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11 The Schwarzschild Spacetime

11.1 [a]The general Schwarzschild metric

The line element

$$ds^2 = e^{2\nu(r)}dt^2 - e^{2\lambda(r)}dr^2 - r^2d\theta^2 - r^2\sin^2\theta\, d\phi^2$$

11.1.1 [b]Geodesic equations and Christoffel symbols.

To find the geodesic we use

$$0 = \frac{d}{ds}\left(\frac{\partial K}{\partial \dot{x}^a}\right) - \frac{\partial K}{\partial x^a}$$

$$K = \frac{1}{2}g_{ab}\dot{x}^a\dot{x}^b$$

$$= \frac{1}{2}e^{2\nu(r)}\dot{t}^2 - \frac{1}{2}e^{2\lambda(r)}\dot{r}^2 - \frac{1}{2}r^2\dot{\theta}^2 - \frac{1}{2}r^2\sin^2\theta\,\dot{\phi}^2$$

<u>$x^a = t$:</u>

$$\frac{\partial K}{\partial t} = 0$$
$$\frac{\partial K}{\partial \dot{t}} = e^{2\nu(r)}\dot{t}$$
$$\frac{d}{ds}\left(\frac{\partial K}{\partial \dot{t}}\right) = 2e^{2\nu(r)}\frac{d\nu}{dr}\dot{r}\dot{t} + e^{2\nu(r)}\ddot{t}$$
$$\Rightarrow \quad 0 = 2e^{2\nu(r)}\frac{d\nu}{dr}\dot{r}\dot{t} + e^{2\nu(r)}\ddot{t}$$
$$\Leftrightarrow \quad 0 = \ddot{t} + 2\frac{d\nu}{dr}\dot{r}\dot{t}$$

<u>$x^a = r$:</u>

$$\frac{\partial K}{\partial r} = e^{2\nu(r)}\frac{d\nu}{dr}\dot{t}^2 - e^{2\lambda(r)}\frac{d\lambda}{dr}\dot{r}^2 - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\phi}^2$$
$$\frac{\partial K}{\partial \dot{r}} = -e^{2\lambda(r)}\dot{r}$$
$$\frac{d}{ds}\left(\frac{\partial K}{\partial \dot{r}}\right) = -2e^{2\lambda(r)}\frac{d\lambda}{dr}\dot{r}^2 - e^{2\lambda(r)}\ddot{r}$$
$$\Rightarrow \quad 0 = -e^{2\lambda(r)}\frac{d\lambda}{dr}\dot{r}^2 - e^{2\lambda(r)}\ddot{r} - e^{2\nu(r)}\frac{d\nu}{dr}\dot{t}^2 + r\dot{\theta}^2 + r\sin^2\theta\,\dot{\phi}^2$$
$$\Leftrightarrow \quad 0 = \ddot{r} + \frac{d\lambda}{dr}\dot{r}^2 + e^{2(\nu(r)-\lambda(r))}\frac{d\nu}{dr}\dot{t}^2 - re^{-2\lambda(r)}\dot{\theta}^2 - r\sin^2\theta\,e^{-2\lambda(r)}\dot{\phi}^2$$

<u>$x^a = \theta$:</u>

$$\frac{\partial K}{\partial \theta} = -r^2\cos\theta\sin\theta\,\dot{\phi}^2$$
$$\frac{\partial K}{\partial \dot{\theta}} = -r^2\dot{\theta}$$
$$\frac{d}{ds}\left(\frac{\partial K}{\partial \dot{\theta}}\right) = -2r\dot{r}\dot{\theta} - r^2\ddot{\theta}$$
$$\Rightarrow \quad 0 = -2r\dot{r}\dot{\theta} - r^2\ddot{\theta} + r^2\cos\theta\sin\theta\,\dot{\phi}^2$$
$$\Leftrightarrow \quad 0 = \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \cos\theta\sin\theta\,\dot{\phi}^2$$

<u>$x^a = \phi$:</u>

$$\frac{\partial K}{\partial \phi} = 0$$
$$\frac{\partial K}{\partial \dot{\phi}} = -r^2\sin^2\theta\,\dot{\phi}$$
$$\frac{d}{ds}\left(\frac{\partial K}{\partial \dot{\phi}}\right) = -2r\sin^2\theta\,\dot{r}\dot{\phi} - 2r^2\cos\theta\sin\theta\,\dot{\theta}\dot{\phi} - 2r^2\sin^2\theta\,\ddot{\phi}$$
$$\Rightarrow \quad 0 = -2r\sin^2\theta\,\dot{r}\dot{\phi} - 2r^2\cos\theta\sin\theta\,\dot{\theta}\dot{\phi} - r^2\sin^2\theta\,\ddot{\phi}$$
$$\Leftrightarrow \quad 0 = \ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot\theta\,\dot{\theta}\dot{\phi}$$

11.1.1.1 Collecting the results

$$0 = \ddot{t} + 2\frac{d\nu}{dr}\dot{r}\dot{t}$$
$$0 = \ddot{r} + \frac{d\lambda}{dr}\dot{r}^2 + e^{2(\nu(r)-\lambda(r))}\frac{d\nu}{dr}\dot{t}^2 - re^{-2\lambda(r)}\dot{\theta}^2 - r\sin^2\theta\,e^{-2\lambda(r)}\dot{\phi}^2$$
$$0 = \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \cos\theta\sin\theta\,\dot{\phi}^2$$

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$$0 \quad = \ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot\theta\,\dot{\theta}\dot{\phi}$$

11.1.2 The Christoffel symbols

Now we can find the Christoffel symbols from the equation

$$0 \quad = \frac{d^2x^a}{ds^2} + \Gamma^a{}_{bc}\frac{dx^b}{ds}\frac{dx^c}{ds}$$

11.1.2.1 Collecting the results

$$\Gamma^t{}_{rt} = \Gamma^t{}_{tr} = \frac{d\nu}{dr} \qquad \Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = \frac{1}{r}$$

$$\Gamma^r{}_{rr} = \frac{d\lambda}{dr} \qquad \Gamma^\theta{}_{\phi\phi} = -\cos\theta\sin\theta$$

$$\Gamma^r{}_{tt} = e^{2(\nu(r)-\lambda(r))}\frac{d\nu}{dr} \qquad \Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} = \frac{1}{r}$$

$$\Gamma^r{}_{\theta\theta} = -re^{-2\lambda(r)} \qquad \Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta} = \cot\theta$$

$$\Gamma^r{}_{\phi\phi} = -r\sin^2\theta\, e^{-2\lambda(r)}$$

11.1.3 [c]The Riemann and Ricci tensor of the general Schwarzschild metric

The line element:

$$ds^2 \quad = e^{2\nu(r)}dt^2 - e^{2\lambda(r)}dr^2 - r^2d\theta^2 - r^2\sin^2\theta\, d\phi^2$$

The Basis one forms

$$\omega^{\hat{t}} = e^{\nu(r)}dt \qquad dt = e^{-\nu(r)}\omega^{\hat{t}}$$

$$\omega^{\hat{r}} = e^{\lambda(r)}dr \qquad dr = e^{-\lambda(r)}\omega^{\hat{r}}$$

$$\omega^{\hat{\theta}} = rd\theta \qquad d\theta = \frac{1}{r}\omega^{\hat{\theta}}$$

$$\omega^{\hat{\phi}} = r\sin\theta\, d\phi \qquad d\phi = \frac{1}{r\sin\theta}\omega^{\hat{\phi}}$$

$$\eta^{ij} \quad = \begin{Bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{Bmatrix}$$

Cartan's First Structure equation and the curvature one-forms:

$$d\omega^{\hat{a}} \quad = -\Gamma^{\hat{a}}{}_{\hat{b}} \wedge \omega^{\hat{b}}$$

$$d\omega^{\hat{t}} \quad = d\left(e^{\nu(r)}dt\right) = \nu' e^{\nu(r)}dr \wedge dt = \nu' e^{-\lambda(r)}\omega^{\hat{r}} \wedge \omega^{\hat{t}} = -\nu' e^{-\lambda(r)}\omega^{\hat{t}} \wedge \omega^{\hat{r}} = -\Gamma^{\hat{t}}{}_{\hat{r}} \wedge \omega^{\hat{r}}$$

$$d\omega^{\hat{r}} \quad = d\left(e^{\lambda(r)}dr\right) = 0$$

$$d\omega^{\hat{\theta}} \quad = d(rd\theta) = dr \wedge d\theta = \frac{1}{r}e^{-\lambda(r)}\omega^{\hat{r}} \wedge \omega^{\hat{\theta}} = -\frac{1}{r}e^{-\lambda(r)}\omega^{\hat{\theta}} \wedge \omega^{\hat{r}} = -\Gamma^{\hat{\theta}}{}_{\hat{r}} \wedge \omega^{\hat{r}}$$

$$d\omega^{\hat{\phi}} \quad = d(r\sin\theta\, d\phi) = \sin\theta\, dr \wedge d\phi + r\cos\theta\, d\theta \wedge d\phi$$

$$= \frac{1}{r}e^{-\lambda(r)}\omega^{\hat{r}} \wedge \omega^{\hat{\phi}} + \frac{\cot\theta}{r}\omega^{\hat{\theta}} \wedge \omega^{\hat{\phi}} = -\frac{1}{r}e^{-\lambda(r)}\omega^{\hat{\phi}} \wedge \omega^{\hat{r}} - \frac{\cot\theta}{r}\omega^{\hat{\phi}} \wedge \omega^{\hat{\theta}}$$

$$= -\Gamma^{\hat{\phi}}{}_{\hat{r}} \wedge \omega^{\hat{r}} - \Gamma^{\hat{\phi}}{}_{\hat{\theta}} \wedge \omega^{\hat{\theta}}$$

11.1.3.1 Collecting the results

$$\Gamma^{\hat{t}}{}_{\hat{r}} = \nu' e^{-\lambda(r)}\omega^{\hat{t}} \qquad \Gamma^{\hat{\phi}}{}_{\hat{r}} = \frac{1}{r}e^{-\lambda(r)}\omega^{\hat{\phi}}$$

$$\Gamma^{\hat{\theta}}{}_{\hat{r}} = \frac{1}{r}e^{-\lambda(r)}\omega^{\hat{\theta}} \qquad \Gamma^{\hat{\phi}}{}_{\hat{\theta}} = \frac{\cot\theta}{r}\omega^{\hat{\phi}}$$

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Summarized in a matrix - Where $\hat{a}$ refers to column and $\hat{b}$ to row:

$$\Gamma^{\hat{a}}{}_{\hat{b}} = \begin{Bmatrix} 0 & \nu' e^{-\lambda(r)}\omega^{\hat{t}} & 0 & 0 \\ \nu' e^{-\lambda(r)}\omega^{\hat{t}} & 0 & \frac{1}{r}e^{-\lambda(r)}\omega^{\hat{\theta}} & \frac{1}{r}e^{-\lambda(r)}\omega^{\hat{\phi}} \\ 0 & -\frac{1}{r}e^{-\lambda(r)}\omega^{\hat{\theta}} & 0 & \frac{\cot\theta}{r}\omega^{\hat{\phi}} \\ 0 & -\frac{1}{r}e^{-\lambda(r)}\omega^{\hat{\phi}} & -\frac{\cot\theta}{r}\omega^{\hat{\phi}} & 0 \end{Bmatrix}$$

11.1.4 The Ricci Rotation coefficients

$$\Gamma^{\hat{a}}{}_{\hat{b}} = \Gamma^{\hat{a}}{}_{\hat{b}\hat{c}}\omega^{\hat{c}}$$

$$\begin{aligned} \Gamma^{\hat{t}}{}_{\hat{r}\hat{t}} &= \nu' e^{-\lambda(r)} & \Gamma^{\hat{\theta}}{}_{\hat{r}\hat{\theta}} &= \frac{1}{r}e^{-\lambda(r)} \\ \Gamma^{\hat{r}}{}_{\hat{t}\hat{t}} &= \nu' e^{-\lambda(r)} & \Gamma^{\hat{\theta}}{}_{\hat{\phi}\hat{\phi}} &= -\frac{\cot\theta}{r} \\ \Gamma^{\hat{r}}{}_{\hat{\theta}\hat{\theta}} &= -\frac{1}{r}e^{-\lambda(r)} & \Gamma^{\hat{\phi}}{}_{\hat{r}\hat{\phi}} &= \frac{1}{r}e^{-\lambda(r)} \\ \Gamma^{\hat{r}}{}_{\hat{\phi}\hat{\phi}} &= -\frac{1}{r}e^{-\lambda(r)} & \Gamma^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}} &= \frac{\cot\theta}{r} \end{aligned}$$

11.1.5 The curvature two forms

$$\Omega^{\hat{a}}{}_{\hat{b}} = d\Gamma^{\hat{a}}{}_{\hat{b}} + \Gamma^{\hat{a}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{b}} = \frac{1}{2}R^{\hat{a}}{}_{\hat{b}\hat{c}\hat{d}}\omega^{\hat{c}} \wedge \omega^{\hat{d}}$$

We need

$$\begin{aligned} d\Gamma^{\hat{r}}{}_{\hat{t}} &= d\left(\nu' e^{-\lambda(r)}\omega^{\hat{t}}\right) = d\left(\nu' e^{\nu(r)-\lambda(r)}dt\right) \\ &= (\nu'' + \nu'(\nu' - \lambda'))e^{\nu(r)-\lambda(r)}\,dr \wedge dt = \left(\nu'' + \nu'(\nu' - \lambda')\right)e^{-2\lambda(r)}\omega^{\hat{r}} \wedge \omega^{\hat{t}} \\ d\Gamma^{\hat{\theta}}{}_{\hat{r}} &= d\left(\frac{1}{r}e^{-\lambda(r)}\omega^{\hat{\theta}}\right) = d\left(e^{-\lambda(r)}d\theta\right) = -\lambda' e^{-\lambda(r)}dr \wedge d\theta = \frac{\lambda'}{r}e^{-2\lambda(r)}\omega^{\hat{\theta}} \wedge \omega^{\hat{r}} \\ d\Gamma^{\hat{\phi}}{}_{\hat{r}} &= d\left(\frac{1}{r}e^{-\lambda(r)}\omega^{\hat{\phi}}\right) = d\left(e^{-\lambda(r)}\sin\theta\,d\phi\right) \\ &= -\lambda' e^{-\lambda(r)}\sin\theta\,dr \wedge d\phi + e^{-\lambda(r)}\cos\theta\,d\theta \wedge d\phi \\ &= -\frac{\lambda'}{r}e^{-2\lambda(r)}\omega^{\hat{r}} \wedge \omega^{\hat{\phi}} + \frac{1}{r^2}e^{-\lambda(r)}\cot\,\theta\,\omega^{\hat{\theta}} \wedge \omega^{\hat{\phi}} \\ d\Gamma^{\hat{\phi}}{}_{\hat{\theta}} &= d\left(\frac{\cot\theta}{r}\omega^{\hat{\phi}}\right) = d(\cos\theta\,d\phi) = -\sin\theta\,d\theta \wedge d\phi = -\frac{1}{r^2}\omega^{\hat{\theta}} \wedge \omega^{\hat{\phi}} \end{aligned}$$

Summerizing:

$$\begin{aligned} d\Gamma^{\hat{r}}{}_{\hat{t}} &= \left(\nu'' + \nu'(\nu' - \lambda')\right)e^{-2\lambda(r)}\omega^{\hat{r}} \wedge \omega^{\hat{t}} \\ d\Gamma^{\hat{\theta}}{}_{\hat{r}} &= \frac{\lambda'}{r}e^{-2\lambda(r)}\omega^{\hat{\theta}} \wedge \omega^{\hat{r}} \\ d\Gamma^{\hat{\phi}}{}_{\hat{r}} &= -\frac{\lambda'}{r}e^{-2\lambda(r)}\omega^{\hat{r}} \wedge \omega^{\hat{\phi}} + \frac{1}{r^2}e^{-\lambda(r)}\cot\,\theta\,\omega^{\hat{\theta}} \wedge \omega^{\hat{\phi}} \\ d\Gamma^{\hat{\phi}}{}_{\hat{\theta}} &= -\frac{1}{r^2}\omega^{\hat{\theta}} \wedge \omega^{\hat{\phi}} \end{aligned}$$

