Susan Larsen 

# Content



General relativity is the description of the smallest changes and the largest entities. The best way to describe this I found in a quote by Roger Penrose:

*Calculus is built from two basic ingredients: differentiation and integration. Differentiation is concerned with velocities, accelerations, the slopes and curvature of curves and surfaces… These are the rates at which things change, and they are quantities defined locally, in terms of structure or behavior in the tiniest neighborhood of single points. Integration on the other hand, is concerned with areas and volumes, with centers of gravity…These are things which involves measures of totality … and they are not defined merely by what is going on in the local or infinitesimal neighborhoods of individual points.*

(Penrose, 2004, s. 103)

# 1 Introduction

We begin with a few important facts:

- The gravitational force is always attractive.
- The gravitational force is a long range force without boundaries.
- A gravitational field is created from all kinds of masses and (because $E = mc^2$) all kinds of energies.
- A mass/energy creates a curvature of the four-dimensional space-time, where masses (test masses) moves along straight lines (geodesics).

<u>Calculus:</u>



http://physicssusan.mono.net logik.susan@gmail.com

Working with GR means working with differential equations at four different levels. It can be very useful - whenever one comes across a GR calculation - to keep in mind, on which level you are working. The four levels of differential equations are:

1. The metric or line-element:

$$ds^2 = g_{ab}dx^a dx^b$$

Example: Gravitational red shift[a]:

$$d\tau = \sqrt{1-\frac{2m}{r}}dt$$

Light emitted upward in a gravitational field, from an observer located at some inner radius $r_1$ to an observer positioned at some outer radius $r_2$

$$\alpha = \frac{\sqrt{1-\frac{2m}{r_2}}}{\sqrt{1-\frac{2m}{r_1}}}$$

2. Killing's equations are conservation equations:

$$\nabla_b X_a + \nabla_a X_b = 0$$

If you move along the direction of a Killing vector, then the metric does not change. This leads to conserved quantities: A free particle moving in a direction where the metric does not change will not fell any forces. If $X$ is a Killing vector,$u = \left(\frac{dt}{d\tau},\frac{dr}{d\tau},\frac{d\theta}{d\tau},\frac{d\phi}{d\tau}\right)$ is the particle four velocity and $p$ is the particle four impulse, then $X \cdot u = g_{ab}X^a u^b = const$ and $X \cdot p = g_{ab}X^a p^b = const$ along a geodesic[b].

Translational symmetry: Whenever $\partial_{\sigma_*} g_{\mu\nu} = 0$ for some fixed $\sigma_*$ (but for all $\mu$ and $\nu$) there will be a symmetry under translation along $x^{\sigma_*}$[c].

Example: Killing vectors in the Schwarzschild metric[d].

The Killing vector that corresponds to the independence of the metric of $t$ is $\xi = (1,0,0,0)$ and of $\phi$ is $\eta = (0,0,0,1)$. The conserved energy per unit rest mass: $e = -\overline{\xi} \cdot \overline{u} = -g_{ab}\xi^a u^b = -g_{tt} \cdot 1 \cdot \frac{dt}{d\tau} = -\left(1-\frac{2m}{r}\right)\frac{dt}{d\tau}$. The conserved angular momentum per unit rest mass $l = \overline{\eta} \cdot \overline{u} = g_{ab}\eta^a u^b = g_{\phi\phi} \cdot 1 \cdot \frac{d\phi}{d\tau} = -r^2 \sin^2\theta \frac{d\phi}{d\tau} = -r^2 \frac{d\phi}{d\tau}$ for $\theta = \frac{\pi}{2}$

3. The Geodesic equation leads to equations of motion:

$$K = \frac{1}{2}g_{ab}\dot{x}^a\dot{x}^b$$

$$\frac{\partial K}{\partial x^a} = \frac{d}{ds}\left(\frac{\partial K}{\partial \dot{x}^a}\right)$$

$$\frac{d^2x^a}{ds^2} + \Gamma^a{}_{bc}\frac{dx^b}{ds}\frac{dx^c}{ds} = 0$$

Example: Planetary orbits[e]

Manipulating the geodesic equations of the Schwarzschild metric leads to the following equation

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{k^2-1}{h^2} + \frac{2m}{h^2}u + 2mu^3$$

Which can be interpreted in terms of elliptic functions, $u = \frac{1}{r}$, and $h$ and $k$ are constants of integration.

