}} &= k \cdot e_{\hat{1}} = k^{\alpha} \cdot e_{\hat{1},\alpha} = \eta_{\alpha\beta} k^{\alpha} \cdot e_{\hat{1}}^{\beta} = -k^{t} \cdot e_{\hat{1}}^{t} + k^{x} \cdot e_{\hat{1}}^{x} \\
&= \omega_*(-\sinh(a\tau) + \cosh(a\tau)) \\
k^{\hat{2}} &= k \cdot e_{\hat{2}} = 0 \\
k^{\hat{3}} &= k \cdot e_{\hat{3}} = 0 \\
\Rightarrow \quad k^{\hat{\alpha}} &= {}^{36}(\omega_*(\cosh(a\tau) - \sinh(a\tau)), \omega_*(-\sinh(a\tau) + \cosh(a\tau)), 0, 0)
\end{aligned}$$

3.5.4 ʸThe four-velocity of an observer in a two-dimensional flat space-time moving – Rindler Space-time

Consider the two-dimensional space-time[37]

$$\begin{aligned}
ds^2 &= -X^2 dT^2 + dX^2 \\
g_{ab} &= \begin{Bmatrix} -X^2 & \\ & 1 \end{Bmatrix} \\
\Rightarrow \quad d\tau^2 &= {}^{38}X^2 dT^2 - dX^2
\end{aligned}$$

An observer moves on a curve – this is the observers world-line

$$X = 2T \; for \; T > 1$$

The four-velocity (or rather two-velocity) of the observer is:

$$u = (u^T, u^X) = \left(\frac{dT}{d\tau}, \frac{dX}{d\tau}\right)$$

Manipulating the metric we get

$$\left(\frac{d\tau}{dT}\right)^2 = X^2 - \left(\frac{dX}{dT}\right)^2 = (2T)^2 - \left(\frac{d(2T)}{dT}\right)^2 = 4(T^2 - 1)$$

$$\Rightarrow \quad \frac{d\tau}{dT} = 2\sqrt{T^2 - 1} \qquad T^2 - 1 > 0$$

$$\Rightarrow \quad \frac{dT}{d\tau} = \frac{1}{2\sqrt{T^2 - 1}}$$

$$\Rightarrow \quad \frac{dX}{d\tau} = \frac{dT}{d\tau}\frac{dX}{dT} = \frac{1}{2\sqrt{T^2 - 1}}\frac{d(2T)}{dT} = \frac{1}{\sqrt{T^2 - 1}} = \frac{1}{\sqrt{\left(\frac{1}{2}X\right)^2 - 1}}$$

$$\Rightarrow \quad u = \left(\frac{1}{2\sqrt{T^2 - 1}}, \frac{1}{\sqrt{T^2 - 1}}\right) = \left(\frac{1}{2\sqrt{T^2 - 1}}, \frac{1}{\sqrt{\left(\frac{1}{2}X\right)^2 - 1}}\right)$$

Is the curve $X = 2T$ of the observer time-like or space-like:

Manipulating the metric we get

$$\begin{aligned}
ds^2 &= \left(-X^2 + \left(\frac{dX}{dT}\right)^2\right) dT^2 = -\left((2T)^2 - \left(\frac{d(2T)}{dT}\right)^2\right) dT^2 = -4(T^2 - 1)dT^2 \\
&< 0
\end{aligned}$$

Which means the curve is timelike.

Another possibility is to look at the square of the four-velocity

---

[36] Notice: $k^2 = k^{\hat{\alpha}} k_{\hat{\alpha}} = \eta_{\alpha\beta} k^{\hat{\alpha}} k^{\hat{\beta}} = -\left(k^{\hat{t}}\right)^2 + \left(k^{\hat{x}}\right)^2 = -\omega_*^2[(\cosh(a\tau) - \sinh(a\tau))^2 - (-\sinh(a\tau) + \cosh(a\tau))^2 = -\omega_*^2[\cosh^2(a\tau) + \sinh^2(a\tau) - 2\cosh(a\tau)\sinh(a\tau) - \sinh^2(a\tau) - \cosh^2(a\tau) + 2\sinh(a\tau)\cosh(a\tau) = 0$ (photon)

[37] This is actually the Rindler metric, which we look at in a later chapter.

[38] Negative time-signature i.e. $d\tau^2 = -ds^2$. See chapter 2

$$
\begin{aligned}
u\cdot u &= u_T u^T + u_X u^X = g_{TT}(u^T)^2 + g_{XX}(u^X)^2\\
&= -X^2\left(\frac{1}{2\sqrt{T^2-1}}\right)^2 + \left(\frac{1}{\sqrt{T^2-1}}\right)^2 = -(2T)^2\frac{1}{4(T^2-1)} + \frac{1}{(T^2-1)}\\
&= (-T^2+1)\frac{1}{(T^2-1)} = -1
\end{aligned}
$$

i.e. the curve is timelike.