The curvature two-forms

$$\begin{aligned} \Omega^{\hat{r}}{}_{\hat{t}} &= d\Gamma^{\hat{r}}{}_{\hat{t}} + \Gamma^{\hat{r}}{}_{\hat{\theta}} \wedge \Gamma^{\hat{\theta}}{}_{\hat{t}} + \Gamma^{\hat{r}}{}_{\hat{\phi}} \wedge \Gamma^{\hat{\phi}}{}_{\hat{t}} = \left(\nu'' + \nu'(\nu' - \lambda')\right)e^{-2\lambda(r)}\omega^{\hat{r}} \wedge \omega^{\hat{t}} \\ \Omega^{\hat{\theta}}{}_{\hat{t}} &= d\Gamma^{\hat{\theta}}{}_{\hat{t}} + \Gamma^{\hat{\theta}}{}_{\hat{r}} \wedge \Gamma^{\hat{r}}{}_{\hat{t}} + \Gamma^{\hat{\theta}}{}_{\hat{\phi}} \wedge \Gamma^{\hat{\phi}}{}_{\hat{t}} = \frac{1}{r}e^{-\lambda(r)}\omega^{\hat{\theta}} \wedge \nu' e^{-\lambda(r)}\omega^{\hat{t}} = \frac{\nu'}{r}e^{-2\lambda(r)}\omega^{\hat{\theta}} \wedge \omega^{\hat{t}} \\ \Omega^{\hat{\phi}}{}_{\hat{t}} &= d\Gamma^{\hat{\phi}}{}_{\hat{t}} + \Gamma^{\hat{\phi}}{}_{\hat{r}} \wedge \Gamma^{\hat{r}}{}_{\hat{t}} + \Gamma^{\hat{\phi}}{}_{\hat{\theta}} \wedge \Gamma^{\hat{\theta}}{}_{\hat{t}} = \frac{\nu'}{r}e^{-2\lambda(r)}\omega^{\hat{\phi}} \wedge \omega^{\hat{t}} \\ \Omega^{\hat{\theta}}{}_{\hat{r}} &= d\Gamma^{\hat{\theta}}{}_{\hat{r}} + \Gamma^{\hat{\theta}}{}_{\hat{t}} \wedge \Gamma^{\hat{t}}{}_{\hat{r}} + \Gamma^{\hat{\theta}}{}_{\hat{\phi}} \wedge \Gamma^{\hat{\phi}}{}_{\hat{r}} = \frac{\lambda'}{r}e^{-2\lambda(r)}\omega^{\hat{\theta}} \wedge \omega^{\hat{r}} \end{aligned}$$

$$
\begin{aligned}
\Omega^{\hat\phi}{}_{\hat r} &= d\Gamma^{\hat\phi}{}_{\hat r} + \Gamma^{\hat\phi}{}_{\hat t} \wedge \Gamma^{\hat t}{}_{\hat r} + \Gamma^{\hat\phi}{}_{\hat\theta} \wedge \Gamma^{\hat\theta}{}_{\hat r} \\
&= -\frac{\lambda'}{r} e^{-2\lambda(r)} \omega^{\hat r} \wedge \omega^{\hat\phi} + \frac{1}{r^2} e^{-\lambda(r)} \cot\,\theta\, \omega^{\hat\theta} \wedge \omega^{\hat\phi} + \frac{\cot\theta}{r} \omega^{\hat\phi} \wedge \frac{1}{r} e^{-\lambda(r)} \omega^{\hat\theta} \\
&= -\frac{\lambda'}{r} e^{-2\lambda(r)} \omega^{\hat r} \wedge \omega^{\hat\phi} \\
\Omega^{\hat\phi}{}_{\hat\theta} &= d\Gamma^{\hat\phi}{}_{\hat\theta} + \Gamma^{\hat\phi}{}_{\hat t} \wedge \Gamma^{\hat t}{}_{\hat\theta} + \Gamma^{\hat\phi}{}_{\hat r} \wedge \Gamma^{\hat r}{}_{\hat\theta} \\
&= -\frac{1}{r^2} \omega^{\hat\theta} \wedge \omega^{\hat\phi} - \frac{1}{r^2} e^{-2\lambda(r)} \omega^{\hat\phi} \wedge \omega^{\hat\theta} \\
&= \frac{\left(1 - e^{-2\lambda(r)}\right)}{r^2} \omega^{\hat\phi} \wedge \omega^{\hat\theta}
\end{aligned}
$$

11.1.5.1 Collecting the results

$$
\begin{aligned}
\Omega^{\hat r}{}_{\hat t} &= \left(\nu'' + \nu'(\nu' - \lambda')\right) e^{-2\lambda(r)} \omega^{\hat r} \wedge \omega^{\hat t} & \Omega^{\hat\theta}{}_{\hat r} &= \frac{\lambda'}{r} e^{-2\lambda(r)} \omega^{\hat\theta} \wedge \omega^{\hat r} \\
\Omega^{\hat\theta}{}_{\hat t} &= \frac{\nu'}{r} e^{-2\lambda(r)} \omega^{\hat\theta} \wedge \omega^{\hat t} & \Omega^{\hat\phi}{}_{\hat r} &= -\frac{\lambda'}{r} e^{-2\lambda(r)} \omega^{\hat r} \wedge \omega^{\hat\phi} \\
\Omega^{\hat\phi}{}_{\hat t} &= \frac{\nu'}{r} e^{-2\lambda(r)} \omega^{\hat\phi} \wedge \omega^{\hat t} & \Omega^{\hat\phi}{}_{\hat\theta} &= \frac{\left(1 - e^{-2\lambda(r)}\right)}{r^2} \omega^{\hat\phi} \wedge \omega^{\hat\theta}
\end{aligned}
$$

Summarized in a matrix - Where $\hat a$ refers to column and $\hat b$ to row

$$
\Omega^{\hat a}{}_{\hat b} = \begin{Bmatrix}
0 & \left(\nu'' + \nu'(\nu' - \lambda')\right) e^{-2\lambda(r)} \omega^{\hat r} \wedge \omega^{\hat t} & \frac{\nu'}{r} e^{-2\lambda(r)} \omega^{\hat\theta} \wedge \omega^{\hat t} & \frac{\nu'}{r} e^{-2\lambda(r)} \omega^{\hat\phi} \wedge \omega^{\hat t} \\
S & 0 & \frac{\lambda'}{r} e^{-2\lambda(r)} \omega^{\hat\theta} \wedge \omega^{\hat r} & \frac{\lambda'}{r} e^{-2\lambda(r)} \omega^{\hat\phi} \wedge \omega^{\hat r} \\
S & AS & 0 & \frac{\left(1 - e^{-2\lambda(r)}\right)}{r^2} \omega^{\hat\phi} \wedge \omega^{\hat\theta} \\
S & AS & AS & 0
\end{Bmatrix}
$$

<u>The independent elements of the Riemann tensor in the non-coordinate basis</u>

$$
\begin{aligned}
R^{\hat r}{}_{\hat t\hat r\hat t} &= \left(\nu'' + \nu'(\nu' - \lambda')\right) e^{-2\lambda(r)} & R^{\hat\phi}{}_{\hat t\hat\phi\hat t} &= \frac{\nu'}{r} e^{-2\lambda(r)} \\
R^{\hat\theta}{}_{\hat t\hat\theta\hat t} &= \frac{\nu'}{r} e^{-2\lambda(r)} & R^{\hat\phi}{}_{\hat r\hat\phi\hat r} &= \frac{\lambda'}{r} e^{-2\lambda(r)} \\
R^{\hat\theta}{}_{\hat r\hat\theta\hat r} &= \frac{\lambda'}{r} e^{-2\lambda(r)} & R^{\hat\phi}{}_{\hat\theta\hat\phi\hat\theta} &= \frac{\left(1 - e^{-2\lambda(r)}\right)}{r^2}
\end{aligned}
$$

11.1.5.2 The Ricci tensor

$$
\begin{aligned}
R_{\hat a\hat b} &= R^{\hat c}{}_{\hat a\hat c\hat b} \\
R_{\hat t\hat t} &= R^{\hat r}{}_{\hat t\hat r\hat t} + R^{\hat\theta}{}_{\hat t\hat\theta\hat t} + R^{\hat\phi}{}_{\hat t\hat\phi\hat t} = \left(\nu'' + \nu'(\nu' - \lambda') + 2\frac{\nu'}{r}\right) e^{-2\lambda(r)} \\
R_{\hat r\hat r} &= R^{\hat c}{}_{\hat r\hat c\hat r} = R^{\hat t}{}_{\hat r\hat t\hat r} + R^{\hat\theta}{}_{\hat r\hat\theta\hat r} + R^{\hat\phi}{}_{\hat r\hat\phi\hat r} \\
&= -R^{\hat r}{}_{\hat t\hat r\hat t} + R^{\hat\theta}{}_{\hat r\hat\theta\hat r} + R^{\hat\phi}{}_{\hat r\hat\phi\hat r} = -\left(\nu'' + \nu'(\nu' - \lambda') - 2\frac{\lambda'}{r}\right) e^{-2\lambda(r)} \\
R_{\hat\theta\hat\theta} &= R^{\hat c}{}_{\hat\theta\hat c\hat\theta} = R^{\hat t}{}_{\hat\theta\hat t\hat\theta} + R^{\hat r}{}_{\hat\theta\hat r\hat\theta} + R^{\hat\phi}{}_{\hat\theta\hat\phi\hat\theta} \\
&= -R^{\hat\theta}{}_{\hat t\hat\theta\hat t} + R^{\hat\theta}{}_{\hat r\hat\theta\hat r} + R^{\hat\phi}{}_{\hat\theta\hat\phi\hat\theta} = -\frac{\nu'}{r} e^{-2\lambda(r)} + \frac{\lambda'}{r} e^{-2\lambda(r)} + \frac{\left(1 - e^{-2\lambda(r)}\right)}{r^2} \\
R_{\hat\phi\hat\phi} &= R^{\hat c}{}_{\hat\phi\hat c\hat\phi} = R^{\hat t}{}_{\hat\phi\hat t\hat\phi} + R^{\hat r}{}_{\hat\phi\hat r\hat\phi} + R^{\hat\theta}{}_{\hat\phi\hat\theta\hat\phi} = -R^{\hat\phi}{}_{\hat t\hat\phi\hat t} + R^{\hat\phi}{}_{\hat r\hat\phi\hat r} + R^{\hat\phi}{}_{\hat\theta\hat\phi\hat\theta}
\end{aligned}
$$

$$= -\frac{\nu'}{r}e^{-2\lambda(r)} + \frac{\lambda'}{r}e^{-2\lambda(r)} + \frac{\left(1 - e^{-2\lambda(r)}\right)}{r^2}$$

The non-diagonal elements:

$$R_{\hat{a}\hat{b}} = R^{\hat{t}}{}_{\hat{a}\hat{t}\hat{b}} + R^{\hat{r}}{}_{\hat{a}\hat{r}\hat{b}} + R^{\hat{\theta}}{}_{\hat{a}\hat{\theta}\hat{b}} + R^{\hat{\phi}}{}_{\hat{a}\hat{\phi}\hat{b}} = 0$$

11.1.5.3 Collecting the results

$$R_{\hat{t}\hat{t}} = \left(\nu'' + \nu'(\nu' - \lambda') + 2\frac{\nu'}{r}\right)e^{-2\lambda(r)}$$

$$R_{\hat{r}\hat{r}} = -\left(\nu'' + \nu'(\nu' - \lambda') - 2\frac{\lambda'}{r}\right)e^{-2\lambda(r)}$$

$$R_{\hat{\theta}\hat{\theta}} = -\frac{\nu'}{r}e^{-2\lambda(r)} + \frac{\lambda'}{r}e^{-2\lambda(r)} + \frac{\left(1 - e^{-2\lambda(r)}\right)}{r^2}$$

$$R_{\hat{\phi}\hat{\phi}} = -\frac{\nu'}{r}e^{-2\lambda(r)} + \frac{\lambda'}{r}e^{-2\lambda(r)} + \frac{\left(1 - e^{-2\lambda(r)}\right)}{r^2}$$

11.1.6 [d] [e]The Vacuum equations

$$G_{\hat{a}\hat{b}} = R_{\hat{a}\hat{b}} - \frac{1}{2}\eta_{\hat{a}\hat{b}}R = 0$$

$\Rightarrow$ [1] $\quad R_{\hat{a}\hat{b}} = 0$

So in order to find the parameters ν and λ we can solve:

$$R_{\hat{t}\hat{t}} = \left(\nu'' + \nu'(\nu' - \lambda') + 2\frac{\nu'}{r}\right)e^{-2\lambda(r)} = 0 \qquad (11.1.)$$

$$R_{\hat{r}\hat{r}} = -\left(\nu'' + \nu'(\nu' - \lambda') - 2\frac{\lambda'}{r}\right)e^{-2\lambda(r)} = 0 \qquad (11.2.)$$

$$R_{\hat{\theta}\hat{\theta}} = R_{\hat{\phi}\hat{\phi}} = -\frac{\nu'}{r}e^{-2\lambda(r)} + \frac{\lambda'}{r}e^{-2\lambda(r)} + \frac{\left(1 - e^{-2\lambda(r)}\right)}{r^2} = 0 \qquad (11.3.)$$

Adding eq. (11.1.) and eq. (11.2.)

$$R_{\hat{t}\hat{t}} + R_{\hat{r}\hat{r}} = \frac{2}{r}(\nu' + \lambda')e^{-2\lambda(r)} = 0$$

$$\Rightarrow \quad \nu' = -\lambda' \qquad (11.4.)$$

Substituting (11.4.) into (11.3.)

$$\Rightarrow \quad 0 = 2\frac{\lambda'}{r}e^{-2\lambda(r)} + \frac{\left(1 - e^{-2\lambda(r)}\right)}{r^2}$$

$$\Rightarrow \quad 0 = (2r\lambda' - 1)e^{-2\lambda(r)} + 1$$

Defining

$$\gamma = e^{-2\lambda(r)}$$

$\Rightarrow \quad \lambda' =$ [2] $-\frac{1}{2}\gamma' e^{2\lambda(r)}$

$$\Rightarrow \quad 0 = (2r\lambda' - 1)e^{-2\lambda(r)} + 1 = 2r\lambda' e^{-2\lambda(r)} - e^{-2\lambda(r)} + 1$$

$$= 2r\left(-\frac{1}{2}\gamma' e^{2\lambda(r)}\right)e^{-2\lambda(r)} - \gamma + 1 = -r\gamma' - \gamma + 1$$

$\Rightarrow \quad \gamma =$ [3] $1 - \frac{k}{r}$

---

[1] See the chapter named: The Vacuum Einstein Equations

[2] $\gamma' = -2\lambda' e^{-2\lambda(r)} \Rightarrow \lambda' = -\frac{1}{2}\gamma' e^{2\lambda(r)}$

[3] $\gamma e^{\int \frac{1}{r}dr} = \int \frac{1}{r} e^{\int \frac{1}{r}dr} dr + k \Rightarrow \gamma e^{\ln(r)} = \int \frac{1}{r} e^{\ln(r)} dr + k \Rightarrow \gamma r = r + k$ (Spiegel, 1990) (18.2)

$$\Rightarrow \qquad e^{2\lambda(r)} = \frac{1}{\gamma} = \frac{1}{1-\frac{k}{r}}$$
$$e^{2\nu(r)} = e^{-2\lambda(r)} = \gamma = 1 - \frac{k}{r}$$

Substituting in the line element

$$ds^2 = e^{2\nu(r)}dt^2 - e^{2\lambda(r)}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2$$
$$= \left(1-\frac{k}{r}\right)dt^2 - \left(\frac{1}{1-\frac{k}{r}}\right)dr^2 - r^2 d\theta^2 - r^2\sin^2\theta\, d\phi^2$$

11.1.7 [f]The meaning of the integration constant: The choice of $k = 2m$

We can use the geodesic equations to justify the choice of $2m$ by investigating the geodesic equations in the classical limit i.e. $r \gg 2m$, $\frac{dt}{d\tau} \to 1$ and $v = \frac{dr}{d\tau} \ll c$, where v is the velocity and τ is the proper time. We want to investigate the case of a radially infalling particle i.e. $d\theta = 0$ and $d\phi = 0$. We also want to work in SI-units so we have to substitute m by $\frac{Gm}{c^2}$. Also remember that $\dot{t} = \frac{dt}{ds} = \frac{dt}{cd\tau} \to \frac{1}{c}$, $\dot{r} = \frac{dr}{ds} = \frac{dr}{cd\tau} = \frac{v}{c}$ and $\ddot{r} = \frac{d^2r}{ds^2} = \frac{d^2r}{c^2d\tau^2} = \frac{a}{c^2}$, where a is the particle acceleration. We use equation (10.38):

$$0 = \ddot{r} - \frac{m}{r(r-2m)}\dot{r}^2 + \frac{m(r-2m)}{r^3}\dot{t}^2 - (r-2m)\dot{\theta}^2 - (r-2m)\sin^2\theta\,\dot{\phi}^2$$

Now before we carry on with the physics we also have to be sure that each term in this equation has the same dimension. It turns out that they don't and therefore the third term has to be multiplied by c^2, in which case each term gets the dimension[4] $\frac{1}{length}$.

$$\Rightarrow \qquad 0 = \ddot{r} - \frac{m}{r(r-2m)}\dot{r}^2 + \frac{m(r-2m)}{r^3}c^2\dot{t}^2 - (r-2m)\dot{\theta}^2 - (r-2m)\sin^2\theta\,\dot{\phi}^2$$

$d\theta = 0, d\phi = 0$:

$$\Rightarrow \qquad 0 = \ddot{r} - \frac{m}{r(r-2m)}\dot{r}^2 + \frac{m(r-2m)}{r^3}c^2\dot{t}^2$$

$r \gg 2m$:

$$\Rightarrow \qquad 0 = \ddot{r} - \frac{m}{r^2}\dot{r}^2 + \frac{m}{r^2}c^2\dot{t}^2$$

$\dot{t} = \frac{1}{c}, \dot{r} = \frac{v}{c}, \ddot{r} = \frac{a}{c^2}$:

$$\Rightarrow \qquad 0 = \frac{a}{c^2} - \frac{m}{r^2}\frac{v^2}{c^2} + \frac{m}{r^2} = \frac{1}{c^2}\left(a - \frac{m}{r^2}v^2 + \frac{mc^2}{r^2}\right)$$

$m \to \frac{Gm}{c^2}$:

$$\Rightarrow \qquad 0 = a - \frac{Gm}{r^2}\left(\frac{v}{c}\right)^2 + \frac{Gm}{r^2}$$

$v \ll c$:

$$\Rightarrow \qquad a = -\frac{Gm}{r^2}$$

Multiplying with M on both sides we get precisely the Newtonian gravitational law

$$\Rightarrow \qquad F = Ma = -\frac{GMm}{r^2}$$

[4] This actually originates from the line element of the Schwarzschild metric itself, because in order to get the same dimension of each term, the first term has to be multiplied by c^2: $ds^2 = \left(1-\frac{2m}{r}\right)c^2dt^2 - \left(1-\frac{2m}{r}\right)^{-1}dr^2 - r^2d\theta^2 - r^2\sin^2\theta\,d\phi^2$

11.1.8 [g]The meaning of the coordinate $\boldsymbol{r}$.