4. The Einstein equations are equations describing the spacetime.

$$G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R$$

$$8\pi G T_{ab} = G_{ab} \pm g_{ab}\Lambda$$

If $n = 4$, $R_{abcd}$ has twenty independent component – ten of which are given by $R_{ab}$ and the remaining ten by the Weyl tensor[f].

Example: The Friedmann equations

A homogenous, isotropic and expanding universe described by the Robertson-Walker space-time[g], in this case the Einstein equations becomes the Friedmann equations:

$$\begin{aligned} 8\pi\rho &= \frac{3}{a^2}(k+\dot{a}^2)-\Lambda \\ -8\pi P &= 2\frac{\ddot{a}}{a}+\frac{1}{a^2}(k+\dot{a}^2)-\Lambda \end{aligned}$$

## 1.1 Space-times

This document includes many different space-time examples. In order to keep track of them I have made this list- in alphabetical order, so that you can see in which chapter you can find the space-time you are looking for.

| Space-time | | Line-element | Chap-ter |
|---|---|---|---|
| Aichelburg-Sexl Solution | $ds^2$ | $= 4\mu\log(x^2+y^2)\,du^2 + 2dudr - \mathrm{dx}^2 - dy^2$ | 14 |
| Alcubierre drive warp drive | $ds^2$ | $= -dt^2 + [dx - V_s(t)f(r_s)dt]^2 + dy^2 + dz^2$ | 4 |
| Anti de Sitter space-time | $ds^2$ | $= -dt^2 + \cos^2(t)\,dr^2 + \cos^2(t)\sinh^2(r)\,d\theta^2 + \cos^2(t)\sinh^2(r)\sin^2\theta\,d\phi^2$ | 8 |
| Brinkman space-time | $ds^2$ | $= H(u,x,y)du^2 + 2dudv - dx^2 - dy^2$ | 14 |
| Classically anti-de-Sitter space-time | $ds^2$ | $= -\cosh^2(r)\,dt^2 + dr^2 + \sinh^2(r)\,d\theta^2 + \sinh^2(r)\sin^2\theta\,d\phi^2$ | 4 |
| Colliding gravitational waves | $ds^2$ | $= \delta(u)(X^2 - Y^2)du^2 + 2dudr - dX^2 - dY^2$ | 14 |
| Collision of a gravitational wave with a electromagnetic wave | $ds^2$ | $= 2dudv - \cos^2 av\,(dx^2 + dy^2)$ | 14 |
| Cylindrical coordinates | $ds^2$ | $= dr^2 + r^2d\phi^2 + dz^2$ | 4 |
| De Sitter space-time | $ds^2$ | $= -dt^2 + a(t)^2\big(d\theta^2 + \sin^2\theta\,(d\phi^2 + \sin^2\phi\,d\psi^2)\big)$ | 8 |
| Eddington-Finkelstein coordinates | $ds^2$ | $= -\left(1 - \frac{2m}{r}\right)dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)$ | 11 |
| Egg geometry | $ds^2$ | $= a^2[(\cos^2\theta + 4\sin^2\theta)d\theta^2 + \sin^2\theta\,d\phi^2]$ | 2 |
| Einstein cylinder | $ds^2$ | $= -dt^2 + (a_0)^2\big(d\theta^2 + \sin^2\theta\,(d\phi^2 + \sin^2\phi\,d\psi^2)\big)$ | 9 (4), 13 |
| Ellipsoid | $ds^2$ | $= adx^2 + bdy^2 + cdz^2$ | 2 |
| Example – three dimensional space | $ds^2$ | $= (u^2+v^2)du^2 + (u^2+v^2)dv^2 + u^2v^2d\theta^2$ | 6 |
| Example – three dimensional space | $ds^2$ | $= dx^2 + 2xdy^2 + 2ydz^2$ | 6 |
| Example – two dimensional space | $ds^2$ | $= y^2\sin x\,dx^2 + x^2\tan y\,dy^2$ | 6 |
| Example: Four-dimensional space-time | $ds^2$ | $= -(1-Ar^2)^2dt^2 + (1-Ar^2)^2dr^2 + r^2d\theta^2 + r^2\sin^2\theta\,d\phi^2$ | 2 |
| Example: Four-dimensional space-time | $ds^2$ | $= -dt^2 + L^2(t,r)dr^2 + B^2(t,r)d\phi^2 + M^2(t,r)dz^2$ | 7 |
| A Fifth dimension | $ds^2$ | $= -dx^2 + dx^2 + dy^2 + dz^2 + R^2d\Omega^2$ | 2 |
| Flat Minkowsky space-time | $ds^2$ | $= -dt^2 + dx^2 + dy^2 + dz^2$ | 2 |
| Flat Minkowsky space-time in polar coordinates | $ds^2$ | $= -dt^2 + dr^2 + r^2d\theta^2 + r^2\sin^2\theta\,d\phi$ | 2, 10 |
| Flat space-time in Eddington Finkelstein coordinates | $ds^2$ | $= -dv^2 + 2dvdr + r^2d\theta^2 + r^2\sin^2\theta\,d\phi$ | 2 |
| General four-dimensional diagonal metric | $ds^2$ | $= g_{00}dx^0dx^0 + g_{11}dx^1dx^1 + g_{22}dx^2dx^2 + g_{33}dx^3dx^3$ | 2 |