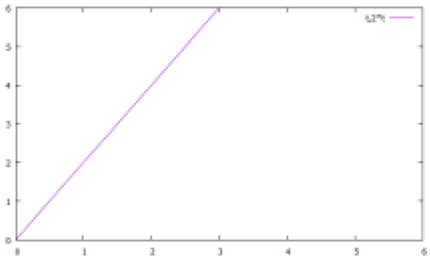
Collecting the results:

The world-line[39]

$$X(\tau) \quad = 2T(\tau)$$

The four-velocity

$$
\begin{aligned}
u^t(T) &= \frac{1}{2\sqrt{T^2-1}}\\
u^x(T) &= \frac{1}{\sqrt{T^2-1}}
\end{aligned}
$$



3.5.5 [z]Observed particle outside a Spherical Symmetric star

An astronaut stationed in a laboratory outside a spherical symmetric star is measuring radially outwards moving protons produced by the star. One particular proton is measured to have the energy E and momentum $\vec{P}$. The mass of the star is M and the laboratory is hovering at a fixed distance R. We want to find the components of the of the energy-momentum vector $\left(p^t, \overrightarrow{p^r}\right)$ in the Schwarzschild space-time of the proton expressed by the measured values.

The Schwarzschild space-time

$$ds^2 \quad = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

The coordinate basis describing the laboratory

$$
\begin{aligned}
(e_{\hat{t}})^\alpha &= \left(\left(1-\frac{2M}{r}\right)^{-\frac{1}{2}}, 0, 0, 0\right)\\
(e_{\hat{1}})^\alpha &= \left(0, \left(1-\frac{2M}{r}\right)^{\frac{1}{2}}, 0, 0\right)\\
(e_{\hat{2}})^\alpha &= \left(0, 0, \frac{1}{r}, 0\right)\\
(e_{\hat{3}})^\alpha &= \left(0, 0, 0, \frac{1}{r\sin\theta}\right)
\end{aligned}
$$

The observed properties

$$
\begin{aligned}
E_{obs} &= -p\cdot e_{\hat{t}} = -p_\alpha \cdot (e_{\hat{t}})^\alpha = -g_{\alpha\beta}p^\beta \cdot (e_{\hat{t}})^\alpha = -g_{tt}p^t\left(1-\frac{2M}{r}\right)^{-\frac{1}{2}}\\
&= p^t\left(1-\frac{2M}{r}\right)^{\frac{1}{2}}\\
|\vec{P}| &= p\cdot e_{\hat{r}} = p_\alpha \cdot (e_r)^\alpha = g_{\alpha\beta}p^\beta \cdot (e_{\hat{r}})^\alpha = g_{rr}p^r\left(1-\frac{2M}{r}\right)^{\frac{1}{2}} = p^r\left(1-\frac{2M}{r}\right)^{-\frac{1}{2}}\\
\Rightarrow \qquad p^t &= E_{obs}\left(1-\frac{2M}{r}\right)^{-\frac{1}{2}}
\end{aligned}
$$

[39] plot t,2*t

$$p^r \quad = |\vec{P}|\left(1-\frac{2M}{r}\right)^{\frac{1}{2}}$$

3.5.6 æThe four force outside a black hole

The four-force outside a black hole on a spacecraft with mass m

$$f^\alpha \quad = m\left(\frac{d^2x^\alpha}{d\tau^2}+\Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}\right)$$

We want to find the four-force observed by a stationary hovering spacecraft ($i.e. \, dr = d\theta = d\phi = 0$).
The four-force in the radial direction:

$$\begin{aligned} f^r \quad &= m\left(\frac{d^2r}{d\tau^2}+\Gamma^r{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}\right) \\ &= m\left(\frac{d^2r}{d\tau^2}+\Gamma^r{}_{tt}\left(\frac{dt}{d\tau}\right)^2+\Gamma^r{}_{rr}\left(\frac{dr}{d\tau}\right)^2+\Gamma^r{}_{\theta\theta}\left(\frac{d\theta}{d\tau}\right)^2+\Gamma^r{}_{\phi\phi}\left(\frac{d\phi}{d\tau}\right)^2\right) = {}^{40} m\Gamma^r{}_{tt}(u^t)^2 \\ &= m\left(\frac{M}{r^2}\right)\left(1-\frac{2M}{r}\right)\left(1-\frac{2M}{r}\right)^{-1} = m\left(\frac{M}{r^2}\right) \end{aligned}$$

The four-force observed in the spacecraft:

$$\begin{aligned} f^{\hat{r}}_{obs} \quad &= f\cdot e_{\hat{r}} = f_\alpha\cdot(e_{\hat{r}})^\alpha = -g_{\alpha\beta}f^\beta\cdot(e_{\hat{r}})^\alpha = g_{rr}f^r(e_{\hat{r}})^r = \left(1-\frac{2M}{r}\right)^{-1} m\left(\frac{M}{r^2}\right)\left(1-\frac{2M}{r}\right)^{\frac{1}{2}} \\ &= m\left(\frac{M}{r^2}\right)\left(1-\frac{2M}{r}\right)^{-\frac{1}{2}} \end{aligned}$$

3.5.7 ø[41]Can an astronaut escape a black hole?