The coordinate r is not a distance from any centre, rather it is related to the area A of a two-dimensional sphere for some fixed t and r.

$$r = \left(\frac{A}{4\pi}\right)^{\frac{1}{2}}$$

How: If you look at the Schwarzschild line-element

$$ds^2 = -\left(1-\frac{2m}{r}\right)dt^2 + \left(1-\frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

You can see that embedded in this is the geometry of a two-dimensional sphere with line-element

$$d\Sigma^2 = r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

The areal element

$$dA = dl^2 dl^3 = r^2 \sin\theta\, d\theta d\phi$$

$$\Rightarrow \quad A = \int_0^\pi r^2 \sin\theta\, d\theta \int_0^{2\pi} d\phi = 4\pi r^2$$

11.2 The Schwarzschild geometry

The line element

$$ds^2 = -\left(1-\frac{2m}{r}\right)dt^2 + \left(1-\frac{2m}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$$

11.2.1 [h]Length and volume of the Schwarzschild geometry

The spatial part of the Schwarzschild line element is

$$dS^2 = -\left(1-\frac{2m}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2\sin^2\theta\, d\phi^2$$

(a) The radial distance between the sphere $r = 2M$ and the sphere $r = 3M$

$$l = \int dl^1 = \int_{2M}^{3M} \frac{1}{\sqrt{1-\frac{2M}{r}}} dr = \int_{2M}^{3M} \sqrt{\frac{r}{r-2M}} dr$$

$$= {}^5\left[\sqrt{(r-2M)r}\right]_{2M}^{3M} + M\int_{2M}^{3M} \frac{dr}{\sqrt{(r-2M)r}}$$

$$= {}^6\left[\sqrt{(r-2M)r} + 2M\ln\left(\sqrt{r}+\sqrt{r-2M}\right)\right]_{2M}^{3M}$$

$$= \sqrt{M*3M} + 2M\ln\left(\sqrt{3M}+\sqrt{M}\right) - 2M\ln\left(\sqrt{2M}\right) = \left(\frac{\sqrt{3}}{2} + \ln\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)\right) * 2M$$

$$\approx 1{,}52 * 2M$$

(b) The spatial volume between the sphere $r = 2M$ and the sphere $r = 3M$

$$\mathcal{V} = \int_{2M}^{3M} dl^1 dl^2 dl^3 = \int_{2M}^{3M} \frac{r^2}{\sqrt{1-\frac{2M}{r}}} dr \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi = 4\pi * 17{,}15 * M^3$$

$$= 6{,}43 * \frac{4\pi}{3}(2M)^3$$

11.2.2 [i]Geodesics in the Schwarzschild Spacetime

To find the geodesic we use the Euler-Lagrange equation

[5] $\int \sqrt{\frac{px+q}{ax+b}}dx = \frac{\sqrt{(ax+b)(px+q)}}{a} + \frac{aq-bp}{2a}\int \frac{dx}{\sqrt{(ax+b)(px+q)}}$ (Spiegel, 1990) (14.123)

[6] $\int \frac{dx}{\sqrt{(ax+b)(px+q)}} = \frac{2}{\sqrt{ap}}\ln\left(\sqrt{a(px+q)} + \sqrt{p(ax+b)}\right)$ (Spiegel, 1990) (14.120)

$$0 = \frac{d}{ds}\left(\frac{\partial F}{\partial \dot{x}^a}\right) - \frac{\partial F}{\partial x^a}$$

where

$$F = \left(1 - \frac{2m}{r}\right)\dot{t}^2 - \left(1 - \frac{2m}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\,\dot{\phi}^2$$

<u>$x^a = t$:</u>

$$\frac{\partial F}{\partial t} = 0$$

$$\frac{\partial F}{\partial \dot{t}} = 2\left(1 - \frac{2m}{r}\right)\dot{t}$$

$$\frac{d}{ds}\left(\frac{\partial F}{\partial \dot{t}}\right) = \frac{4m}{r^2}\dot{r}\dot{t} + 2\left(1 - \frac{2m}{r}\right)\ddot{t}$$

$$\Rightarrow \quad 0 = \frac{4m}{r^2}\dot{r}\dot{t} + 2\left(1 - \frac{2m}{r}\right)\ddot{t}$$

$$\Leftrightarrow \quad 0 = \ddot{t} + \frac{2m}{r(r-2m)}\dot{r}\dot{t}$$

<u>$x^a = r$:</u>

$$\frac{\partial F}{\partial r} = \frac{2m}{r^2}\dot{t}^2 + \frac{2m}{(r-2m)^2}\dot{r}^2 - 2r\dot{\theta}^2 - 2r\sin^2\theta\,\dot{\phi}^2$$

$$\frac{\partial F}{\partial \dot{r}} = -2\left(1 - \frac{2m}{r}\right)^{-1}\dot{r}$$

$$\frac{d}{ds}\left(\frac{\partial F}{\partial \dot{r}}\right) = -2\left(1 - \frac{2m}{r}\right)^{-1}\ddot{r} + 2\left(1 - \frac{2m}{r}\right)^{-2}\frac{2m}{r^2}\dot{r}^2 = -2\left(1 - \frac{2m}{r}\right)^{-1}\ddot{r} + \frac{4m}{(r-2m)^2}\dot{r}^2$$

$$\Rightarrow \quad 0 = -2\left(1 - \frac{2m}{r}\right)^{-1}\ddot{r} + \frac{2m}{(r-2m)^2}\dot{r}^2 - \frac{2m}{r^2}\dot{t}^2 + 2r\dot{\theta}^2 + 2r\sin^2\theta\,\dot{\phi}^2$$

$$\Leftrightarrow \quad 0 = \ddot{r} - \frac{m}{r(r-2m)}\dot{r}^2 + \frac{m(r-2m)}{r^3}\dot{t}^2 - (r-2m)\dot{\theta}^2 - (r-2m)\sin^2\theta\,\dot{\phi}^2$$

<u>$x^a = \theta$:</u>

$$\frac{\partial F}{\partial \theta} = -2r^2\cos\theta\sin\theta\,\dot{\phi}^2$$

$$\frac{\partial F}{\partial \dot{\theta}} = -2r^2\dot{\theta}$$

$$\frac{d}{ds}\left(\frac{\partial F}{\partial \dot{\theta}}\right) = -4r\dot{r}\dot{\theta} - 2r^2\ddot{\theta}$$

$$\Rightarrow \quad 0 = -4r\dot{r}\dot{\theta} - 2r^2\ddot{\theta} + 2r^2\cos\theta\sin\theta\,\dot{\phi}^2$$

$$\Leftrightarrow \quad 0 = \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \cos\theta\sin\theta\,\dot{\phi}^2$$

<u>$x^a = \phi$:</u>

$$\frac{\partial F}{\partial \phi} = 0$$

$$\frac{\partial F}{\partial \dot{\phi}} = -2r^2\sin^2\theta\,\dot{\phi}$$

$$\frac{d}{ds}\left(\frac{\partial F}{\partial \dot{\phi}}\right) = -4r\sin^2\theta\,\dot{r}\dot{\phi} - 4r^2\cos\theta\sin\theta\,\dot{\theta}\dot{\phi} - 2r^2\sin^2\theta\,\ddot{\phi}$$

$$\Rightarrow \quad 0 = -4r\sin^2\theta\,\dot{r}\dot{\phi} - 4r^2\cos\theta\sin\theta\,\dot{\theta}\dot{\phi} - 2r^2\sin^2\theta\,\ddot{\phi}$$

$$\Leftrightarrow \quad 0 = \ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot\theta\,\dot{\theta}\dot{\phi}$$

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11.2.2.1 Collecting the results

$$
\begin{aligned}
0 &= \ddot{t} + \frac{2m}{r(r-2m)}\dot{r}\dot{t} \\
0 &= \ddot{r} - \frac{m}{r(r-2m)}\dot{r}^2 + \frac{m(r-2m)}{r^3}\dot{t}^2 - (r-2m)\dot{\theta}^2 - (r-2m)\sin^2\theta\,\dot{\phi}^2 \\
0 &= \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \cos\theta\sin\theta\,\dot{\phi}^2 \\
0 &= \ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot\theta\,\dot{\theta}\dot{\phi}
\end{aligned}
$$

11.2.3 [j]The Riemann tensor of the Schwarzschild metric

Solving the vacuum equations we find $\nu = -\lambda$ and $e^{2\nu} = 1 - \frac{2m}{r}$, $e^{2\lambda} = \left(1 - \frac{2m}{r}\right)^{-1}$

The line element:

$$
ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2\sin^2\theta\, d\phi^2
$$

Now we can find the independent elements of the Riemann tensor in the non-coordinate basis:

$$
\begin{aligned}
\nu' &= \frac{1}{2}e^{-2\nu}\frac{d}{dr}(e^{2\nu}) = \frac{1}{2}e^{-2\nu}\frac{d}{dr}\left(1 - \frac{2m}{r}\right) = \frac{1}{2}e^{-2\nu}\left(\frac{2m}{r^2}\right) = \frac{m}{r^2 - r2m} \\
\nu'' &= \frac{d}{dr}\left(\frac{1}{2}e^{-2\nu}\frac{d}{dr}(e^{2\nu})\right) = -\nu' e^{-2\nu}\frac{d}{dr}(e^{2\nu}) + \frac{1}{2}e^{-2\nu}\frac{d^2}{dr^2}(e^{2\nu}) \\
&= -2\nu'\nu' + \frac{1}{2}e^{-2\nu}\frac{d}{dr}\left(\frac{2m}{r^2}\right) = -2\nu'\nu' - \frac{2m}{r^3}e^{-2\nu} \\
\Rightarrow \quad R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}} &= (\nu'' + \nu'(\nu' - \lambda'))e^{-2\lambda(r)} = (\nu'' + 2\nu'\nu')e^{2\nu(r)} \\
&= \left(-2\nu'\nu' - \frac{2m}{r^3}e^{-2\nu} + 2\nu'\nu'\right)e^{2\nu(r)} = -\frac{2m}{r^3} \\
R^{\hat{\theta}}{}_{\hat{t}\hat{\theta}\hat{t}} &= \frac{\nu'}{r}e^{-2\lambda(r)} = \frac{\nu'}{r}e^{2\nu(r)} = \frac{1}{r}\frac{1}{2}e^{-2\nu}\left(\frac{2m}{r^2}\right)e^{2\nu(r)} = \frac{m}{r^3} \\
R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} &= \frac{\lambda'}{r}e^{-2\lambda(r)} = -\frac{m}{r^3} \\
R^{\hat{\phi}}{}_{\hat{t}\hat{\phi}\hat{t}} &= \frac{\nu'}{r}e^{-2\lambda(r)} = \frac{m}{r^3} \\
R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{r}} &= \frac{\lambda'}{r}e^{-2\lambda(r)} = -\frac{m}{r^3} \\
R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{\theta}} &= \frac{\left(1 - e^{-2\lambda(r)}\right)}{r^2} = \frac{\left(1 - e^{2\nu(r)}\right)}{r^2} = \frac{\left(1 - \left(1 - \frac{2m}{r}\right)\right)}{r^2} = \frac{2m}{r^3}
\end{aligned}
$$

11.2.3.1 Collecting the results

$$
\begin{aligned}
R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}} &= -\frac{2m}{r^3} \\
R^{\hat{\theta}}{}_{\hat{t}\hat{\theta}\hat{t}} &= R^{\hat{\phi}}{}_{\hat{t}\hat{\phi}\hat{t}} = \frac{m}{r^3} \\
R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} &= R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{r}} = -\frac{m}{r^3} \\
R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{\theta}} &= \frac{2m}{r^3}
\end{aligned}
$$

11.3 The general time-dependent Schwarzschild space-time

11.3.1 [k]The Ricci tensor, Ricci scalar and Einstein tensor

The line element:

$$ds^2 = e^{2\nu(t,r)}dt^2 - e^{2\lambda(t,r)}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2$$

The Basis one forms

$$\begin{aligned} \omega^{\hat{t}} &= e^{\nu(t,r)}dt & dt &= e^{-\nu(t,r)}\omega^{\hat{t}} \\ \omega^{\hat{r}} &= e^{\lambda(t,r)}dr & dr &= e^{-\lambda(t,r)}\omega^{\hat{r}} \\ \omega^{\hat{\theta}} &= r d\theta & d\theta &= \frac{1}{r}\omega^{\hat{\theta}} \\ \omega^{\hat{\phi}} &= r\sin\theta\, d\phi & d\phi &= \frac{1}{r\sin\theta}\omega^{\hat{\phi}} \end{aligned}$$

$$\eta^{ij} = \begin{Bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{Bmatrix}$$

Cartan's First Structure equation and the calculation of the Cartan structure coefficients $\Gamma^{\hat{a}}{}_{\hat{b}}$:

$$\begin{aligned} d\omega^{\hat{a}} &= -\Gamma^{\hat{a}}{}_{\hat{b}} \wedge \omega^{\hat{b}} \\ d\omega^{\hat{t}} &= d\left(e^{\nu(t,r)}dt\right) = \nu' e^{\nu(t,r)} dr \wedge dt = \nu' e^{-\lambda(t,r)}\omega^{\hat{r}} \wedge \omega^{\hat{t}} = -\nu' e^{-\lambda(t,r)}\omega^{\hat{t}} \wedge \omega^{\hat{r}} \\ &= -\Gamma^{\hat{t}}{}_{\hat{r}} \wedge \omega^{\hat{r}} \\ d\omega^{\hat{r}} &= d\left(e^{\lambda(t,r)}dr\right) = \dot{\lambda} e^{\lambda(t,r)} dt \wedge dr = \dot{\lambda} e^{-\nu(t,r)}\omega^{\hat{t}} \wedge \omega^{\hat{r}} = -\dot{\lambda} e^{-\nu(t,r)}\omega^{\hat{r}} \wedge \omega^{\hat{t}} \\ &= -\Gamma^{\hat{r}}{}_{\hat{t}} \wedge \omega^{\hat{t}} \\ d\omega^{\hat{\theta}} &= d(r d\theta) = dr \wedge d\theta = \frac{1}{r}e^{-\lambda(r,t)}\omega^{\hat{r}} \wedge \omega^{\hat{\theta}} = -\frac{1}{r}e^{-\lambda(r,t)}\omega^{\hat{\theta}} \wedge \omega^{\hat{r}} = -\Gamma^{\hat{\theta}}{}_{\hat{r}} \wedge \omega^{\hat{r}} \\ d\omega^{\hat{\phi}} &= d(r\sin\theta\, d\phi) = \sin\theta\, dr \wedge d\phi + r\cos\theta\, d\theta \wedge d\phi \\ &= \frac{1}{r}e^{-\lambda(r,t)}\omega^{\hat{r}} \wedge \omega^{\hat{\phi}} + \frac{\cot\theta}{r}\omega^{\hat{\theta}} \wedge \omega^{\hat{\phi}} = -\frac{1}{r}e^{-\lambda(r,t)}\omega^{\hat{\phi}} \wedge \omega^{\hat{r}} - \frac{\cot\theta}{r}\omega^{\hat{\phi}} \wedge \omega^{\hat{\theta}} \\ &= -\Gamma^{\hat{\phi}}{}_{\hat{r}} \wedge \omega^{\hat{r}} - \Gamma^{\hat{\phi}}{}_{\hat{\theta}} \wedge \omega^{\hat{\theta}} \end{aligned}$$

11.3.1.1 Collecting the results

In this case we have to be particular careful in reading off the curvature one forms. The curvature one-forms are antisymmetric: $\Gamma_{\hat{a}\hat{b}} = -\Gamma_{\hat{b}\hat{a}}$.This means that $\Gamma^{\hat{t}}{}_{\hat{r}} = \eta^{\hat{t}\hat{t}}\Gamma_{\hat{t}\hat{r}} = -\eta^{\hat{t}\hat{t}}\Gamma_{\hat{r}\hat{t}} = -\eta^{\hat{t}\hat{t}}\eta_{\hat{r}\hat{r}}\Gamma^{\hat{r}}{}_{\hat{t}} = \Gamma^{\hat{r}}{}_{\hat{t}}$. But in the former calculation we found that $\Gamma^{\hat{t}}{}_{\hat{r}} = \nu' e^{-\lambda(t,r)}\omega^{\hat{t}}$ and $\Gamma^{\hat{r}}{}_{\hat{t}} = \dot{\lambda} e^{-\nu(t,r)}\omega^{\hat{r}}$, which means that we in order to fulfill the antisymmetric properties need to require that $\Gamma^{\hat{t}}{}_{\hat{r}} = \Gamma^{\hat{r}}{}_{\hat{t}} = \dot{\lambda} e^{-\nu(t,r)}\omega^{\hat{r}} + \nu' e^{-\lambda(t,r)}\omega^{\hat{t}}$, because $\Gamma^{\hat{t}}{}_{\hat{r}} = \nu' e^{-\lambda(t,r)}\omega^{\hat{t}} +$ (something that makes $\Gamma_{\hat{t}\hat{r}}$ antisymmetric), and $\Gamma^{\hat{r}}{}_{\hat{t}} = \dot{\lambda} e^{-\nu(t,r)}\omega^{\hat{r}} +$ (something that makes $\Gamma_{\hat{r}\hat{t}}$ antisymmetric).

$$\begin{aligned} \Gamma^{\hat{t}}{}_{\hat{r}} &= \nu' e^{-\lambda(t,r)}\omega^{\hat{t}} + Symmetric &&= \nu' e^{-\lambda(t,r)}\omega^{\hat{t}} + \dot{\lambda} e^{-\nu(t,r)}\omega^{\hat{r}} \\ \Gamma^{\hat{r}}{}_{\hat{t}} &= \dot{\lambda} e^{-\nu(t,r)}\omega^{\hat{r}} + Symmetric &&= \nu' e^{-\lambda(t,r)}\omega^{\hat{t}} + \dot{\lambda} e^{-\nu(t,r)}\omega^{\hat{r}} \\ \Gamma^{\hat{\theta}}{}_{\hat{r}} &= \frac{1}{r}e^{-\lambda(r,t)}\omega^{\hat{\theta}} \\ \Gamma^{\hat{\phi}}{}_{\hat{r}} &= \frac{1}{r}e^{-\lambda(r,t)}\omega^{\hat{\phi}} \\ \Gamma^{\hat{\phi}}{}_{\hat{\theta}} &= \frac{\cot\theta}{r}\omega^{\hat{\phi}} \end{aligned}$$

Summarizing the curvature one forms in a matrix - Where $\hat{a}$ refers to column and $\hat{b}$ to row:

$$\Gamma^{\hat{a}}{}_{\hat{b}} = \begin{Bmatrix} 0 & \dot{\lambda} e^{-\nu(t,r)}\omega^{\hat{r}} + \nu' e^{-\lambda(t,r)}\omega^{\hat{t}} & 0 & 0 \\ S & 0 & \frac{1}{r}e^{-\lambda(t,r)}\omega^{\hat{\theta}} & \frac{1}{r}e^{-\lambda(t,r)}\omega^{\hat{\phi}} \\ 0 & AS & 0 & \frac{\cot\theta}{r}\omega^{\hat{\phi}} \\ 0 & AS & AS & 0 \end{Bmatrix}$$