http://physicssusan.mono.net logik.susan@gmail.com

| | | | |
|---|---|---|---|
| Gödel metric | $ds^2$ | $= \frac{1}{2\omega^2}\left((dt + e^x dz)^2 - dx^2 - dy^2 - \frac{1}{2}e^{2x}dz^2\right)$ | 2,9 |
| Gravitationally collapse of an inhomogeneous spherically symmetric dust cloud | $ds^2$ | $= -dt^2 + e^{2b(t,r)}dr^2 + R^2(t,r)d\phi^2$ | 8 |
| Homogenous closed universe | $dS^2$ | $= \frac{1}{1-\left(\frac{r}{a}\right)^2}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$ | 2 |
| Hyperbolsk plane – Poincaré Half-plane | $ds^2$ | $= \frac{1}{y^2}(dx^2 + dy^2)$ | 4,7 |
| Impulsive gravitational wave | $ds^2$ | $= 2dudv - [1 - v\Theta(v)]^2 dx^2 - [1 + v\Theta(v)]^2 dy^2$ | 14 |
| Kerr Spinning black hole | $ds^2$ | $= \left(1 - \frac{2mr}{\Sigma}\right)dt^2 + \frac{4amr\sin^2\theta}{\Sigma}dtd\phi - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2 - \left(r^2 + a^2 + \frac{2a^2mr\sin^2\theta}{\Sigma}\right)\sin^2\theta\, d\phi^2$ | 2, 11 |
| Kruskal coordinates | $ds^2$ | $= \frac{32m^3}{r}e^{-\frac{r}{2m}}(dv^2 - du^2) - r^2(d\theta + \sin^2\theta\, d\phi)$ | 11 |
| Linearized metric | $ds^2$ | $= (\eta_{ab} + \epsilon h_{ab})dx^a dx^b$ | 9, 12, 14 |
| Lorentz hyperboloid | $ds^2$ | $= d\psi^2 + \sinh^2\psi\ d\theta^2 + \sinh^2\psi \sin^2\theta\, d\phi^2$ | 7, 13 |
| Nariai space-time | $ds^2$ | $= -\Lambda v^2 du^2 + 2dudv - \frac{1}{\Omega^2}(dx^2 + dy^2)$ | 14 |
| North pole | $ds^2$ | $= \left(1 - \frac{y^2}{3a^2}\right)dx^2 + \frac{xy}{3a^2}dxdy + \left(1 - \frac{x^2}{3a^2}\right)dy^2$ | 2 |
| Peanut geometry | $ds^2$ | $= a^2 d\theta^2 + a^2 f^2(\theta)d\phi^2 f(\theta)$ | 2 |
| Penrose-Kahn metric | $ds^2$ | $= 2dudv - (1-u)^2 dx^2 - (1+u)^2 dy^2$ | 14 |
| Plane waves: $h_{ab} = h_{ab}(t - z)$ | $ds^2$ | $= (\eta_{ab} + \epsilon h_{ab})dx^a dx^b$ | 14 |
| Poincaré metric | $ds^2$ | $= \frac{4}{1 - x^2 - y^2}(dx^2 + dy^2)$ | 7 |
| Reissner-Nordström spacetime | $ds^2$ | $= \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)dt^2 - \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2\sin^2\theta\, d\phi^2$ | 10 |
| Rindler metric | $ds^2$ | $= -X^2 dT^2 + dX^2$ | 2,3,4,7 |
| Robertson Walker space-time | $ds^2$ | $= -dt^2 + \frac{a^2(t)}{1 - kr^2}dr^2 + a^2(t)r^2 d\theta^2 + a^2(t)r^2\sin^2\theta\, d\phi^2$ | 13 |
| Rosen line element | $ds^2$ | $= dUdV - a^2(U)dx^2 - b^2(U)dy^2$ | 14 |
| Schwarzschild metric | $ds^2$ | $= -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$ | 10, 11, 12 |
| Schwarzschild metric: general solution | $ds^2$ | $= e^{2v(r)}dt^2 - e^{2\lambda(r)}dr^2 - r^2 d\theta^2 - r^2\sin^2\theta\, d\phi^2$ | 11 |