Imagine an astronaut stuck outside a black hole wants to go home. He has brought some kind of energy device and wants to know the escape velocity needed in order to reach his home far away, slowly decelerating arriving with zero velocity. The astronaut, mass m, is at rest in the distance R from the center of the black hole, mass M.
We treat the problem as an observer problem. The route back home is a radial geodesic as we found above where $\frac{dt}{d\tau}=\frac{1}{1-\frac{2M}{r}}$ and [42]the energy the astronaut needs to escape and fly home is $E_{escape} = m\gamma = m\frac{1}{\sqrt{1-V_{escape}^2}}$ which corresponds to the observed energy, E_{obs}.
The observer equation:

$$\begin{aligned} p^{\hat{0}} \quad &= E_{obs} = -p_{escape}\cdot e_{\hat{t}} = -(p_{escape})_\alpha\cdot(e_{\hat{t}})^\alpha \\ &= -g_{\alpha\beta}(p_{escape})^\beta\cdot(e_{\hat{t}})^\alpha = {}^{43} - g_{tt}(p_{escape})^t\left(1-\frac{2M}{r}\right)^{-\frac{1}{2}} \end{aligned}$$

[40] In the following chapters we will find: $\Gamma^r{}_{tt} = 1-\frac{2M}{r}$, $u^t = \left(1-\frac{2M}{r}\right)^{-1/2}$

[41] I have presented this example in another chapter (chapter 12) too because I think it is a good example concerning the observer problem as well as a black hole problem.

[42] The situation is equivalent to a particle with four-impulse $p = \left(M,\vec{V_o}\right)$ passing a stationary observer with four velocity $u_o = \left(\frac{dt}{d\tau},\frac{dr}{d\tau},\frac{d\theta}{d\tau},\frac{d\phi}{d\tau}\right)$. The particle in this case has the energy $E = -p\cdot u_o$. The escape energy has the opposite sign. (Hartle, 2003, s. 98)

[43] Recall: $(e_{\hat{t}})^\alpha = \left(\left(1-\frac{2M}{r}\right)^{-\frac{1}{2}},0,0,0\right)$

$$= ^{44} -g_{tt} m \frac{dt}{d\tau}\left(1-\frac{2M}{r}\right)^{-\frac{1}{2}}$$

$$= ^{45} \left(1-\frac{2M}{r}\right) m \left(\frac{1}{1-\frac{2M}{r}}\right)\left(1-\frac{2M}{r}\right)^{-\frac{1}{2}} = m\left(1-\frac{2M}{r}\right)^{-\frac{1}{2}}$$

$$\Rightarrow \qquad m\frac{1}{\sqrt{1-V_{escape}^2}} = m\left(1-\frac{2M}{r}\right)^{-\frac{1}{2}}$$

$$\Rightarrow \qquad V_{escape} = \sqrt{\frac{2M}{R}}$$

Referencer

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[a] (Hartle, 2003, s. 99)
[b] (Hartle, 2003, s. 99)
[c] (McMahon, 2006, s. 324)
[d] (Hartle, 2003, s. 99)
[e] (McMahon, 2006, s. 323)
[f] (Hartle, 2003, s. 82)
[g] (Hartle, 2003, s. 100)
[h] (Hartle, 2003, s. 85)
[i] (Hartle, 2003, s. 87)
[j] (McMahon, 2006, s. 324), (Hartle, 2003, s. 86)
[k] (Hartle, 2003, s. 88)
[l] (Hartle, 2003, s. 100)
[m] (Hartle, 2003, s. 100)
[n] (Hartle, 2003, s. 100)
[o] (Hartle, 2003, s. 85)
[p] (Hartle, 2003, s. 86)
[q] (Hartle, 2003, s. 100)
[r] (Hartle, 2003, s. 88)
[s] (Hartle, 2003, s. 95)
[t] (Hartle, 2003, s. 156)
[u] (Hartle, 2003, s. 98)
[v] (Hartle, 2003, s. 102)
[w] (Hartle, 2003, s. 99)
[x] (Hartle, 2003, s. 102)
[y] (McMahon, 2006, s. 84), (Hartle, 2003, s. 143, 165, 184), (Kay, 1988, s. 126)
[z] (Hartle, 2003, s. 215)
[æ] (Hartle, 2003, s. 261, 278)
[ø] (Hartle, 2003, s. 277)

[44] Recall: $p^t = m_{astronaut}\frac{dt}{d\tau}$
[45] Recall: $g_{tt} = -\left(1-\frac{2M}{r}\right)$