11.3.2 The curvature two forms

$$\Omega^{\hat{a}}{}_{\hat{b}} \;= d\Gamma^{\hat{a}}{}_{\hat{b}} + \Gamma^{\hat{a}}{}_{\hat{c}} \wedge \Gamma^{\hat{c}}{}_{\hat{b}} = \frac{1}{2} R^{\hat{a}}{}_{\hat{b}\hat{c}\hat{d}} \omega^{\hat{c}} \wedge \omega^{\hat{d}}$$

We need

$$\begin{aligned}
d\Gamma^{\hat{r}}{}_{\hat{t}} &= d\left(\dot{\lambda} e^{-\nu(t,r)} \omega^{\hat{r}} + \nu' e^{-\lambda(t,r)} \omega^{\hat{t}}\right) \\
&= d\left(\dot{\lambda} e^{\lambda(t,r)-\nu(t,r)} dr + \nu' e^{\nu(t,r)-\lambda(t,r)} dt\right) \\
&= \left(\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\nu}\right) e^{\lambda(t,r)-\nu(t,r)} dt \wedge dr + (\nu'' + \nu'(\nu' - \lambda')) e^{\nu(t,r)-\lambda(t,r)}\, dr \wedge dt \\
&= \left[-\left(\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\nu})\right) e^{-2\nu(t,r)} + \left(\nu'' + \nu'(\nu' - \lambda')\right) e^{-2\lambda(t,r)}\right] \omega^{\hat{r}} \wedge \omega^{\hat{t}} \\
d\Gamma^{\hat{\theta}}{}_{\hat{t}} &= 0 \\
d\Gamma^{\hat{\phi}}{}_{\hat{t}} &= 0 \\
d\Gamma^{\hat{\theta}}{}_{\hat{r}} &= d\left(\frac{1}{r} e^{-\lambda(t,r)} \omega^{\hat{\theta}}\right) = d\left(e^{-\lambda(t,r)} d\theta\right) = -\dot{\lambda} e^{-\lambda(t,r)} dt \wedge d\theta - \lambda' e^{-\lambda(t,r)} dr \wedge d\theta \\
&= \frac{\dot{\lambda}}{r} e^{-\nu(t,r)-\lambda(t,r)} \omega^{\hat{\theta}} \wedge \omega^{\hat{t}} + \frac{\lambda'}{r} e^{-2\lambda(t,r)} \omega^{\hat{\theta}} \wedge \omega^{\hat{r}} \\
d\Gamma^{\hat{\phi}}{}_{\hat{r}} &= d\left(\frac{1}{r} e^{-\lambda(t,r)} \omega^{\hat{\phi}}\right) = d\left(e^{-\lambda(t,r)} \sin\theta\, d\phi\right) \\
&= -\dot{\lambda} e^{-\lambda(t,r)} \sin\theta\, dt \wedge d\phi - \lambda' e^{-\lambda(t,r)} \sin\theta\, dr \wedge d\phi + e^{-\lambda(t,r)} \cos\theta\, d\theta \wedge d\phi \\
&= \frac{\dot{\lambda}}{r} e^{-\nu(t,r)-\lambda(t,r)} \omega^{\hat{\phi}} \wedge \omega^{\hat{t}} + \frac{\lambda'}{r} e^{-2\lambda(t,r)} \omega^{\hat{\phi}} \wedge \omega^{\hat{r}} + \frac{1}{r^2} e^{-\lambda(t,r)} \cot\,\theta\, \omega^{\hat{\theta}} \wedge \omega^{\hat{\phi}} \\
d\Gamma^{\hat{\phi}}{}_{\hat{\theta}} &= d\left(\frac{\cot\theta}{r} \omega^{\hat{\phi}}\right) = d(\cos\theta\, d\phi) = -\sin\theta\, d\theta \wedge d\phi = -\frac{1}{r^2} \omega^{\hat{\theta}} \wedge \omega^{\hat{\phi}}
\end{aligned}$$

11.3.2.1 Collecting the results

Summerized i an matrix where $\hat{a}$ corresponds to column and $\hat{b}$ to row

$$d\Gamma^{\hat{a}}{}_{\hat{b}} \;= \left\{\begin{array}{cccc}
 & \begin{bmatrix} -\left(\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\nu})\right) e^{-2\nu(t,r)} \\ +\left(\nu'' + \nu'(\nu' - \lambda')\right) e^{-2\lambda(t,r)} \end{bmatrix} \omega^{\hat{r}} \wedge \omega^{\hat{t}} & & \\
S & & \begin{bmatrix} \frac{\dot{\lambda}}{r} e^{-\nu(t,r)-\lambda(t,r)} \omega^{\hat{\theta}} \wedge \omega^{\hat{t}} \\ +\frac{\lambda'}{r} e^{-2\lambda(t,r)} \omega^{\hat{\theta}} \wedge \omega^{\hat{r}} \end{bmatrix} & \begin{bmatrix} \frac{\dot{\lambda}}{r} e^{-\nu(t,r)-\lambda(t,r)} \omega^{\hat{\phi}} \wedge \omega^{\hat{t}} \\ +\frac{\lambda'}{r} e^{-2\lambda(t,r)} \omega^{\hat{\phi}} \wedge \omega^{\hat{r}} \\ +\frac{1}{r^2} e^{-\lambda(t,r)} \cot\,\theta\, \omega^{\hat{\theta}} \wedge \omega^{\hat{\phi}} \end{bmatrix} \\
 & AS & & -\frac{1}{r^2} \omega^{\hat{\theta}} \wedge \omega^{\hat{\phi}} \\
 & AS & AS &
\end{array}\right\}$$

$$\begin{aligned}
\Rightarrow \qquad \Omega^{\hat{r}}{}_{\hat{t}} &= d\Gamma^{\hat{r}}{}_{\hat{t}} + \Gamma^{\hat{r}}{}_{\hat{\theta}} \wedge \Gamma^{\hat{\theta}}{}_{\hat{t}} + \Gamma^{\hat{r}}{}_{\hat{\phi}} \wedge \Gamma^{\hat{\phi}}{}_{\hat{t}} \\
&= \left[-\left(\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\nu})\right) e^{-2\nu(t,r)} + \left(\nu'' + \nu'(\nu' - \lambda')\right) e^{-2\lambda(t,r)}\right] \omega^{\hat{r}} \wedge \omega^{\hat{t}} \\
\Omega^{\hat{\theta}}{}_{\hat{t}} &= d\Gamma^{\hat{\theta}}{}_{\hat{t}} + \Gamma^{\hat{\theta}}{}_{\hat{r}} \wedge \Gamma^{\hat{r}}{}_{\hat{t}} + \Gamma^{\hat{\theta}}{}_{\hat{\phi}} \wedge \Gamma^{\hat{\phi}}{}_{\hat{t}} = \Gamma^{\hat{\theta}}{}_{\hat{r}} \wedge \Gamma^{\hat{r}}{}_{\hat{t}} \\
&= \frac{1}{r} e^{-\lambda(t,r)} \omega^{\hat{\theta}} \wedge \left(\dot{\lambda} e^{-\nu(t,r)} \omega^{\hat{r}} + \nu' e^{-\lambda(t,r)} \omega^{\hat{t}}\right) \\
&= \frac{\dot{\lambda}}{r} e^{-\lambda(t,r)-\nu(t,r)} \omega^{\hat{\theta}} \wedge \omega^{\hat{r}} + \frac{\nu'}{r} e^{-2\lambda(t,r)} \omega^{\hat{\theta}} \wedge \omega^{\hat{t}} \\
\Omega^{\hat{\phi}}{}_{\hat{t}} &= d\Gamma^{\hat{\phi}}{}_{\hat{t}} + \Gamma^{\hat{\phi}}{}_{\hat{r}} \wedge \Gamma^{\hat{r}}{}_{\hat{t}} + \Gamma^{\hat{\phi}}{}_{\hat{\theta}} \wedge \Gamma^{\hat{\theta}}{}_{\hat{t}} = \Gamma^{\hat{\phi}}{}_{\hat{r}} \wedge \Gamma^{\hat{r}}{}_{\hat{t}} \\
&= \frac{1}{r} e^{-\lambda(t,r)} \omega^{\hat{\phi}} \wedge \left(\dot{\lambda} e^{-\nu(t,r)} \omega^{\hat{r}} + \nu' e^{-\lambda(t,r)} \omega^{\hat{t}}\right) \\
&= \frac{\dot{\lambda}}{r} e^{-\lambda(t,r)-\nu(t,r)} \omega^{\hat{\phi}} \wedge \omega^{\hat{r}} + \frac{\nu'}{r} e^{-2\lambda(t,r)} \omega^{\hat{\phi}} \wedge \omega^{\hat{t}}
\end{aligned}$$

$$\begin{aligned}
\Omega^{\hat\theta}{}_{\hat r} &= d\Gamma^{\hat\theta}{}_{\hat r} + \Gamma^{\hat\theta}{}_{\hat t} \wedge \Gamma^{\hat t}{}_{\hat r} + \Gamma^{\hat\theta}{}_{\hat\phi} \wedge \Gamma^{\hat\phi}{}_{\hat r} \\
&= \frac{\dot\lambda}{r} e^{-\nu(t,r)-\lambda(t,r)} \omega^{\hat\theta} \wedge \omega^{\hat t} + \frac{\lambda'}{r} e^{-2\lambda(t,r)} \omega^{\hat\theta} \wedge \omega^{\hat r} \\
\Omega^{\hat\phi}{}_{\hat r} &= d\Gamma^{\hat\phi}{}_{\hat r} + \Gamma^{\hat\phi}{}_{\hat t} \wedge \Gamma^{\hat t}{}_{\hat r} + \Gamma^{\hat\phi}{}_{\hat\theta} \wedge \Gamma^{\hat\theta}{}_{\hat r} \\
\Omega^{\hat\phi}{}_{\hat\theta} &= d\Gamma^{\hat\phi}{}_{\hat\theta} + \Gamma^{\hat\phi}{}_{\hat t} \wedge \Gamma^{\hat t}{}_{\hat\theta} + \Gamma^{\hat\phi}{}_{\hat r} \wedge \Gamma^{\hat r}{}_{\hat\theta} \\
&= {}^{7}\frac{\dot\lambda}{r} e^{-\nu(t,r)-\lambda(t,r)} \omega^{\hat\phi} \wedge \omega^{\hat t} + \frac{\lambda'}{r} e^{-2\lambda(t,r)} \omega^{\hat\phi} \wedge \omega^{\hat r} \\
&= -\frac{1}{r^2} \omega^{\hat\theta} \wedge \omega^{\hat\phi} - \frac{1}{r^2} e^{-2\lambda(t,r)} \omega^{\hat\phi} \wedge \omega^{\hat\theta} = \frac{\left(1 - e^{-2\lambda(t,r)}\right)}{r^2} \omega^{\hat\phi} \wedge \omega^{\hat\theta}
\end{aligned}$$

The diagonal elements:

$$\Omega^{\hat a}{}_{\hat a} = d\Gamma^{\hat a}{}_{\hat a} + \Gamma^{\hat a}{}_{\hat c} \wedge \Gamma^{\hat c}{}_{\hat a} = 0$$

11.3.2.2 Collecting the results

Summerized i an matrix where $\hat a$ corresponds to column and $\hat b$ to row

$$\Omega^{\hat a}{}_{\hat b} = \left\{ \begin{matrix}
S & \begin{bmatrix} -\left(\ddot\lambda + \dot\lambda(\dot\lambda - \dot\nu)\right) e^{-2\nu(t,r)} \\ +\left(\nu'' + \nu'(\nu' - \lambda')\right) e^{-2\lambda(t,r)} \end{bmatrix} \omega^{\hat r} \wedge \omega^{\hat t} & \begin{bmatrix} \frac{\dot\lambda}{r} e^{-\lambda(t,r)-\nu(t,r)} \omega^{\hat\theta} \wedge \omega^{\hat r} \\ + \frac{\nu'}{r} e^{-2\lambda(t,r)} \omega^{\hat\theta} \wedge \omega^{\hat t} \end{bmatrix} & \begin{bmatrix} \frac{\dot\lambda}{r} e^{-\lambda(t,r)-\nu(t,r)} \omega^{\hat\phi} \wedge \omega^{\hat r} \\ + \frac{\nu'}{r} e^{-2\lambda(t,r)} \omega^{\hat\phi} \wedge \omega^{\hat t} \end{bmatrix} \\
 & S & \begin{bmatrix} \frac{\dot\lambda}{r} e^{-\nu(t,r)-\lambda(t,r)} \omega^{\hat\theta} \wedge \omega^{\hat t} \\ + \frac{\lambda'}{r} e^{-2\lambda(t,r)} \omega^{\hat\theta} \wedge \omega^{\hat r} \end{bmatrix} & \begin{bmatrix} \frac{\dot\lambda}{r} e^{-\nu(t,r)-\lambda(t,r)} \omega^{\hat\phi} \wedge \omega^{\hat t} \\ + \frac{\lambda'}{r} e^{-2\lambda(t,r)} \omega^{\hat\phi} \wedge \omega^{\hat r} \end{bmatrix} \\
 & AS & & \frac{\left(1 - e^{-2\lambda(t,r)}\right)}{r^2} \omega^{\hat\phi} \wedge \omega^{\hat\theta} \\
 & AS & AS &
\end{matrix} \right\}$$

11.3.3 The independent elements of the Riemann tensor in the coordinate basis

$$\begin{aligned}
R^{\hat r}{}_{\hat t\hat r\hat t} &= -\left(\ddot\lambda + \dot\lambda(\dot\lambda - \dot\nu)\right) e^{-2\nu(t,r)} + \left(\nu'' + \nu'(\nu' - \lambda')\right) e^{-2\lambda(t,r)} \\
R^{\hat\theta}{}_{\hat t\hat\theta\hat t} &= \frac{\nu'}{r} e^{-2\lambda(t,r)} \\
R^{\hat\theta}{}_{\hat t\hat\theta\hat r} &= \frac{\dot\lambda}{r} e^{-\lambda(t,r)-\nu(t,r)} \\
R^{\hat\theta}{}_{\hat r\hat\theta\hat r} &= \frac{\lambda'}{r} e^{-2\lambda(t,r)} \\
R^{\hat\phi}{}_{\hat t\hat\phi\hat t} &= \frac{\nu'}{r} e^{-2\lambda(t,r)} \\
R^{\hat\phi}{}_{\hat t\hat\phi\hat r} &= \frac{\dot\lambda}{r} e^{-\lambda(t,r)-\nu(t,r)} \\
R^{\hat\phi}{}_{\hat r\hat\phi\hat r} &= \frac{\lambda'}{r} e^{-2\lambda(t,r)} \\
R^{\hat\phi}{}_{\hat\theta\hat\phi\hat\theta} &= \frac{\left(1 - e^{-2\lambda(t,r)}\right)}{r^2}
\end{aligned}$$

11.3.4 The Ricci tensor:

$$\begin{aligned}
R_{\hat a\hat b} &= R^{\hat c}{}_{\hat a\hat c\hat b} \\
R_{\hat t\hat t} &= R^{\hat c}{}_{\hat t\hat c\hat t} = R^{\hat r}{}_{\hat t\hat r\hat t} + R^{\hat\theta}{}_{\hat t\hat\theta\hat t} + R^{\hat\phi}{}_{\hat t\hat\phi\hat t}
\end{aligned}$$

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[7] $= \frac{\dot\lambda}{r} e^{-\nu(t,r)-\lambda(t,r)} \omega^{\hat\phi} \wedge \omega^{\hat t} + \frac{\lambda'}{r} e^{-2\lambda(t,r)} \omega^{\hat\phi} \wedge \omega^{\hat r} + \frac{1}{r^2} e^{-\lambda(t,r)} \cot\theta\, \omega^{\hat\theta} \wedge \omega^{\hat\phi} + \frac{\cot\theta}{r} \omega^{\hat\phi} \wedge \frac{1}{r} e^{-\lambda(r)} \omega^{\hat\theta} =$

$$= -\left(\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\nu})\right)e^{-2\nu(t,r)} + \left(\nu'' + \nu'(\nu' - \lambda') + 2\frac{\nu'}{r}\right)e^{-2\lambda(t,r)}$$

$$R_{\hat{r}\hat{t}} = R^{\hat{c}}{}_{\hat{r}\hat{c}\hat{t}} = R^{\hat{t}}{}_{\hat{r}\hat{t}\hat{t}} + R^{\hat{r}}{}_{\hat{r}\hat{r}\hat{t}} + R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{t}} + R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{t}} = R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{t}} + R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{t}} = 2\frac{\dot{\lambda}}{r}e^{-\lambda(t,r)-\nu(t,r)}$$

$$= R_{\hat{t}\hat{r}}$$

$$R_{\hat{\theta}\hat{t}} = R^{\hat{c}}{}_{\hat{\theta}\hat{c}\hat{t}} = R^{\hat{t}}{}_{\hat{\theta}\hat{t}\hat{t}} + R^{\hat{r}}{}_{\hat{\theta}\hat{r}\hat{t}} + R^{\hat{\theta}}{}_{\hat{\theta}\hat{\theta}\hat{t}} + R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{t}} = 0$$

$$R_{\hat{\phi}\hat{t}} = R^{\hat{c}}{}_{\hat{\phi}\hat{c}\hat{t}} = R^{\hat{t}}{}_{\hat{\phi}\hat{t}\hat{t}} + R^{\hat{r}}{}_{\hat{\phi}\hat{r}\hat{t}} + R^{\hat{\theta}}{}_{\hat{\phi}\hat{\theta}\hat{t}} + R^{\hat{\phi}}{}_{\hat{\phi}\hat{\phi}\hat{t}} = 0$$

$$R_{\hat{r}\hat{r}} = R^{\hat{c}}{}_{\hat{r}\hat{c}\hat{r}} = R^{\hat{t}}{}_{\hat{r}\hat{t}\hat{r}} + R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} + R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{r}} = -R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}} + R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} + R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{r}}$$

$$= \left(\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\nu})\right)e^{-2\nu(t,r)} - \left(\nu'' + \nu'(\nu' - \lambda') - 2\frac{\lambda'}{r}\right)e^{-2\lambda(t,r)}$$

$$R_{\hat{\theta}\hat{r}} = R^{\hat{c}}{}_{\hat{\theta}\hat{c}\hat{r}} = R^{\hat{t}}{}_{\hat{\theta}\hat{t}\hat{r}} + R^{\hat{r}}{}_{\hat{\theta}\hat{r}\hat{r}} + R^{\hat{\theta}}{}_{\hat{\theta}\hat{\theta}\hat{r}} + R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{r}} = 0$$

$$R_{\hat{\phi}\hat{r}} = R^{\hat{c}}{}_{\hat{\phi}\hat{c}\hat{r}} = 0$$

$$R_{\hat{\theta}\hat{\theta}} = R^{\hat{c}}{}_{\hat{\theta}\hat{c}\hat{\theta}} = R^{\hat{t}}{}_{\hat{\theta}\hat{t}\hat{\theta}} + R^{\hat{r}}{}_{\hat{\theta}\hat{r}\hat{\theta}} + R^{\hat{\theta}}{}_{\hat{\theta}\hat{\theta}\hat{\theta}} + R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{\theta}} = -R^{\hat{\theta}}{}_{\hat{t}\hat{\theta}\hat{t}} + R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} + R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{\theta}}$$

$$= \left(-\frac{\nu'}{r} + \frac{\lambda'}{r}\right)e^{-2\lambda(t,r)} + \frac{\left(1 - e^{-2\lambda(t,r)}\right)}{r^2}$$

$$R_{\hat{\phi}\hat{\theta}} = R^{\hat{c}}{}_{\hat{\phi}\hat{c}\hat{\theta}} = 0$$

$$R_{\hat{\phi}\hat{\phi}} = R^{\hat{c}}{}_{\hat{\phi}\hat{c}\hat{\phi}} = R^{\hat{t}}{}_{\hat{\phi}\hat{t}\hat{\phi}} + R^{\hat{r}}{}_{\hat{\phi}\hat{r}\hat{\phi}} + R^{\hat{\theta}}{}_{\hat{\phi}\hat{\theta}\hat{\phi}} + R^{\hat{\phi}}{}_{\hat{\phi}\hat{\phi}\hat{\phi}} = -R^{\hat{\phi}}{}_{\hat{t}\hat{\phi}\hat{t}} + R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{r}} + R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{\theta}}$$

$$= \left(-\frac{\nu'}{r} + \frac{\lambda'}{r}\right)e^{-2\lambda(t,r)} + \frac{\left(1 - e^{-2\lambda(t,r)}\right)}{r^2}$$