| | | | |
|---|---|---|---|
| Schwarzschild metric: general time dependent | $ds^2$ | $= e^{2\nu(t,r)}dt^2 - e^{2\lambda(t,r)}dr^2 - r^2d\theta^2 - r^2\sin^2\theta\, d\phi^2$ | 11 |
| Schwarzschild metric: The effect of the cosmological constant over the scale of the solar system | $ds^2$ | $= \left(1+\frac{1}{3}\Lambda r^2\right)dt^2 - \frac{dr^2}{1+\frac{1}{3}\Lambda r^2}$ | 11 |
| Schwarzschild metric: The general Schwarzschild metric with nonzero cosmological constant. | $ds^2$ | $= f(r)dt^2 - \frac{1}{f(r)}dr^2 - r^2d\theta^2 - r^2\sin^2\theta\, d\phi^2$ | 11 |
| Schwarzschild metric: With two different masses | $ds^2$ | $= \left(1-\frac{2m_1}{r}\right)dt^2 - \left(1-\frac{2m_2}{r}\right)^{-1}dr^2 - r^2d\theta^2 - r^2\sin^2\theta\, d\phi^2$ | 12 |
| Schwarzschild metric: $\theta = \frac{\pi}{2}$ | $ds^2$ | $= -\left(1-\frac{2m}{r}\right)dt^2 + \left(1-\frac{2m}{r}\right)^{-1}dr^2 + r^2d\phi^2$ | 11 |
| Spatial part of a homogenous, isotropic metric | $d\sigma^2$ | $= \frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2\theta\, d\phi$ | 13 |
| Static Weak field | $ds^2$ | $= -\left(1+\frac{2\Phi(x^i)}{c^2}\right)(cdt)^2 + \left(1-\frac{2\Phi(x^i)}{c^2}\right)(dx^2+dy^2 + dz^2)$ | 2 |
| Taub-nut space-time | $ds^2$ | $= -\frac{dt^2}{U^2(t)} + (2l)^2U^2(t)(dr+\cos\theta\, d\phi)^2 + V^2(t)(d\theta^2+\sin^2\theta\, d\phi^2)$ | 7 |
| Three dimensional flat space-time | $ds^2$ | $= -(cdt)^2 + dx^2 + dy^2$ | 3 |
| Three-dimensional flat space in polar coordinates | $ds^2$ | $= dr^2 + r^2d\theta^2 + r^2\sin^2\theta\, d\phi^2$ | 2,7, 13 |
| Three-dimensional flat space-time in polar coordinates | $ds^2$ | $= -dt^2 + dr^2 + r^2d\phi^2$ | 4 |
| Tolman-Bondi-de Sitter space-time | $ds^2$ | $= dt^2 - e^{-2\psi(t,r)}dr^2 - R^2(t,r)d\theta^2 - R^2(t,r)\sin^2\theta\, d\phi^2$ | 7 |
| Two interacting waves | $ds^2$ | $= 2dudv - \cos^2 av\, dx^2 - \cosh^2 av\, dy^2$ | 14 |
| Two-dimensional flat space | $dS^2$ | $= dx^2 + dy^2$ | 4 |