Collecting the results: Summerized i an matrix where $\hat{a}$ corresponds to column and $\hat{b}$ to row

$$R_{\hat{a}\hat{b}} = \left\{\begin{matrix} \left[\begin{matrix} -\left(\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\nu})\right)e^{-2\nu(t,r)} \\ + \left(\nu'' + \nu'(\nu' - \lambda') + 2\frac{\nu'}{r}\right)e^{-2\lambda(t,r)} \end{matrix}\right] & 2\frac{\dot{\lambda}}{r}e^{-\lambda(t,r)-\nu(t,r)} & & \\ S & \left[\begin{matrix} \left(\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\nu})\right)e^{-2\nu(t,r)} \\ - \left(\nu'' + \nu'(\nu' - \lambda') - 2\frac{\lambda'}{r}\right)e^{-2\lambda(t,r)} \end{matrix}\right] & & \\ & & \left[\begin{matrix} \left(-\frac{\nu'}{r} + \frac{\lambda'}{r}\right)e^{-2\lambda(t,r)} \\ + \frac{\left(1 - e^{-2\lambda(t,r)}\right)}{r^2} \end{matrix}\right] & \\ & & & \left[\begin{matrix} \left(-\frac{\nu'}{r} + \frac{\lambda'}{r}\right)e^{-2\lambda(t,r)} \\ + \frac{\left(1 - e^{-2\lambda(t,r)}\right)}{r^2} \end{matrix}\right] \end{matrix}\right\}$$

Notice:

$$R_{\hat{t}\hat{t}} + R_{\hat{r}\hat{r}} = 2\left(\frac{\nu'}{r} + \frac{\lambda'}{r}\right)e^{-2\lambda(t,r)}$$

$$R_{\hat{\theta}\hat{\theta}} = R_{\hat{\phi}\hat{\phi}}$$

11.3.5 The Ricci scalar

$$R = \eta^{\hat{a}\hat{b}}R_{\hat{a}\hat{b}} = \eta^{\hat{t}\hat{t}}R_{\hat{t}\hat{t}} + \eta^{\hat{r}\hat{r}}R_{\hat{r}\hat{r}} + \eta^{\hat{\theta}\hat{\theta}}R_{\hat{\theta}\hat{\theta}} + \eta^{\hat{\phi}\hat{\phi}}R_{\hat{\phi}\hat{\phi}} = R_{\hat{t}\hat{t}} - R_{\hat{r}\hat{r}} - 2R_{\hat{\theta}\hat{\theta}}$$

$$= [8] - 2\left(\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\nu})\right)e^{-2\nu(t,r)} + 2\left(\nu'' + \nu'(\nu' - \lambda') + \frac{1}{r^2} + 2\left(\frac{\nu'}{r} - \frac{\lambda'}{r}\right)\right)e^{-2\lambda(t,r)} - 2\frac{1}{r^2}$$

11.3.6 The Einstein tensor

$$G_{\hat{a}\hat{b}} = R_{\hat{a}\hat{b}} - \frac{1}{2}\eta_{\hat{a}\hat{b}}R$$

$$\Rightarrow \quad G_{\hat{t}\hat{t}} = R_{\hat{t}\hat{t}} - \frac{1}{2}\eta_{\hat{t}\hat{t}}R = [9] \left(2\frac{\lambda'}{r} - \frac{1}{r^2}\right)e^{-2\lambda(t,r)} + \frac{1}{r^2}$$

$$G_{\hat{r}\hat{t}} = R_{\hat{r}\hat{t}} - \frac{1}{2}\eta_{\hat{r}\hat{t}}R = R_{\hat{r}\hat{t}} = 2\frac{\dot{\lambda}}{r}e^{-\lambda(t,r)-\nu(t,r)} = G_{\hat{t}\hat{r}}$$

$$G_{\hat{r}\hat{r}} = R_{\hat{r}\hat{r}} - \frac{1}{2}\eta_{\hat{r}\hat{r}}R = R_{\hat{r}\hat{r}} + \frac{1}{2}R = [10] \left(\frac{1}{r^2} + 2\frac{\nu'}{r}\right)e^{-2\lambda(t,r)} - \frac{1}{r^2}$$

$$G_{\hat{\theta}\hat{\theta}} = R_{\hat{\theta}\hat{\theta}} - \frac{1}{2}\eta_{\hat{\theta}\hat{\theta}}R$$

$$= R_{\hat{\theta}\hat{\theta}} + \frac{1}{2}R$$

$$= [11] - \left(\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\nu})\right)e^{-2\nu(t,r)} + \left(\nu'' + \nu'(\nu' - \lambda') + \left(\frac{\nu'}{r} - \frac{\lambda'}{r}\right)\right)e^{-2\lambda(t,r)}$$

$$= G_{\hat{\phi}\hat{\phi}}$$

Collecting the results: Summerized i an matrix where $\hat{a}$ corresponds to column and $\hat{b}$ to row

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[8] $= -\left(\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\nu})\right)e^{-2\nu(t,r)} + \left(\nu'' + \nu'(\nu' - \lambda') + 2\frac{\nu'}{r}\right)e^{-2\lambda(t,r)} - \left(\left(\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\nu})\right)e^{-2\nu(t,r)} - \left(\nu'' + \nu'(\nu' - \lambda') - 2\frac{\lambda'}{r}\right)e^{-2\lambda(t,r)}\right) - 2\left(\left(-\frac{\nu'}{r} + \frac{\lambda'}{r}\right)e^{-2\lambda(t,r)} + \frac{(1 - e^{-2\lambda(t,r)})}{r^2}\right) =$

[9] $= -\left(\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\nu})\right)e^{-2\nu(t,r)} + \left(\nu'' + \nu'(\nu' - \lambda') + 2\frac{\nu'}{r}\right)e^{-2\lambda(t,r)} - \frac{1}{2}\left(-2\left(\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\nu})\right)e^{-2\nu(t,r)} + 2\left(\nu'' + \nu'(\nu' - \lambda') + \frac{1}{r^2} + 2\left(\frac{\nu'}{r} - \frac{\lambda'}{r}\right)\right)e^{-2\lambda(t,r)} - 2\frac{1}{r^2}\right) = \left(\nu'' + \nu'(\nu' - \lambda') + 2\frac{\nu'}{r}\right)e^{-2\lambda(t,r)} - \left(\nu'' + \nu'(\nu' - \lambda') + \frac{1}{r^2} + 2\left(\frac{\nu'}{r} - \frac{\lambda'}{r}\right)\right)e^{-2\lambda(t,r)} + \frac{1}{r^2} =$

[10] $= \left(\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\nu})\right)e^{-2\nu(t,r)} - \left(\nu'' + \nu'(\nu' - \lambda') - 2\frac{\lambda'}{r}\right)e^{-2\lambda(t,r)} + \frac{1}{2}\left(-2\left(\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\nu})\right)e^{-2\nu(t,r)} + 2\left(\nu'' + \nu'(\nu' - \lambda') + \frac{1}{r^2} + 2\left(\frac{\nu'}{r} - \frac{\lambda'}{r}\right)\right)e^{-2\lambda(t,r)} - 2\frac{1}{r^2}\right) = -\left(\nu'' + \nu'(\nu' - \lambda') - 2\frac{\lambda'}{r}\right)e^{-2\lambda(t,r)} + \left(\nu'' + \nu'(\nu' - \lambda') + \frac{1}{r^2} + 2\left(\frac{\nu'}{r} - \frac{\lambda'}{r}\right)\right)e^{-2\lambda(t,r)} - \frac{1}{r^2} =$

[11] $= \left(-\frac{\nu'}{r} + \frac{\lambda'}{r}\right)e^{-2\lambda(t,r)} + \frac{(1 - e^{-2\lambda(t,r)})}{r^2} + \frac{1}{2}\left(-2\left(\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\nu})\right)e^{-2\nu(t,r)} + 2\left(\nu'' + \nu'(\nu' - \lambda') + \frac{1}{r^2} + 2\left(\frac{\nu'}{r} - \frac{\lambda'}{r}\right)\right)e^{-2\lambda(t,r)} - 2\frac{1}{r^2}\right) =$

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$$G_{\hat{a}\hat{b}} = \begin{Bmatrix} \left[\begin{matrix}\left(2\frac{\lambda'}{r}-\frac{1}{r^2}\right)e^{-2\lambda(t,r)} \\ +\frac{1}{r^2}\end{matrix}\right] & 2\frac{\dot{\lambda}}{r}e^{-\lambda(t,r)-\nu(t,r)} & & \\ S & \left[\begin{matrix}\left(\frac{1}{r^2}+2\frac{\nu'}{r}\right)e^{-2\lambda(t,r)} \\ -\frac{1}{r^2}\end{matrix}\right] & & \\ & & \left[\begin{matrix}-\left(\ddot{\lambda}+\dot{\lambda}\left(\dot{\lambda}-\dot{\nu}\right)\right)e^{-2\nu(t,r)} \\ +\left(\nu''+\nu'(\nu'-\lambda')+\left(\frac{\nu'}{r}-\frac{\lambda'}{r}\right)\right)e^{-2\lambda(t,r)}\end{matrix}\right] & \\ & & & \left[\begin{matrix}-\left(\ddot{\lambda}+\dot{\lambda}\left(\dot{\lambda}-\dot{\nu}\right)\right)e^{-2\nu(t,r)} \\ +\left(\nu''+\nu'(\nu'-\lambda')+\left(\frac{\nu'}{r}-\frac{\lambda'}{r}\right)\right)e^{-2\lambda(t,r)}\end{matrix}\right] \end{Bmatrix}$$

11.3.7 The Ricci and Einstein tensor in the coordinate basis:

The Ricci tensor

$$R_{ab} = \Lambda^{\hat{c}}{}_a \Lambda^{\hat{d}}{}_b R_{\hat{c}\hat{d}}$$

$$\Lambda^{\hat{c}}{}_a = \begin{Bmatrix} e^{\nu(t,r)} & & & \\ & e^{\lambda(t,r)} & & \\ & & r & \\ & & & r\sin\theta \end{Bmatrix}$$

$$\Rightarrow \quad R_{tt} = \Lambda^{\hat{c}}{}_t \Lambda^{\hat{d}}{}_t R_{\hat{c}\hat{d}} = \left(\Lambda^{\hat{t}}{}_t\right)^2 R_{\hat{t}\hat{t}}$$

$$= e^{2\nu(t,r)}\left(-\left(\ddot{\lambda}+\dot{\lambda}\left(\dot{\lambda}-\dot{\nu}\right)\right)e^{-2\nu(t,r)} + \left(\nu''+\nu'(\nu'-\lambda')+2\frac{\nu'}{r}\right)e^{-2\lambda(t,r)}\right)$$

$$= -\left(\ddot{\lambda}+\dot{\lambda}\left(\dot{\lambda}-\dot{\nu}\right)\right) + \left(\nu''+\nu'(\nu'-\lambda')+2\frac{\nu'}{r}\right)e^{2\nu(t,r)-2\lambda(t,r)}$$

$$R_{rt} = \Lambda^{\hat{c}}{}_r \Lambda^{\hat{d}}{}_t R_{\hat{c}\hat{d}} = \Lambda^{\hat{r}}{}_r \Lambda^{\hat{t}}{}_t R_{\hat{r}\hat{t}} = e^{\lambda(t,r)}e^{\nu(t,r)}2\frac{\dot{\lambda}}{r}e^{-\lambda(t,r)-\nu(t,r)} = 2\frac{\dot{\lambda}}{r}$$

$$R_{rr} = \Lambda^{\hat{c}}{}_r \Lambda^{\hat{d}}{}_r R_{\hat{c}\hat{d}} = \left(\Lambda^{\hat{r}}{}_r\right)^2 R_{\hat{r}\hat{r}}$$

$$= e^{2\lambda(t,r)}\left(\left(\ddot{\lambda}+\dot{\lambda}\left(\dot{\lambda}-\dot{\nu}\right)\right)e^{-2\nu(t,r)} - \left(\nu''+\nu'(\nu'-\lambda')-2\frac{\lambda'}{r}\right)e^{-2\lambda(t,r)}\right)$$

$$= \left(\ddot{\lambda}+\dot{\lambda}\left(\dot{\lambda}-\dot{\nu}\right)\right)e^{-2\nu(t,r)+2\lambda(t,r)} - \left(\nu''+\nu'(\nu'-\lambda')-2\frac{\lambda'}{r}\right)$$

$$R_{\theta\theta} = \Lambda^{\hat{c}}{}_\theta \Lambda^{\hat{d}}{}_\theta R_{\hat{c}\hat{d}} = \left(\Lambda^{\hat{\theta}}{}_\theta\right)^2 R_{\hat{\theta}\hat{\theta}}$$

$$= r^2\left(\left(-\frac{\nu'}{r}+\frac{\lambda'}{r}\right)e^{-2\lambda(t,r)} + \frac{\left(1-e^{-2\lambda(t,r)}\right)}{r^2}\right)$$

$$= \left((-\nu'+\lambda')r-1\right)e^{-2\lambda(t,r)} + 1$$

$$R_{\phi\phi} = \Lambda^{\hat{c}}{}_\phi \Lambda^{\hat{d}}{}_\phi R_{\hat{c}\hat{d}} = \left(\Lambda^{\hat{\phi}}{}_\phi\right)^2 R_{\hat{\phi}\hat{\phi}}$$

$$= r^2\sin^2\theta\left(\left(-\frac{\nu'}{r}+\frac{\lambda'}{r}\right)e^{-2\lambda(t,r)} + \frac{\left(1-e^{-2\lambda(t,r)}\right)}{r^2}\right)$$

$$= \left(\left((-\nu'+\lambda')r-1\right)e^{-2\lambda(t,r)} + 1\right)\sin^2\theta$$

Collecting the results:

$$R_{tt} = -\left(\ddot{\lambda}+\dot{\lambda}\left(\dot{\lambda}-\dot{\nu}\right)\right) + \left(\nu''+\nu'(\nu'-\lambda')+2\frac{\nu'}{r}\right)e^{2\nu(t,r)-2\lambda(t,r)}$$

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$$R_{rt} = R_{tr} = 2\frac{\dot{\lambda}}{r}$$
$$R_{rr} = \left(\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\nu})\right)e^{-2\nu(t,r)+2\lambda(t,r)} - \left(\nu'' + \nu'(\nu' - \lambda') - 2\frac{\lambda'}{r}\right)$$
$$R_{\theta\theta} = ((-\nu' + \lambda')r - 1)e^{-2\lambda(t,r)} + 1$$
$$R_{\phi\phi} = \left(((-\nu' + \lambda')r - 1)e^{-2\lambda(t,r)} + 1\right)\sin^2\theta = R_{\theta\theta}\sin^2\theta$$

11.3.8 The Einstein tensor

$$G_{ab} = \Lambda^{\hat{c}}{}_a \Lambda^{\hat{d}}{}_b G_{\hat{c}\hat{d}}$$
$$\Lambda^{\hat{c}}{}_a = \begin{Bmatrix} e^{\nu(t,r)} & & & \\ & e^{\lambda(t,r)} & & \\ & & r & \\ & & & r\sin\theta \end{Bmatrix}$$
$$\Rightarrow \quad G_{tt} = \Lambda^{\hat{c}}{}_t \Lambda^{\hat{d}}{}_t G_{\hat{c}\hat{d}} = \left(\Lambda^{\hat{t}}{}_t\right)^2 G_{\hat{t}\hat{t}} = e^{2\nu(t,r)}\left(\left(2\frac{\lambda'}{r} - \frac{1}{r^2}\right)e^{-2\lambda(t,r)} + \frac{1}{r^2}\right)$$
$$G_{rt} = \Lambda^{\hat{c}}{}_r \Lambda^{\hat{d}}{}_t G_{\hat{c}\hat{d}} = \Lambda^{\hat{r}}{}_r \Lambda^{\hat{t}}{}_t G_{\hat{r}\hat{t}} = e^{\lambda(t,r)}e^{\nu(t,r)}2\frac{\dot{\lambda}}{r}e^{-\lambda(t,r)-\nu(t,r)} = 2\frac{\dot{\lambda}}{r}$$
$$G_{rr} = \Lambda^{\hat{c}}{}_r \Lambda^{\hat{d}}{}_r G_{\hat{c}\hat{d}} = \left(\Lambda^{\hat{r}}{}_r\right)^2 G_{\hat{r}\hat{r}}$$
$$= e^{2\lambda(t,r)}\left(\left(\frac{1}{r^2} + 2\frac{\nu'}{r}\right)e^{-2\lambda(t,r)} - \frac{1}{r^2}\right) = \left(\frac{1}{r^2} + 2\frac{\nu'}{r}\right) - \frac{e^{2\lambda(t,r)}}{r^2}$$
$$G_{\theta\theta} = \Lambda^{\hat{c}}{}_\theta \Lambda^{\hat{d}}{}_\theta G_{\hat{c}\hat{d}} = \left(\Lambda^{\hat{\theta}}{}_\theta\right)^2 G_{\hat{\theta}\hat{\theta}}$$
$$= r^2\left[-\left(\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\nu})\right)e^{-2\nu(t,r)} + \left(\nu'' + \nu'(\nu' - \lambda') + \left(\frac{\nu'}{r} - \frac{\lambda'}{r}\right)\right)e^{-2\lambda(t,r)}\right]$$
$$G_{\phi\phi} = \Lambda^{\hat{c}}{}_\phi \Lambda^{\hat{d}}{}_\phi G_{\hat{c}\hat{d}} = \left(\Lambda^{\hat{\phi}}{}_\phi\right)^2 G_{\hat{\phi}\hat{\phi}}$$
$$= r^2\sin^2\theta\left[-\left(\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\nu})\right)e^{-2\nu(t,r)} + \left(\nu'' + \nu'(\nu' - \lambda') + \left(\frac{\nu'}{r} - \frac{\lambda'}{r}\right)\right)e^{-2\lambda(t,r)}\right]$$

Collecting the results:

$$G_{tt} = e^{2\nu(t,r)}\left(\left(2\frac{\lambda'}{r} - \frac{1}{r^2}\right)e^{-2\lambda(t,r)} + \frac{1}{r^2}\right)$$
$$G_{rt} = G_{tr} = 2\frac{\dot{\lambda}}{r}$$
$$G_{rr} = \frac{1}{r^2} + 2\frac{\nu'}{r} - \frac{e^{2\lambda(t,r)}}{r^2}$$
$$G_{\theta\theta} = r^2\left[-\left(\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\nu})\right)e^{-2\nu(t,r)} + \left(\nu'' + \nu'(\nu' - \lambda') + \left(\frac{\nu'}{r} - \frac{\lambda'}{r}\right)\right)e^{-2\lambda(t,r)}\right]$$
$$G_{\phi\phi} = \sin^2\theta\, G_{\theta\theta}$$

11.3.9 The Einstein tensor of second kind

$$G^a_{\ b} = g^{ac}G_{bc}$$

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$$g^{ab} = \begin{Bmatrix} e^{-2\nu(t,r)} & & & \\ & -e^{-2\lambda(t,r)} & & \\ & & -\frac{1}{r^2} & \\ & & & -\frac{1}{r^2\sin^2\theta} \end{Bmatrix}$$

$\Rightarrow$

$$G^t{}_t = g^{tc}G_{tc} = g^{tt}G_{tt} = e^{-2\nu(t,r)}e^{2\nu(t,r)}\left(\left(2\frac{\lambda'}{r}-\frac{1}{r^2}\right)e^{-2\lambda(t,r)}+\frac{1}{r^2}\right)$$

$$= \left(2\frac{\lambda'}{r}-\frac{1}{r^2}\right)e^{-2\lambda(t,r)}+\frac{1}{r^2}$$

$$G^t{}_r = g^{tc}G_{rc} = g^{tt}G_{rt} = e^{-2\nu(t,r)}2\frac{\dot{\lambda}}{r}$$

$$G^r{}_t = g^{rc}G_{tc} = g^{rr}G_{tr} = -e^{-2\lambda(t,r)}2\frac{\dot{\lambda}}{r}$$

$$G^r{}_r = g^{rr}G_{rr} = -e^{-2\lambda(t,r)}\left(\left(\frac{1}{r^2}+2\frac{\nu'}{r}\right)-\frac{e^{2\lambda(t,r)}}{r^2}\right) = \frac{1}{r^2}-e^{-2\lambda(t,r)}\left(\frac{1}{r^2}+2\frac{\nu'}{r}\right)$$

$$G^\theta{}_\theta = g^{\theta\theta}G_{\theta\theta}$$

$$= -\frac{1}{r^2}r^2\left[-\left(\ddot{\lambda}+\dot{\lambda}(\dot{\lambda}-\dot{\nu})\right)e^{-2\nu(t,r)}+\left(\nu''+\nu'(\nu'-\lambda')+\left(\frac{\nu'}{r}-\frac{\lambda'}{r}\right)\right)e^{-2\lambda(t,r)}\right]$$

$$= \left(\ddot{\lambda}+\dot{\lambda}(\dot{\lambda}-\dot{\nu})\right)e^{-2\nu(t,r)}-\left(\nu''+\nu'(\nu'-\lambda')+\left(\frac{\nu'}{r}-\frac{\lambda'}{r}\right)\right)e^{-2\lambda(t,r)}$$

$$G^\phi{}_\phi = g^{\phi\phi}G_{\phi\phi}$$

$$= [12]\left(\ddot{\lambda}+\dot{\lambda}(\dot{\lambda}-\dot{\nu})\right)e^{-2\nu(t,r)}-\left(\nu''+\nu'(\nu'-\lambda')+\left(\frac{\nu'}{r}-\frac{\lambda'}{r}\right)\right)e^{-2\lambda(t,r)}$$

Collecting the results

$$G^t{}_t = \left(2\frac{\lambda'}{r}-\frac{1}{r^2}\right)e^{-2\lambda(t,r)}+\frac{1}{r^2}$$

$$G^t{}_r = e^{-2\nu(t,r)}2\frac{\dot{\lambda}}{r}$$

$$G^r{}_t = -e^{-2\lambda(t,r)}2\frac{\dot{\lambda}}{r}$$

$$G^r{}_r = \frac{1}{r^2}-e^{-2\lambda(t,r)}\left(\frac{1}{r^2}+2\frac{\nu'}{r}\right)$$

$$G^\theta{}_\theta = \left(\ddot{\lambda}+\dot{\lambda}(\dot{\lambda}-\dot{\nu})\right)e^{-2\nu(t,r)}-\left(\nu''+\nu'(\nu'-\lambda')+\left(\frac{\nu'}{r}-\frac{\lambda'}{r}\right)\right)e^{-2\lambda(t,r)}$$

$$G^\phi{}_\phi = G^\theta{}_\theta$$

[12] $= -\frac{1}{r^2\sin^2\theta}r^2\sin^2\theta\left[-\left(\ddot{\lambda}+\dot{\lambda}(\dot{\lambda}-\dot{\nu})\right)e^{-2\nu(t,r)}+\left(\nu''+\nu'(\nu'-\lambda')+\left(\frac{\nu'}{r}-\frac{\lambda'}{r}\right)\right)e^{-2\lambda(t,r)}\right] =$

11.4 Other Examples

11.4.1 [m]Calculation of the scalar $R_{abcd}R^{abcd}$ in the Schwarzschild metric

$$
\begin{aligned}
R_{abcd}R^{abcd} &= R_{\hat{r}\hat{t}\hat{r}\hat{t}}R^{\hat{r}\hat{t}\hat{r}\hat{t}} + R_{\hat{r}\hat{t}\hat{t}\hat{r}}R^{\hat{r}\hat{t}\hat{t}\hat{r}} + R_{\hat{t}\hat{r}\hat{t}\hat{r}}R^{\hat{t}\hat{r}\hat{t}\hat{r}} + R_{\hat{t}\hat{r}\hat{r}\hat{t}}R^{\hat{t}\hat{r}\hat{r}\hat{t}} + R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}}R^{\hat{\theta}\hat{t}\hat{\theta}\hat{t}} + R_{\hat{\theta}\hat{t}\hat{t}\hat{\theta}}R^{\hat{\theta}\hat{t}\hat{t}\hat{\theta}} \\
&\quad + R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}}R^{\hat{t}\hat{\theta}\hat{t}\hat{\theta}} + R_{\hat{t}\hat{\theta}\hat{\theta}\hat{t}}R^{\hat{t}\hat{\theta}\hat{\theta}\hat{t}} + R_{\hat{\theta}\hat{r}\hat{\theta}\hat{r}}R^{\hat{\theta}\hat{r}\hat{\theta}\hat{r}} + R_{\hat{\theta}\hat{r}\hat{r}\hat{\theta}}R^{\hat{\theta}\hat{r}\hat{r}\hat{\theta}} \\
&\quad + R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}}R^{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} + R_{\hat{r}\hat{\theta}\hat{\theta}\hat{r}}R^{\hat{r}\hat{\theta}\hat{\theta}\hat{r}} + R_{\hat{\phi}\hat{t}\hat{\phi}\hat{t}}R^{\hat{\phi}\hat{t}\hat{\phi}\hat{t}} + R_{\hat{\phi}\hat{t}\hat{t}\hat{\phi}}R^{\hat{\phi}\hat{t}\hat{t}\hat{\phi}} \\
&\quad + R_{\hat{t}\hat{\phi}\hat{t}\hat{\phi}}R^{\hat{t}\hat{\phi}\hat{t}\hat{\phi}} + R_{\hat{t}\hat{\phi}\hat{\phi}\hat{t}}R^{\hat{t}\hat{\phi}\hat{\phi}\hat{t}} + R_{\hat{\phi}\hat{r}\hat{\phi}\hat{r}}R^{\hat{\phi}\hat{r}\hat{\phi}\hat{r}} + R_{\hat{\phi}\hat{r}\hat{r}\hat{\phi}}R^{\hat{\phi}\hat{r}\hat{r}\hat{\phi}} \\
&\quad + R_{\hat{r}\hat{\phi}\hat{r}\hat{\phi}}R^{\hat{r}\hat{\phi}\hat{r}\hat{\phi}} + R_{\hat{r}\hat{\phi}\hat{\phi}\hat{r}}R^{\hat{r}\hat{\phi}\hat{\phi}\hat{r}} + R_{\hat{\phi}\hat{\theta}\hat{\phi}\hat{\theta}}R^{\hat{\phi}\hat{\theta}\hat{\phi}\hat{\theta}} + R_{\hat{\phi}\hat{\theta}\hat{\theta}\hat{\phi}}R^{\hat{\phi}\hat{\theta}\hat{\theta}\hat{\phi}} \\
&\quad + R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}}R^{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} + R_{\hat{\theta}\hat{\phi}\hat{\phi}\hat{\theta}}R^{\hat{\theta}\hat{\phi}\hat{\phi}\hat{\theta}} \\
&= \left(\frac{2m}{r^3}\right)\left(\frac{2m}{r^3}\right) + \left(-\frac{2m}{r^3}\right)\left(-\frac{2m}{r^3}\right) + \left(\frac{2m}{r^3}\right)\left(\frac{2m}{r^3}\right) + \left(-\frac{2m}{r^3}\right)\left(-\frac{2m}{r^3}\right) \\
&\quad + \left(-\frac{m}{r^3}\right)\left(-\frac{m}{r^3}\right) + \left(\frac{m}{r^3}\right)\left(\frac{m}{r^3}\right) + \left(\frac{m}{r^3}\right)\left(\frac{m}{r^3}\right) + \left(-\frac{m}{r^3}\right)\left(-\frac{m}{r^3}\right) \\
&\quad + \left(\frac{m}{r^3}\right)\left(\frac{m}{r^3}\right) + \left(-\frac{m}{r^3}\right)\left(-\frac{m}{r^3}\right) + \left(\frac{m}{r^3}\right)\left(\frac{m}{r^3}\right) + \left(-\frac{m}{r^3}\right)\left(-\frac{m}{r^3}\right) \\
&\quad + \left(-\frac{m}{r^3}\right)\left(-\frac{m}{r^3}\right) + \left(\frac{m}{r^3}\right)\left(\frac{m}{r^3}\right) + \left(-\frac{m}{r^3}\right)\left(-\frac{m}{r^3}\right) + \left(\frac{m}{r^3}\right)\left(\frac{m}{r^3}\right) \\
&\quad + \left(\frac{m}{r^3}\right)\left(\frac{m}{r^3}\right) + \left(-\frac{m}{r^3}\right)\left(-\frac{m}{r^3}\right) + \left(\frac{m}{r^3}\right)\left(\frac{m}{r^3}\right) + \left(-\frac{m}{r^3}\right)\left(-\frac{m}{r^3}\right) \\
&\quad + \left(-\frac{2m}{r^3}\right)\left(-\frac{2m}{r^3}\right) + \left(\frac{2m}{r^3}\right)\left(\frac{2m}{r^3}\right) + \left(-\frac{2m}{r^3}\right)\left(-\frac{2m}{r^3}\right) + \left(\frac{2m}{r^3}\right)\left(\frac{2m}{r^3}\right) \\
&= \frac{48m^2}{r^6}
\end{aligned}
$$

11.4.2 [n]Geodesics and Christoffel symbols of the Schwarzschild metric with $\theta = \frac{\pi}{2}$

The line element

$$ds^2 = -\left(1-\frac{2m}{r}\right)dt^2 + \left(1-\frac{2m}{r}\right)^{-1} dr^2 + r^2 d\phi^2$$

To find the geodesic we use the Euler-Lagrange equation:

$$0 = \frac{d}{ds}\left(\frac{\partial F}{\partial \dot{x}^a}\right) - \frac{\partial F}{\partial x^a}$$

where

$$F = -\left(1-\frac{2m}{r}\right)\dot{t}^2 + \left(1-\frac{2m}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\phi}^2$$

<u>$x^a = t$:</u>

$$
\begin{aligned}
\frac{\partial F}{\partial t} &= 0 \\
\frac{\partial F}{\partial \dot{t}} &= -2\left(1-\frac{2m}{r}\right)\dot{t} \\
\frac{d}{ds}\left(\frac{\partial F}{\partial \dot{t}}\right) &= -\frac{4m}{r^2}\dot{r}\dot{t} - 2\left(1-\frac{2m}{r}\right)\ddot{t} \\
\Rightarrow \qquad 0 &= \frac{4m}{r^2}\dot{r}\dot{t} + 2\left(1-\frac{2m}{r}\right)\ddot{t} \\
\Leftrightarrow \qquad 0 &= \ddot{t} + \frac{2m}{r(r-2m)}\dot{r}\dot{t}
\end{aligned}
$$

<u>$x^a = r$:</u>

$$\begin{aligned}
\frac{\partial F}{\partial r} &= {}^{13} -\frac{2m}{r^2}\dot{t}^2 - \frac{2m}{r^2}\left(1-\frac{2m}{r}\right)^{-2}\dot{r}^2 + 2r\dot{\phi}^2 \\
\frac{\partial F}{\partial \dot{r}} &= 2\left(1-\frac{2m}{r}\right)^{-1}\dot{r} \\
\frac{d}{ds}\left(\frac{\partial F}{\partial \dot{r}}\right) &= 2\left(1-\frac{2m}{r}\right)^{-1}\ddot{r} - \frac{4m}{r^2}\left(1-\frac{2m}{r}\right)^{-2}\dot{r}^2 \\
\Rightarrow \qquad 0 &= 2\left(1-\frac{2m}{r}\right)^{-1}\ddot{r} - \frac{2m}{r^2}\left(1-\frac{2m}{r}\right)^{-2}\dot{r}^2 + \frac{2m}{r^2}\dot{t}^2 - 2r\dot{\phi}^2 \\
\Leftrightarrow \qquad 0 &= \ddot{r} - \frac{m}{r(r-2m)}\dot{r}^2 + (r-2m)\frac{m}{r^3}\dot{t}^2 - (r-2m)\dot{\phi}^2
\end{aligned}$$

<u>$x^a = \phi$:</u>

$$\begin{aligned}
\frac{\partial F}{\partial \phi} &= 0 \\
\frac{\partial F}{\partial \dot{\phi}} &= 2r^2\dot{\phi} \\
\frac{d}{ds}\left(\frac{\partial F}{\partial \dot{\phi}}\right) &= 4r\dot{r}\dot{\phi} + 2r^2\ddot{\phi} \\
\Rightarrow \qquad 0 &= 4r\dot{r}\dot{\phi} + 2r^2\ddot{\phi} \\
\Leftrightarrow \qquad 0 &= \ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi}
\end{aligned}$$

Collecting the results

$$\begin{aligned}
0 &= \ddot{t} + \frac{2m}{r(r-2m)}\dot{r}\dot{t} \\
0 &= \ddot{r} - \frac{m}{r(r-2m)}\dot{r}^2 + (r-2m)\frac{m}{r^3}\dot{t}^2 - (r-2m)\dot{\phi}^2 \\
0 &= \ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi}
\end{aligned}$$

We can now find the Christoffel symbols:

$$\begin{aligned}
\Gamma^t{}_{rt} &= \frac{m}{r(r-2m)} \\
\Gamma^r{}_{rr} &= -\frac{m}{r(r-2m)} \\
\Gamma^r{}_{tt} &= \frac{m(r-2m)}{r^3} \\
\Gamma^r{}_{\phi\phi} &= -(r-2m) \\
\Gamma^\phi{}_{r\phi} &= \Gamma^\phi{}_{\phi r} = \frac{1}{r}
\end{aligned}$$

11.4.3 The general Schwarzschild metric with nonzero cosmological constant.

11.4.3.1 °The Ricci rotation coefficients and Ricci tensor for the Schwarzschild metric with nonzero cosmological constant.