| Two-dimensional flat space in polar coordinates | $dS^2$ | $= dr^2 + r^2d\phi^2$ | 2,5 |
| Two-dimensional flat space-time | $ds^2$ | $= -dt^2 + dx^2$ | 3,4 |
| Two-dimensional sphere with radius $a$ | $ds^2$ | $= a^2d\theta^2 + a^2\sin^2\theta\, d\phi^2$ | 5 |
| Two-dimensional unit sphere | $ds^2$ | $= d\theta^2 + \sin^2\theta\, d\phi^2$ | 7 |
| Worm hole geometry | $ds^2$ | $= -dt^2 + dr^2 + (b^2+r^2)(d\theta^2 + \sin^2\theta\, d\phi^2)$ | 4 |

## 2 The Metric tensor and Vector Transformations.

See separate document

## 3 Four vectors and four velocity

See separate document

# 4 Christoffel Symbols, Geodesic Equation and Killing Vectors
See separate document

# 5 Covariant Derivative, Lie Derivative and Killings Equation
See separate document

# 6 The Riemann tensor
See separate document

# 7 Cartan's Structure Equations – a Shortcut to the Einstein equation
See separate document

# 8 The Einstein Field Equations
See separate document

# 9 The Energy-Momentum Tensor
See separate document

# 10 Null Tetrads and the Petrov Classification
See separate document

# 11 The Schwarzschild Solution
See separate document

# 12 Black Holes
See separate document

# 13 Cosmology
See separate document

# 14 Gravitational Waves
See separate document

# 15 Appendix A: Tensor Calculus
See separate document

## 16 Appendix B: Collection of results.

See separate document

## Bibliografi

A.S.Eddington. (1924). *The Mathematical Theory of Relativity.* Cambridge: At the University Press.

Carroll, S. M. (2004). *An Introduction to General Relativity, Spacetime and Geometry.* San Fransisco, CA: Addison Wesley.

d'Inverno, R. (1992). *Introducing Einstein's Relativity.* Oxford: Clarendon Press.

McMahon, D. (2006). *Relativity Demystified.* McGraw-Hill Companies, Inc.

Penrose, R. (2004). *The Road to Reality.* New York: Vintage Books.

---

[a] (McMahon, p. 234)
[b] (McMahon, p. 168)
[c] (Carroll, 2004)
[d] (McMahon, p. 220)
[e] (A.S.Eddington, pp. 85-86)
[f] (d'Inverno, p. 87)
[g] (McMahon, p. 161)