The line element

$$ds^2 = f(r)dt^2 - \frac{1}{f(r)}dr^2 - r^2 d\theta^2 - r^2\sin^2\theta\, d\phi^2$$

[13] $\frac{\partial}{\partial t}\left(\left(1-\frac{2m}{r}\right)^{-1}\right) = -\frac{2m}{r^2}\left(1-\frac{2m}{r}\right)^{-2} = -\frac{2m}{(r-2m)^2}$

$$f(r) = 1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2$$

Now we can compare with the line element of the Schwarzschild metric with zero cosmological constant, where the primes should not be mistaken for the derivative d/dr.

$$ds^2 = e^{2\nu(r')}dt'^2 - e^{2\lambda(r')}dr'^2 - r'^2 d\theta'^2 - r'^2 \sin^2\theta'\, d\phi'^2$$

And choose:

$$\begin{aligned} e^{\nu(r')}dt' &= \sqrt{f(r)}dt \\ e^{\lambda(r')}dr' &= \frac{1}{\sqrt{f(r)}}dr \\ r'd\theta' &= rd\theta \\ r'\sin\theta'\, d\phi' &= r\sin\theta\, d\phi \end{aligned}$$

Comparing the two metrics we see: $\phi' = \phi, \theta' = \theta, r' = r, e^{\nu(r')} = \sqrt{f(r)}, \nu = -\lambda, t' = t$

Next we can use the former calculations of the Schwarzschild metric with zero cosmological constant to find the Ricci rotation coefficients and the Ricci tensor for the Schwarzschild metric with non-zero cosmological constant.

$$\begin{aligned} \Gamma^{\hat{t}}{}_{\hat{r}\hat{t}} &= \Gamma^{\hat{r}}{}_{\hat{t}\hat{t}} = \frac{d\nu(r')}{dr'}e^{-\lambda(r')} = \frac{1}{2f(r)}\frac{df(r)}{dr}\sqrt{f(r)} = \frac{1}{2\sqrt{f(r)}}\frac{df(r)}{dr} \\ &=^{14} \frac{\frac{2m}{r^2} - \frac{2}{3}\Lambda r}{2\sqrt{1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2}} = \frac{3m - \Lambda r^3}{r^2\sqrt{9 - \frac{18m}{r} - 3\Lambda r^2}} \\ &= \frac{3m - \Lambda r^3}{r^{3/2}\sqrt{9r - 18m - 3\Lambda r^3}} \\ \Gamma^{\hat{r}}{}_{\hat{\theta}\hat{\theta}} &= \Gamma^{\hat{r}}{}_{\hat{\phi}\hat{\phi}} = -\Gamma^{\hat{\theta}}{}_{\hat{r}\hat{\theta}} = -\Gamma^{\hat{\phi}}{}_{\hat{r}\hat{\phi}} = -\frac{1}{r'}e^{-\lambda(r')} = -\frac{1}{r}\sqrt{f(r)} \\ &= -\frac{1}{r}\sqrt{1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2} \\ \Gamma^{\hat{\theta}}{}_{\hat{\phi}\hat{\phi}} &= -\Gamma^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}} = -\frac{\cot\theta'}{r'} = -\frac{\cot\theta}{r} \end{aligned}$$

The Ricci tensor

We have $R_{\hat{a}\hat{b}} = \eta_{\hat{a}\hat{b}}\Lambda$ valid in vacuum systems with a cosmological constant, from which we immediately can see that $(\eta_{\hat{a}\hat{b}} = diag(1,-1,-1,-1))$ $R_{\hat{t}\hat{t}} = -R_{\hat{r}\hat{r}} = -R_{\hat{\theta}\hat{\theta}} = -R_{\hat{\phi}\hat{\phi}} = \Lambda$

11.4.3.2 ᵖThe general Schwarzschild metric in vacuum with a cosmological constant: The Ricci scalar

The line element

$$ds^2 = e^{2\nu(r)}dt^2 - e^{2\lambda(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

In this case we can write the Einstein equation in the local frame (non-coordinate basis):

$$\begin{aligned} 0 &= R_{\hat{a}\hat{b}} - \frac{1}{2}\eta_{\hat{a}\hat{b}}R + \eta_{\hat{a}\hat{b}}\Lambda \\ \Rightarrow \quad 0 &= \eta^{\hat{a}\hat{b}}R_{\hat{a}\hat{b}} - \frac{1}{2}\eta^{\hat{a}\hat{b}}\eta_{\hat{a}\hat{b}}R + \eta^{\hat{a}\hat{b}}\eta_{\hat{a}\hat{b}}\Lambda = R - \frac{1}{2}4R + 4\Lambda \\ \Leftrightarrow \quad R &= 4\Lambda \end{aligned}$$

11.4.3.3 ᑫThe general Schwarzschild metric in vacuum with a cosmological constant: Integration constants

It can be shown that[15]

[14] Multiply by $\frac{\frac{3r^2}{2}}{\frac{3r^2}{2}}$

[15] From the vacuum equations

$$\begin{aligned} \lambda(r) &= \ln k - \nu(r) \\ \Rightarrow \quad \lambda'(r) &= -\nu'(r) \\ e^{-2\lambda(r)} &= \frac{1}{k^2} e^{2\nu(r)} \end{aligned}$$

We need

$$\begin{aligned} \left(re^{2\nu(r)}\right)' &= e^{2\nu(r)} + 2r\nu'(r)e^{2\nu(r)} \\ \Leftrightarrow \quad \nu'(r) &= \frac{\left(re^{2\nu(r)}\right)'}{2re^{2\nu(r)}} - \frac{1}{2r} \end{aligned}$$

We use the Ricci tensor in the non-coordinate basis for the coordinate $\hat{\theta}$

$$\begin{aligned} R_{\hat{\theta}\hat{\theta}} &= -\frac{\nu'}{r} e^{-2\lambda(r)} + \frac{\lambda'}{r} e^{-2\lambda(r)} + \frac{\left(1 - e^{-2\lambda(r)}\right)}{r^2} \\ \Rightarrow \quad \Lambda &= \frac{\nu'}{r} e^{-2\lambda(r)} - \frac{\lambda'}{r} e^{-2\lambda(r)} - \frac{\left(1 - e^{-2\lambda(r)}\right)}{r^2} = 2\frac{\nu'}{k^2 r} e^{2\nu(r)} - \frac{1}{r^2} + \frac{e^{2\nu(r)}}{k^2 r^2} \\ &= 2\left(\frac{\left(re^{2\nu(r)}\right)'}{2re^{2\nu(r)}} - \frac{1}{2r}\right)\frac{e^{2\nu(r)}}{k^2 r} - \frac{1}{r^2} + \frac{e^{2\nu(r)}}{k^2 r^2} \end{aligned}$$

Renaming

$$\begin{aligned} g(r) &= re^{2\nu(r)} \\ \Rightarrow \quad \Lambda &= \left(\frac{g'(r)}{g(r)} - \frac{1}{r}\right)\frac{g(r)}{k^2 r^2} - \frac{1}{r^2} + \frac{g(r)}{k^2 r^3} \\ \Leftrightarrow \quad g'(r) &= -k^2 + k^2 \Lambda r^2 \end{aligned}$$

We guess the solution (polynomials with exponents higher than 3 cannot contribute):

$$\begin{aligned} g(r) &= re^{2\nu(r)} = A + Br + Cr^2 + Dr^3 \\ \Rightarrow \quad g'(r) &= B + 2Cr + 3Dr^2 \\ \Rightarrow \quad -k^2 + k^2\Lambda r^2 &= B + 2Cr + 3Dr^2 \end{aligned}$$

Now comparing the coefficients we find

$$\begin{aligned} B &= -k^2 \\ C &= 0 \\ D &= \frac{1}{3} k^2 \Lambda \end{aligned}$$

and we can conclude that

$$\begin{aligned} re^{2\nu(r)} &= A - k^2 r + \frac{1}{3} k^2 \Lambda r^3 \\ \Rightarrow \quad e^{2\nu(r)} &= \frac{A}{r} - k^2 + \frac{1}{3} k^2 \Lambda r^2 \end{aligned}$$

and the line element

$$ds^2 = e^{2\nu(r)} dt^2 - e^{2\lambda(r)} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2$$

becomes

$$ds^2 = \left(\frac{A}{r} - k^2 + \frac{1}{3} k^2 \Lambda r^2\right) dt^2 - \left(\frac{A}{r} - k^2 + \frac{1}{3} k^2 \Lambda r^2\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2$$

If $k^2 = -1$ and $\Lambda = 0$ this should be identical to the ordinary Schwarzschild vacuum metric, which means that A has to be equal to: $A = -2m$

11.4.3.4 'The general Schwarzschild metric in vacuum with a cosmological constant: The spatial part of the line element.

The line element

$$ds^2 = e^{2\nu(r)} dt^2 - e^{2\lambda(r)} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2$$

can in Gaussian normal coordinates be written as

$$ds^2 = dt^2 - a^2(t) d\sigma^2$$

In this case we want to find the spatial part of the line element

$$d\sigma^2 = g_{ij} dx_i dx_j$$

and to do that we will use the method[s] where the metric is found from the Ricci-tensor:

$$R_{ij} = 2K g_{ij}$$

$$\Leftrightarrow \quad g_{ij} = \frac{1}{2K} R_{ij}$$

$$\Rightarrow \quad g_{rr} = \frac{1}{2K} R_{rr} = \frac{1}{2K} \left(\Lambda^{\hat{r}}{}_{r}\right)^2 R_{\hat{r}\hat{r}} = -\frac{1}{2K} \left(e^{\lambda(r)}\right)^2 \Lambda = -\frac{\Lambda}{2K} e^{2\lambda(r)}$$

In another chapter we found that

$$e^{-2\lambda(r)} = \frac{1}{k^2} e^{2\nu(r)} = \frac{1}{k^2} \left(\frac{A}{r} - k^2 + \frac{1}{3} k^2 \Lambda r^2\right) = \frac{A}{k^2 r} - 1 + \frac{1}{3} \Lambda r^2$$

$$\Rightarrow \quad g_{rr} = -\frac{\Lambda}{2K} \frac{1}{\frac{A}{k^2 r} - 1 + \frac{1}{3} \Lambda r^2}$$

$$g_{\theta\theta} = \frac{1}{2K} R_{\theta\theta} = \frac{1}{2K} \left(\Lambda^{\hat{\theta}}{}_{\theta}\right)^2 R_{\hat{\theta}\hat{\theta}} = -\frac{1}{2K} (r)^2 \Lambda = -\frac{\Lambda}{2K} r^2$$

$$g_{\phi\phi} = \frac{1}{2K} R_{\phi\phi} = \frac{1}{2K} \left(\Lambda^{\hat{\phi}}{}_{\phi}\right)^2 R_{\hat{\phi}\hat{\phi}} = -\frac{1}{2K} (r \sin\theta)^2 \Lambda = -\frac{\Lambda}{2K} r^2 \sin^2\theta$$

$$\Rightarrow \quad d\sigma^2 = -\frac{\Lambda}{2K} \frac{dr^2}{\frac{A}{k^2 r} - 1 + \frac{1}{3} \Lambda r^2} - \frac{\Lambda}{2K} r^2 d\theta^2 - \frac{\Lambda}{2K} r^2 \sin^2\theta \, d\phi^2$$

where we can omit the common factor $= -\frac{\Lambda}{2K}$ and finally get if we choose as before $A = -2m$ and $k^2 = -1$

$$d\sigma^2 = \frac{dr^2}{1 - \frac{2m}{r} - \frac{1}{3} \Lambda r^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

11.5 [t] [u]The Schwarzschild space-time in Kruskal Coordinates.

11.5.1 The Kruskal coordinates $r > 2m$

$$u = e^{\frac{r}{4m}} \sqrt{\frac{r}{2m} - 1} \cosh\frac{t}{4m}$$

$$v = e^{\frac{r}{4m}} \sqrt{\frac{r}{2m} - 1} \sinh\frac{t}{4m}$$

where

$$u^2 - v^2 = e^{\frac{r}{2m}} \left(\frac{r}{2m} - 1\right)$$

We calculate

$$\frac{\partial u}{\partial t} = \frac{1}{4m} e^{\frac{r}{4m}} \sqrt{\frac{r}{2m} - 1} \sinh\frac{t}{4m} = \frac{v}{4m}$$

$$\frac{\partial v}{\partial t} = \frac{1}{4m} e^{\frac{r}{4m}} \sqrt{\frac{r}{2m} - 1} \cosh\frac{t}{4m} = \frac{u}{4m}$$

$$\frac{\partial u}{\partial r} = \frac{1}{4m} e^{\frac{r}{4m}} \sqrt{\frac{r}{2m} - 1} \cosh\frac{t}{4m} + e^{\frac{r}{4m}} \frac{1}{2m} \frac{1}{2\sqrt{\frac{r}{2m} - 1}} \cosh\frac{t}{4m} = \frac{u}{4m} \left(\frac{1}{1 - \frac{2m}{r}}\right)$$

$$\frac{\partial v}{\partial r} = \frac{1}{4m} e^{\frac{r}{4m}} \sqrt{\frac{r}{2m} - 1} \sinh\frac{t}{4m} + e^{\frac{r}{4m}} \frac{1}{2m} \frac{1}{2\sqrt{\frac{r}{2m} - 1}} \sinh\frac{t}{4m} = \frac{v}{4m} \left(\frac{1}{1 - \frac{2m}{r}}\right)$$

Now we can use the chain rule

$$du = \frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial r} dr$$

$$dv = \frac{\partial v}{\partial t}dt + \frac{\partial v}{\partial r}dr$$

Written as a matrix

$$\begin{Bmatrix} du \\ dv \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial t} & \frac{\partial u}{\partial r} \\ \frac{\partial v}{\partial t} & \frac{\partial v}{\partial r} \end{Bmatrix} \begin{Bmatrix} dt \\ dr \end{Bmatrix}$$

With the inverse

$$\begin{Bmatrix} dt \\ dr \end{Bmatrix} =^{16} \begin{Bmatrix} \frac{\partial u}{\partial t} & \frac{\partial u}{\partial r} \\ \frac{\partial v}{\partial t} & \frac{\partial v}{\partial r} \end{Bmatrix}^{-1} \begin{Bmatrix} du \\ dv \end{Bmatrix} = \frac{1}{\frac{\partial u}{\partial t}\frac{\partial v}{\partial r} - \frac{\partial u}{\partial r}\frac{\partial v}{\partial t}} \begin{Bmatrix} \frac{\partial v}{\partial r} & -\frac{\partial u}{\partial r} \\ -\frac{\partial v}{\partial t} & \frac{\partial u}{\partial t} \end{Bmatrix} \begin{Bmatrix} du \\ dv \end{Bmatrix}$$

$$= \frac{1}{\left(\frac{v}{4m}\right)^2 \left(\frac{1}{1-\frac{2m}{r}}\right) - \left(\frac{u}{4m}\right)^2 \left(\frac{1}{1-\frac{2m}{r}}\right)} \begin{Bmatrix} \frac{v}{4m}\left(\frac{1}{1-\frac{2m}{r}}\right) & -\frac{u}{4m}\left(\frac{1}{1-\frac{2m}{r}}\right) \\ -\frac{u}{4m} & \frac{v}{4m} \end{Bmatrix} \begin{Bmatrix} du \\ dv \end{Bmatrix}$$

$$= \frac{4m}{v^2-u^2} \begin{Bmatrix} v & -u \\ -u\left(1-\frac{2m}{r}\right) & v\left(1-\frac{2m}{r}\right) \end{Bmatrix} \begin{Bmatrix} du \\ dv \end{Bmatrix}$$

$$\Rightarrow \quad dt = \frac{4m}{v^2-u^2}(vdu - udv)$$

$$\Rightarrow \quad dt^2 = \frac{16m^2}{(v^2-u^2)^2}(v^2du^2 + u^2dv^2 - 2uvdudv)$$

$$\Rightarrow \quad dr = \frac{4m}{v^2-u^2}\left(1-\frac{2m}{r}\right)(-udu + vdv)$$

$$\Rightarrow \quad dr^2 = \frac{16m^2}{(v^2-u^2)^2}\left(1-\frac{2m}{r}\right)^2 (u^2du^2 + v^2dv^2 - 2uvdudv)$$

Next we find

$$\left(1-\frac{2m}{r}\right)dt^2 = \left(1-\frac{2m}{r}\right)\frac{16m^2}{(v^2-u^2)^2}(v^2du^2 + u^2dv^2 - 2uvdudv\,)$$

$$\left(1-\frac{2m}{r}\right)^{-1} dr^2 = \left(1-\frac{2m}{r}\right)\frac{16m^2}{(v^2-u^2)^2}(u^2du^2 + v^2dv^2 - 2uvdudv)$$

Substituting into the Schwarzschild line element

$$\begin{aligned} ds^2 &= \left(1-\frac{2m}{r}\right)dt^2 - \left(1-\frac{2m}{r}\right)^{-1} dr^2 - r^2(d\theta + \sin^2\theta\, d\phi) \\ &= \left(1-\frac{2m}{r}\right)\frac{16m^2}{(v^2-u^2)^2}\left(v^2du^2 + u^2dv^2 - (u^2du^2 + v^2dv^2)\right) - r^2(d\theta + \sin^2\theta\, d\phi) \\ &= \left(1-\frac{2m}{r}\right)\frac{16m^2}{(u^2-v^2)}(dv^2 - dr^2) - r^2(d\theta + \sin^2\theta\, d\phi) \\ &= 16m^2\left(1-\frac{2m}{r}\right)e^{-\frac{r}{2m}}\left(\frac{r}{2m}-1\right)^{-1}(du^2 - dv^2) - r^2(d\theta + \sin^2\theta\, d\phi) \\ &= \frac{32m^3}{r}e^{-\frac{r}{2m}}(dv^2 - du^2) - r^2(d\theta + \sin^2\theta\, d\phi) \end{aligned}$$

11.5.2 The Kruskal coordinates $r < 2m$

We use the same method as before

[16] $\begin{Bmatrix} a & b \\ c & d \end{Bmatrix}^{-1} = \frac{1}{ad-bc}\begin{Bmatrix} d & -b \\ -c & a \end{Bmatrix}$

$$
\begin{aligned}
u &= e^{\frac{r}{4m}}\sqrt{1-\frac{r}{2m}}\sinh\frac{t}{4m} \\
v &= e^{\frac{r}{4m}}\sqrt{1-\frac{r}{2m}}\cosh\frac{t}{4m} \\
u^2-v^2 &= e^{\frac{r}{2m}}\left(\frac{r}{2m}-1\right) \\
\frac{\partial u}{\partial t} &= \frac{1}{4m}e^{\frac{r}{4m}}\sqrt{1-\frac{r}{2m}}\cosh\frac{t}{4m} = \frac{v}{4m} \\
\frac{\partial v}{\partial t} &= \frac{1}{4m}e^{\frac{r}{4m}}\sqrt{1-\frac{r}{2m}}\sinh\frac{t}{4m} = \frac{u}{4m} \\
\frac{\partial u}{\partial r} &= \frac{1}{4m}e^{\frac{r}{4m}}\sqrt{1-\frac{r}{2m}}\sinh\frac{t}{4m} - e^{\frac{r}{4m}}\frac{1}{2m}\frac{1}{2\sqrt{1-\frac{r}{2m}}}\sinh\frac{t}{4m} = \frac{u}{4m}\left(\frac{1}{\frac{2m}{r}-1}\right) \\
\frac{\partial v}{\partial r} &= \frac{1}{4m}e^{\frac{r}{4m}}\sqrt{1-\frac{r}{2m}}\cosh\frac{t}{4m} - e^{\frac{r}{4m}}\frac{1}{2m}\frac{1}{2\sqrt{1-\frac{r}{2m}}}\cosh\frac{t}{4m} = \frac{v}{4m}\left(\frac{1}{\frac{2m}{r}-1}\right) \\
\begin{Bmatrix} dt \\ dr \end{Bmatrix} &= \begin{Bmatrix} \frac{\partial u}{\partial t} & \frac{\partial u}{\partial r} \\ \frac{\partial v}{\partial t} & \frac{\partial v}{\partial r} \end{Bmatrix}^{-1} \begin{Bmatrix} du \\ dv \end{Bmatrix} = \frac{1}{\frac{\partial u}{\partial t}\frac{\partial v}{\partial r}-\frac{\partial u}{\partial r}\frac{\partial v}{\partial t}} \begin{Bmatrix} \frac{\partial v}{\partial r} & -\frac{\partial u}{\partial r} \\ -\frac{\partial v}{\partial t} & \frac{\partial u}{\partial t} \end{Bmatrix} \begin{Bmatrix} du \\ dv \end{Bmatrix} \\
&= \frac{1}{\left(\frac{v}{4m}\right)^2\left(\frac{1}{\frac{2m}{r}-1}\right) - \left(\frac{u}{4m}\right)^2\left(\frac{1}{\frac{2m}{r}-1}\right)} \begin{Bmatrix} \frac{v}{4m}\left(\frac{1}{\frac{2m}{r}-1}\right) & -\frac{u}{4m}\left(\frac{1}{\frac{2m}{r}-1}\right) \\ -\frac{u}{4m} & \frac{v}{4m} \end{Bmatrix} \begin{Bmatrix} du \\ dv \end{Bmatrix} \\
&= \frac{4m}{v^2-u^2}\begin{Bmatrix} v & -u \\ -u\left(\frac{2m}{r}-1\right) & v\left(\frac{2m}{r}-1\right) \end{Bmatrix} \begin{Bmatrix} du \\ dv \end{Bmatrix} \\
\Rightarrow \quad dt &= \frac{4m}{v^2-u^2}(vdu-udv) \\
\Rightarrow \quad dt^2 &= \frac{16m^2}{(v^2-u^2)^2}(v^2du^2+u^2dv^2-2uvdudv) \\
\Rightarrow \quad dr &= \frac{4m}{v^2-u^2}\left(\frac{2m}{r}-1\right)(-udu+vdv) \\
\Rightarrow \quad dr^2 &= \frac{16m^2}{(v^2-u^2)^2}\left(\frac{2m}{r}-1\right)^2(u^2du^2+v^2dv^2-2uvdudv)
\end{aligned}
$$

$$
\left(1-\frac{2m}{r}\right)dt^2 = \left(1-\frac{2m}{r}\right)\frac{16m^2}{(v^2-u^2)^2}(v^2du^2+u^2dv^2-2uvdudv\ )
$$

$$
\left(1-\frac{2m}{r}\right)^{-1}dr^2 = \left(1-\frac{2m}{r}\right)\frac{16m^2}{(v^2-u^2)^2}(u^2du^2+v^2dv^2-2uvdudv)
$$

$$
\begin{aligned}
\Rightarrow \quad ds^2 &= \left(1-\frac{2m}{r}\right)dt^2 - \left(1-\frac{2m}{r}\right)^{-1}dr^2 - r^2(d\theta+\sin^2\theta\, d\phi) \\
&= \frac{32m^3}{r}e^{-\frac{r}{2m}}(dv^2-du^2) - r^2(d\theta+\sin^2\theta\, d\phi)
\end{aligned}
$$

11.6 [v]Eddington-Finkelstein Coordinates – Avoiding the singularity

In order to avoid problems in the $r = 2m$ singularity in the Schwarzschild space-time we can use a transformation of coordinates: The Eddington-Finkelstein coordinates, which introduces a new coordinate $v(t, r)$ defined by

$$t = v - r - 2m\ln\left|\frac{r}{2m} - 1\right|$$

Where the numerical value ensures that the transformation is valid both outside and inside the horizon $r = 2m$. Also notice that $t \to \infty$ when $r \to 2m$ which we saw in the Schwarzschild solution.

11.6.1 [w]The Line-element

$$t = v - r - 2m\ln\left|\frac{r}{2m} - 1\right|$$

$$\Rightarrow \quad dt = dv - dr - d\left(2m\ln\left|\frac{r}{2m} - 1\right|\right) = dv - dr - \frac{1}{\frac{r}{2m} - 1}dr = dv - \frac{1}{1 - \frac{2m}{r}}dr$$

$$\Rightarrow \quad dt^2 = \left(dv - \frac{1}{1 - \frac{2m}{r}}dr\right)^2 = dv^2 + \left(\frac{1}{1 - \frac{2m}{r}}\right)^2 dr^2 - 2\frac{1}{1 - \frac{2m}{r}}dvdr$$

Substituting this into the Schwarzschild line-element

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$= {}^{17\ 18} - \left(1 - \frac{2m}{r}\right)dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

11.6.2 [x]The Radial Null Geodesics

The radial null geodesics implies that $ds^2 = d\theta = d\phi = 0$

$$\Rightarrow \quad 0 = -\left(1 - \frac{2m}{r}\right)dv^2 + 2dvdr = \left(-\left(1 - \frac{2m}{r}\right)dv + 2dr\right)dv$$

$dv = 0$:

This solution leads to the familiar ingoing null rays in the Schwarzschild solution

$$dt^2 = \left(\frac{1}{1 - \frac{2m}{r}}\right)^2 dr^2$$

$r = 2m$:

$$\Rightarrow \quad \frac{dv}{dr} = 0$$

which are light rays that are neither ingoing nor outgoing.

Next we solve the differential equation:

$$0 = -\left(1 - \frac{2m}{r}\right)dv + 2dr$$

$$\Rightarrow \quad \frac{dv}{dr} = 2\left(1 - \frac{2m}{r}\right)^{-1} = \frac{2r}{r - 2m}$$

Notice if $r > 2m$ then $\frac{dv}{dr} > 0$ and the radial light rays are outgoing

Notice if $r < 2m$ then $\frac{dv}{dr} < 0$ and the radial light rays are ingoing

---

[17] $= -\left(1 - \frac{2m}{r}\right)\left(dv^2 + \left(\frac{1}{1 - \frac{2m}{r}}\right)^2 dr^2 - 2\frac{1}{1 - \frac{2m}{r}}dvdr\right) + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) =$

[18] Notice: For $r \to \infty$ the Eddington-Finkelstein space-time approaches the flat space-time: $ds^2 = -dv^2 + 2dvdr + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi$ which is the same as flat Minkowsky space-time.

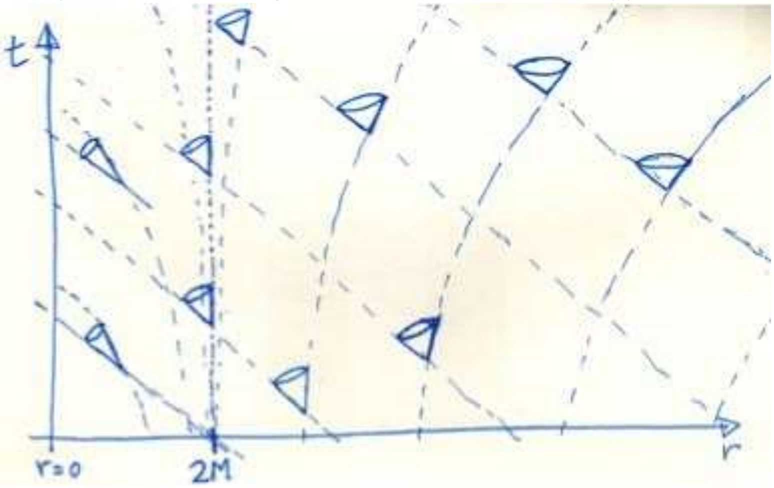
Solving

$$dv = \frac{2r}{r-2m}dr$$

$$\Rightarrow \quad v - v_0 \overset{[19]}{=} 2(r + 2m\ln|r-2m|)$$

In summary: For $r > 2m$ we have both ingoing and outgoing radial light rays, the ingoing light rays can either pass the event horizon or stop at the event horizon. The event horizon is so to speak a one way membrane and no light can go out again. If $r < 2m$ all the light rays are ingoing.

As illustrated by the drawing below[y] the radial null-geodesic points both outward and inward outside the event horizon but only inward inside the event horizon. Also notice that for an astronaut his trajectory always lies inside the light cone so the same rules applies for him.



11.7 The Kerr metric

11.7.1 [z]The Kerr-Newman geometry

A more general metric is the Kerr-Newman geometry, corresponding to a simultaneously rotating and electrically charged black hole of mass m, charge Q and angular momentum S.

$$ds^2 \overset{[20\ 21\ 22\ 23\ 24\ 25]}{=} \frac{\Delta}{\Sigma}(dt - a\sin^2\theta\, d\phi)^2 - \frac{\sin^2\theta}{\Sigma}\left((r^2+a^2)d\phi - adt\right)^2 - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2$$

$$= \left(1 - \frac{2mr}{\Sigma} + \frac{Q^2}{\Sigma}\right)dt^2 + \frac{4amr\sin^2\theta}{\Sigma}dtd\phi - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2 - \left(r^2 + a^2 + \frac{2a^2mr\sin^2\theta}{\Sigma}\right)\sin^2\theta\, d\phi^2$$

$$\Delta = r^2 - 2mr + a^2 + Q^2$$

---

[19] (Spiegel, 1990) (14.60) $\int \frac{xdx}{ax+b} = \frac{x}{a} - \frac{b}{a^2}\ln(ax+b)$

[20] $= \frac{\Delta}{\Sigma}[dt^2 + (a\sin^2\theta)^2 d\phi^2 - 2a\sin^2\theta\, dtd\phi] - \frac{\sin^2\theta}{\Sigma}[a^2dt^2 + (r^2+a^2)^2d\phi^2 - 2a(r^2+a^2)dtd\phi] - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2 = \frac{1}{\Sigma}[\Delta - a^2\sin^2\theta]dt^2 + \frac{1}{\Sigma}[-\Delta 2a\sin^2\theta + 2\sin^2\theta\, a(r^2+a^2)]dtd\phi - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2 + \frac{1}{\Sigma}[\Delta(a\sin^2\theta)^2 - \sin^2\theta\,(r^2+a^2)^2]d\phi^2 =$

[21] $= \frac{1}{\Sigma}[\Delta - a^2\sin^2\theta]dt^2 + \frac{2a\sin^2\theta}{\Sigma}[-\Delta + (r^2+a^2)]dtd\phi - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2 + \frac{\sin^2\theta}{\Sigma}[\Delta a^2\sin^2\theta - (r^2+a^2)^2]d\phi^2 =$

[22] $= \frac{1}{\Sigma}[(\Sigma - 2mr + a^2\sin^2\theta + Q^2) - a^2\sin^2\theta]dt^2 + \frac{2a\sin^2\theta}{\Sigma}[-(\Sigma - 2mr + a^2\sin^2\theta) + (\Sigma + a^2\sin^2\theta)]dtd\phi - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2 + \frac{\sin^2\theta}{\Sigma}[(\Sigma - 2mr + a^2\sin^2\theta)a^2\sin^2\theta - (\Sigma + a^2\sin^2\theta)^2]d\phi^2 =$

[23] $= \left(1 - \frac{2mr}{\Sigma} + \frac{Q^2}{\Sigma}\right)dt^2 + \frac{4amr\sin^2\theta}{\Sigma}dtd\phi - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2 + \frac{\sin^2\theta}{\Sigma}[\Sigma a^2\sin^2\theta - 2mra^2\sin^2\theta + (a^2\sin^2\theta)^2 - (\Sigma^2 + (a^2\sin^2\theta)^2 + 2\Sigma a^2\sin^2\theta)]d\phi^2$

[24] $= \left(1 - \frac{2mr}{\Sigma} + \frac{Q^2}{\Sigma}\right)dt^2 + \frac{4amr\sin^2\theta}{\Sigma}dtd\phi - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2 + \frac{\sin^2\theta}{\Sigma}[-\Sigma a^2\sin^2\theta - 2mra^2\sin^2\theta - \Sigma^2]d\phi^2 =$

[25] $= \left(1 - \frac{2mr}{\Sigma} + \frac{Q^2}{\Sigma}\right)dt^2 + \frac{4amr\sin^2\theta}{\Sigma}dtd\phi - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2 - \sin^2\theta\left[a^2\sin^2\theta + \frac{2mra^2\sin^2\theta}{\Sigma} + \Sigma\right]d\phi^2 =$

$$\Sigma = r^2 + a^2\cos^2\theta = r^2 + a^2 - a^2\sin^2\theta = \Delta - Q^2 + 2mr - a^2\sin^2\theta$$
$$a = \frac{S}{m}$$

11.7.1.1 $Q = 0$

In the case of $Q = 0$ we see immediately that the Kerr-Newman geometry reduces to the Kerr geometry describing a non-charged rotating black hole.

$$ds^2 = \left(1 - \frac{2mr}{\Sigma}\right)dt^2 + \frac{4amr\sin^2\theta}{\Sigma}dtd\phi - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2 - \left(r^2 + a^2 + \frac{2a^2mr\sin^2\theta}{\Sigma}\right)\sin^2\theta\, d\phi^2$$
$$\Delta = r^2 - 2mr + a^2$$
$$\Sigma = r^2 + a^2\cos^2\theta = r^2 + a^2 - a^2\sin^2\theta = \Delta + 2mr - a^2\sin^2\theta$$
$$a = \frac{S}{m}$$

11.7.1.2 $S = 0$

In the case of $S = 0$ we see immediately that the Kerr-Newman geometry reduces to the Reissner-Nordstrøm geometry describing a charged non-rotating black hole.

$$ds^2 =^{26} \left(1 - \frac{2mr}{\Sigma} + \frac{Q^2}{\Sigma}\right)dt^2 - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2 - r^2\sin^2\theta\, d\phi^2$$
$$= \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)dt^2 - \frac{1}{1 - \frac{2m}{r} + \frac{Q^2}{r^2}}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2$$
$$\Delta = r^2 - 2mr + Q^2$$
$$\Sigma = r^2$$

11.7.1.3 $Q = 0$ and $S = 0$

In the case of $Q = 0$ and $S = 0$ we see immediately that the Kerr-Newman geometry reduces to the Schwarzschild geometry describing a non-charged non-rotating black hole.

$$ds^2 =^{27} \left(1 - \frac{2m}{r}\right)dt^2 - \frac{1}{1 - \frac{2m}{r}}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2$$
$$\Delta = r^2 - 2mr$$
$$\Sigma = r^2$$

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[26] $= \left(1 - \frac{2mr}{\Sigma} + \frac{Q^2}{\Sigma}\right)dt^2 + \frac{4amr\sin^2\theta}{\Sigma}dtd\phi - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2 - \left(r^2 + a^2 + \frac{2a^2mr\sin^2\theta}{\Sigma}\right)\sin^2\theta\, d\phi^2 =$

[27] $= \left(1 - \frac{2mr}{\Sigma}\right)dt^2 + \frac{4amr\sin^2\theta}{\Sigma}dtd\phi - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2 - \left(r^2 + a^2 + \frac{2a^2mr\sin^2\theta}{\Sigma}\right)\sin^2\theta\, d\phi^2 =$

[a] (A.S.Eddington, 1924, p. 83)
[b] (McMahon, 2006, s. 231)
[c] (McMahon, 2006, s. 204)
[d] (McMahon, 2006, s. 204)
[e] (McMahon, 2006, s. 211)
[f] (McMahon, 2006, s. 218)
[g] (Hartle, 2003, p. 187)
[h] (Hartle, 2003, p. 166)
[i] (McMahon, 2006, s. 216)
[j] (McMahon, 2006, s. 215), (Hartle, 2003, p. 546)
[k] (McMahon, 2006, s. 231), (Hartle, 2003, s. 546)
[l] (d'Inverno, 1992, p. 90)
[m] (McMahon, 2006, s. 216), equation (10.35)
[n] (Hartle, 2003, p. 183)
[o] (McMahon, 2006, s. 231-32)
[p] (McMahon, 2006, s. 277)
[q] (McMahon, 2006, s. 277)
[r] (McMahon, 2006, s. 277)
[s] (McMahon, 2006, s. 260)
[t] (McMahon, 2006, s. 242)
[u] (Hartle, An Introduction to Einstein's General Relativity, 2003, s. 279)
[v] (McMahon, 2006, s. 239), (Hartle, An Introduction to Einstein's General Relativity, 2003, s. 256), (Carroll, 2004, p. 221)
[w] (Hartle, An Introduction to Einstein's General Relativity, 2003, s. 277)
[x] (Hartle, An Introduction to Einstein's General Relativity, 2003, s. 259)
[y] lhttps://physics.stackexchange.com/questions/57726/future-light-cones-inside-black-hole
[z] (C.W.Misner, 1973) chapter 33

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